GREGARIOUS AND RECLUSIVE TRIPLES

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1. Triples, products and squares

For the triple of numbers $(1, 3, 8)$, the product of any pair of them is one less than a square. Similarly, the product of any two numbers in the triple $(1, 2, 5)$ is one more than a square. The reader may recognize the numbers in these triples as alternate terms of the Fibonacci sequence, defined by the recursion $f_0 = 0$, $f_1 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for each integer n. The terms with nonnegative even indices are

$$0, 1, 3, 8, 21, 55, 144, 377, 987, \ldots;$$

we find that for each three consecutive terms (x, y, z) in this sequence $xy+1$, $xz+1$ and $yz + 1$ are all squares. Likewise, for each three consecutive terms (x, y, z) in the sequence of Fibonacci numbers with positive odd indices,

$$1, 2, 5, 13, 34, 89, 233, 610, 1597, \ldots,$$

$xy - 1$, $xz - 1$ and $yz - 1$ are all squares. These are familiar Fibonacci properties.

Define a vector (x, y, z) of three integers to be a $k-$**triple** if $xy + k = c^2$, $yz + k = a^2$ and $zx + k = b^2$ for integers k, a, b, c. We have provided examples of $1-$triples and $(-1)-$triples. Both of these are parts of a larger scheme. Consider this table, whose $k = 1$ row is familiar.

$k \downarrow n \rightarrow$	-5	-4	-3	-2	-1	0	1	2	3	4	5
-2	54	19	9	2	3	1	6	11	33	82	219
-1	29	10	5	1	2	1	5	10	29	73	194
0	4	1	1	0	1	1	4	9	25	64	169
1	-21	-8	-3	-1	0	1	3	8	21	55	144
2	-46	-17	-7	-2	-1	1	2	7	17	46	119
3	-71	-26	-11	-3	-2	1	1	6	13	37	94
4	-96	-35	-15	-4	-3	1	0	5	9	28	69

Any three consecutive entries in the row labelled k are a $k-$triple. Suppose that the nth terms in this row is given by $u(k, n)$. You will observe that for these rows, any three consecutive entries constitute a $k-$ triple,

$$u(k, -2) = -k; \qquad u(k, -1) = -k + 1; \qquad u(k, 0) = 1;$$

and also that

$$u(k, n + 3) = 2u(k, n + 2) + 2u(k, n + 1) - u(k, n).$$

The reader is invited to conjecture a general formula for $u(k, n)$ and check out the $k-$triples. (A good place to start is with row $k = 0$ and look at the value of $u(k, n)$ as n increases or decreases by 1.)

In a similar way, row $k = -1$ in the table below reproduced the $(-1)-$triples we have already seen.

$k \downarrow n \rightarrow$	-5	-4	-3	-2	-1	0	1	2	3	4	5
-4	164	61	25	8	5	1	4	5	17	40	109
-3	139	52	21	7	4	1	3	4	13	31	84
-2	114	43	17	6	3	1	2	3	9	22	59
-1	89	34	13	5	2	1	1	2	5	13	34
0	64	25	9	4	1	1	0	1	1	4	9
1	39	16	5	3	0	1	-1	0	-3	-5	-16
2	14	7	1	2	-1	1	-2	-1	-7	-14	-41
3	-11	-2	-3	1	-2	1	-3	-2	-11	-23	-66

Let $v(k,n)$ be the nth element in the kth row. In this extract, we note that the kth row consists of $k-$triples, that

$v(k,-2) = -k+4; \qquad v(k,-1) = v(k,2) = -k+1; \qquad v(k,0) = 0; \qquad v(k,1) = -k;$

and that

$$v(k,n+3) = 2v(k,n+2) + 2v(k,n+1) - v(k,n).$$

Again, the reader is invited to conjecture a general formula for $v(k,n)$ and check out the occurrence of $k-$triples.

We need some definitions. Motivted by the recursion satisfied by $u(k,n)$ and $v(k,n)$, we define the **right associate** of (x,y,z) to be the triple (y,z,w) where $w = 2(y+z) - x$, the **left associate** of the triple (x,y,z) to be $(2(x+y)-z,x,y)$ and the **central associate** of (x,y,z) to be $(x, 2(x+z)-y, z)$.

A $k-$triple is **gregarious** if all its associates are $k-$triples (with the same value of k). A sequence $\{u_n\}$ satisfying the **gregarious** recursion $u_{n+3} = 2u_{n+2}+2u_{n+1}-u_n$ is $k-$**gregarious** if each three consecutive terms constitute a $k-$triple. Each line in the tables is a gregarious sequence.

In what follows, we shall secure our assertions about $\{u(k,n)\}$, $\{v(k,n)\}$, consider the construction of other $k-$ triples and find that not all of them are gregarious. A $k-$triple whose associate are not all $k-$triples is said to be **reclusive**.

2. A general constuction of $k-$triples and $k-$quadruples.

Suppose that x, y and c are arbitrary integers. Define $z = x+y+2c$. Let $k = c^2 - xy$. Then, (x,y,z) is a $k-$triple, since $xz+k = (x+c)^2$ and $yz+k = (y+c)^2$. Let us look at its right associate $(X,Y,Z) = (y,z,w)$, where $w = 2(y+z)-x$ Then $w-(y+z) = y+z-x = 2(y+c)$ and $yz+k = (y+c)^2$. So the triple (X,Y,Z) is formed similarly to (x,y,z) with the role of c played by $y+c$. A similar situation holds with the left associate. $(2(x+y)-z,x,y)$. Thus we can embed (x,y,z) into a $k-$gregarious sequence. The sequences $\{u(k,n)\}$ and $\{v(k,n)\}$ are examples of this construction (check that the value of c works for each triple). $k-$gregarious sequences formed in this way are **super** $k-$**gregarious**.

If we permute the terms of (x,y,z) to (x,z,y), we find that $y = x+z-2(c+x)$ and $xz = [-(c+x)]^2$ and we can embed this triple in another sequence of $k-$triples.

Any triple (x, y, z) of integers for which $z - (x + y)$ is an even number $2c$, is $k-$gregarious with $k = c^2 - xy$. So such triples are as prolific as Fibonacci's rabbits. This table gives some general examples:

	(x, y, z)	(a, b, c)
k		
$r^2 + s^2 + t^2 - 2(rs + st + rt)$	$(2r, 2s, 2t)$	$(-r + s + t, r - s + t, r + s - t)$
$r^2 + s^2 + t^2 - 2(rs + st + rt) - 2r$	$(2r, 2s + 1, 2t + 1)$	$(s + t - r + 1, r - s + t, r + s - t)$

It is natural to ask whether, for any value of k, there are quadruples of numbers for which the product of any pair plus k is a square. The construction just described makes it quite straightforward to answer this in the affirmative. If we extend the triple $(x, y, x+y+2c)$ to the left, we get the quadruple $(x+y-2c, x, y, x+y+2c)$. Since $(x + y - 2c, x, y)$ and $(x, y, x + y + 2c)$ are $k-$triples, it is necessary only to arrange that

$$(x + y - 2c)(x + y + 2c) + k = (x + y)^2 - 4c^2 + (c^2 - xy) = (x^2 + xy + y^2) - 3c^2$$

is equal to d^2 for some integer d. In other words, we need to find numbers expressible in each of the forms $x^2 + xy + y^2$ and $3c^2 + d^2$.

It turns out that the forms $\phi(x, y) = x^2+xy+y^2$ and $\psi(c, d) = 3c^2+d^2$ represent the same set of numbers. When x and y have the same parity, then we can select c and d so that $x = c + d$ and $y = c - d$ to find that $f(x, y) = g(c, d)$. This leads us to the d^2-quadruple $(0, c + d, c - d, 4c)$.

If x and y have different parity, then, wolog, suppose that y is even. Then $\phi(x + y, -x) = \phi(x, y)$ and the relations $x + y = c + d$, $-x = c - d$ lead us to the $(3c^2-2cd)-$quadruple $(d-c, d-c, 2c, d+3c)$. It may appear that these $k-$quadruples may be too facile in the sense that they involve either one entry that vanishes or two entries that are equal, both of which reduce the number of distinct products involved. However, where there are more than one way to represent a number either in either of the forms $\phi(x, y)$ or $\psi(c, d)$, we can get numerous examples of $k-$quadruples by using each c with each of the pairs (x, y). Some of the quadruples will have all its entries distinct. This table gives some examples.

$\phi(x,y)=\psi(c,d)$	(x,y)	(c,d)	(c,k)	$k-$quadruple
49	$(-3,8)$, $(7,0)$	$(4,1)$, $(0,7)$	$(4,40)$	$(-3,-3,8,13)$
			$(4,16)$	$(-1,7,0,15)$
			$(0,24)$	$(5,-3,8,5)$
			$(0,0)$	$(7,7,0,7)$
91	$(-1,10)$, $(5,6)$	$(3,8)$, $(5,4)$	$(3,-21)$	$(5,5,6,17)$
			$(3,19)$	$(3,-1,10,15)$
			$(5,-5)$	$(1,5,6,21)$
			$(5,35)$	$(-1,-1,10,19)$
133	$(-1,12)$, $(9,4)$	$(2,11)$, $(6,5)$	$(2,-32)$	$(9,9,4,17)$
			$(2,16)$	$(9,-1,12,15)$
			$(6,0)$	$(1,9,4,25)$
			$(6,48)$	$(-1,-1,12,23)$

When $k=1$, there are various quadruples:

$$(r-1, r+1, 4r, 4r(4r^2-1));$$
$$(1, r^2-1, r(r+2), 4r(r^3+2r^2-1));$$
$$(r, s^2-1+(r-1)(s-1)^2, s(rs+2), 4r^3s^4+8r^2(2-r)s^3+4r(r-1)(r-5)s^2+4(2r-1)(r-2)s+4(r-1));$$
$$(r, 4(r-1), r-2, 4(2r-3)(2r-1)(r-1);$$
$$(r, s, r+s+2c, 2c(r+c)(s+c)).$$

3. The tables involving $u(k,n)$ and $v(k,n)$.

Recall that

$$u(k,-2) = -k = f_0^2 - kf_{-2}^2$$
$$u(k,-1) = 1-k = f_1^2 - kf_{-1}^2$$
$$u(k,0) = 1 = f_2^2 - kf_0^2.$$

The $k-$triple $(u(k,-2), u(k,-1), u(k,0))$ is gregarious with $c=-k$. It will be shown by induction that, in general,

$$u(k,n) = f_{n+2}^2 - kf_n^2.$$

(The particular case of $k=1$ is already well known: $f_{2(n+1)} = u(1,n) = f_{n+2}^2 - f_n^2$.) We have already established the base equations, so all that is necessary is to establish that $\{f_n^2\}$ is a gregarious sequence.

As for $\{v(k,n)\}$, it can be shown by induction that

$$v(k,n) = f_{n-1}^2 - kf_n^2,$$

a fact which is easily checked when $n=-1,0,1$. The rest of the argument hinges on checking that $\{f_n^2\}$ is a gregarious sequence. In particular,

$$f_{2n-1} = v(-1,n) = f_{n-1}^2 + f_n^2.$$

A number of familiar properties of the Fibonacci numbers are relevant:

$$f_{2n+2} = f_{2n-2} + f_{2n} + 2f_{2n-1}; \tag{1}$$

$$f_{2n-2}f_{2n} + 1 = f_{2n-1}^2; \tag{2}$$

$$f_{2n+3} = f_{2n-1} + f_{2n+1} + 2f_{2n}; \tag{3}$$

$$f_{2n-1}f_{2n+1} - 1 = f_{2n}^2. \tag{4}$$

$$f_{n+1}f_{n-1} - f_n^2 = (-1)^n; \tag{5}$$

$$f_{n+2}f_{n-2} - f_n^2 = (-1)^{n-1}; \tag{6}$$

$$f_{n+2}f_{n-1} - f_{n+1}f_n = (-1)^n; \tag{7}$$

$$f_{n+1}^2 f_{n-1}^2 + f_n^4 = 2f_{n+1}f_n^2 f_{n-1} + 1; \tag{8}$$

$$f_{n+2}^2 f_{n-2}^2 + f_n^4 = 2f_{n+2}f_n^2 f_{n-2} + 1; \tag{9}$$

$$f_{n+2}^2 f_{n-1}^2 + f_{n+1}^2 f_n^2 = 2f_{n+2}f_{n+1}f_n f_{n-1} + 1. \tag{10}$$

Equations (8), (9), (10) result from squaring (5), (6), (7) and rearranging the terms.

In addition, alternate squares of the Fibonacci numbers satisfy the recursion

$$f_{n+3}^2 = 2(f_{n+2}^2 + f_{n+1}^2) - f_n^2,$$

whose characteristic polynomial is $t^3 - 2t^2 - 2t + 1 = (t+1)(t^2 - 3t + 1)$.

Two other Fibonacci results play a role in analyzing the sequences $\{u(k,n)\}$ and $\{v(k,n)\}$.

$$f_{n-1}^2 - 3f_n^2 + f_{n+1}^2 = 2(-1)^n;$$

$$f_{n+3}^2 = 2(f_{n+2}^2 + f_{n+1}^2) - f_n^2.$$

To see the first of these, note that

$$\begin{aligned} f_{n-1}^2 - 3f_n^2 + f_{n+1}^2 &= f_{n-2}f_n + (-1)^n - 3f_n^2 + f_{n+2}f_n + (-1)^n \\ &= f_n[(f_{n-2} - f_n) - f_n - (-f_n + f_{n+2})] + 2(-1)^n \\ &= f_n[-f_{n+1} - f_n + f_{n+1}] + 2(-1)^n = 2(-1)^n. \end{aligned}$$

For the second, note that

$$f_{n+3}^2 - 2(f_{n+2}^2 + f_{n+1}^2) + f_n^2 = (f_{n+3}^2 - 3f_{n+2}^2 + f_{n+1}^2) + (f_{n+2}^2 - 3f_{n+1}^2 + f_n^2) = 0.$$

Therefore,

$$\begin{aligned} u(k, n-1) - 3u(k, n) + u(k, n+1) &= 2(-1)^n - 2k(-1)^n = -2(k-1)(-1)^{n-1}; \\ v(k, n-1) - 3v(k, n) + v(k, n-1) &= 2(-1)^{n-1} - 2k(-1)^n = 2(k+1)(-1)^{n-1}; \\ u(k, n+3) &= 2(u(k, n+2) + u(k, n+1)) - u(k, n); \\ v(k, n+3) &= 2(v(k, n+2) + v(k, n+1)) - v(k, n). \end{aligned}$$

To verify that $\{u(k, n)\}$ and $\{v(k, n)\}$ are $k-$sequences, we have that

$$\begin{aligned} (f_{n+1}^2 - kf_{n-1}^2)(f_{n+2}^2 - kf_n^2) + k &= f_{n+2}^2 f_{n+1}^2 - k[f_{n+1}^2 f_n^2 + f_{n+2}^2 f_{n-1}^2 - 1] + k^2 f_n^2 f_{n-1}^2 \\ &= f_{n+2}^2 f_{n+1}^2 - 2kf_{n+2}f_{n+1}f_n f_{n-1} + k^2 f_n^2 f_{n-1}^2 = (f_{n+2}f_{n+1} - kf_{n-1}f_n)^2; \end{aligned}$$

and

$$\begin{aligned} (f_{n+2}^2 - kf_n^2)(f_n^2 - kf_{n-2}^2) + k &= f_{n+2}^2 f_n^2 - k[f_{n+2}^2 f_{n-2}^2 + f_n^4 + 1] + k^2 f_n^2 f_{n-2}^2 \\ &= f_{n+2}^2 f_n^2 - 2k[f_{n+2}^2 f_n^4 f_{n-2}^2] + k^2 f_n^2 f_{n-2}^2 = f_n^2 (f_{n+2} - f_{n-2})^2. \end{aligned}$$

It is interesting to note that

$$f_{2n+2} - 3f_{2n} + f_{2n-2} = 0$$

and

$$f_{2n+3} - 3f_{2n+1} + f_{2n-1} = 0.$$

4. Reclusive $k-$triples and their families

Not every $k-$triple generates a succession of $k-$triples when embeded in a sequence satisfying the congenial recurrence. For example, when $x = y$, there are triples for which (x, x, z) is a $k-$triple, but its right associate $(x, z, x + 2z)$ is not. With $xy + k = c^2$, $zx + k = b^2$, $yz + k = a^2$, we have the examples:

k	(x, y, z)	(a, b, c)
$4r^4 + 8r^3 - 4r + 1$	$(2r + 1, 2r + 1, 2(2r + 1))$	$(2r^2 + 2r + 1, 2r^2 + 2r + 1, 2r^2 + 2r)$
$r^4 - 6r^2s^2 + s^4$	$(2rs, 2rs, 4rs)$	$(r^2 + s^2, r^2 + s^2, r^2 - s^2)$
$-(12r^3 - 16r^2 - 3r)$	$(3r, 4r^2 - 1, 4r^2 + 3r - 1)$	$(4r^2 + 1, 5r, 4r)$

The last one is an example of using a process involving Pythagorean triples. Suppose that $z = x + y$, $x^2 + k = c^2$. Then $a^2 = yz + k = x^2 + xy + k = x^2 + c^2$ and $b^2 = xz + k = y^2 + c^2$. Then (x, c, a) and (y, c, b) are both Pythagorean triples sharing the value of a leg. These triples allow us to isolate the values of a, b, c, x, y.

For example, three three Pythagorean triples $(5,12,13)$, $(9,12,15)$ and $(35,12,37)$ share the term $c = 12$. Using the three pairs of them, arrive at the reclusive $k-$triples $(5,9,14)$, $(5,35,40)$ and $(9,35,44)$ with values of k respectively equal to 99, -31 and -171. The associate of $(5,35,40)$ to the right is $(35,40,145)$ and we note that $35 \times 145 - 31 = 71^2 + 3$, a near miss. This is not the only occurrence of this.

The $k-$triple $(2r+1, r^2(r+1)^2 - 1, r^2(r+1)^2 + 2r)$ with $k = -(2r^5 + r^4 - 4r^3 - 3r^2 - 2r - 1)$ has right associate

$$(r^4 + 2r^3 + r^2 - 1, r^2 + 2r^3 + r^2 + 2r, 4r^4 + 8r^3 + 4r^2 + 2r - 3).$$

We find that

$$\begin{aligned}(r^4 + 2r^3 + r^2 - 1)&(4r^4 + 8r^3 + 4r^3 + 2r - 3) - (2r^5 + r^4 - 4r^3 - 3r^2 - 2r - 1)\\ &= (4r^8 + 16r^7 + 24r^6 + 18r^5 + r^4 - 12r^3 - 7r^2 - 2r + 3)\\ &\quad + (-2r^5 - r^4 + 4r^3 + 3r^2 + 2r + 1)\\ &= 4r^8 + 16r^7 + 24r^6 + 16r^5 - 8r^3 - 4r^2 + 1 + 3\\ &= (2r^4 + 4r^3 + 2r^2 - 1)^2 + 3 = [2r^2(r+1)^2 - 1]^2 + 3.\end{aligned}$$

Once we start with a $k-$triple, (x,y,z), we can generate an infinite family of $k-$triples with the same values of x and y. We will suppose that xy is not a square. These new $k-$triples will be a isolated set of reclusive triples. Suppose that we know that $xy + k = c^2$. Then, we wish to find a value z for which $yz + k = a^2$ and $xz + k = b^2$. Then we want to have

$$xa^2 - yb^2 = (x - y)k.$$

If this has one solution (a,b), then it has infinitely many obtained by combining it with solutions (u,v) of $u^2 - (xy)v^2 = 1$. Note that, if $(A,B) = (au + ybv, bu + xav)$, then

$$\begin{aligned}xA^2 - yB^2 &= (xa^2u^2 + 2xyuvab + xy^2b^2v^2) - (yb^2u^2 + 2xyuvab + yx^2a^2v^2)\\ &= xa^2(u^2 - xyv^2) - yb^2(u^2 - xyv^2) = xa^2 - yb^2 = (x-y)k.\end{aligned}$$

We can use $(x,y) = (2,4)$ as a case study. In this case, we start with a congenial triple and use Pell's equation to derive reclusive triples. Solution of the equation $a^2 - 2b^2 = \pm 1$:

$$(a,b) = (1,0), (1,1), (3,2), (7,5), (17,12), (41,29), (99,70), (239,169), (577,408), (1393,985).$$

If $(a,b) = (a_n, b_n)$ is a solution, then the next solution is

$$(a_{n+1}, b_{n+1}) = (3a_n + 4b_n, 2a_n + 3b_n).$$

(We could also use $a_{n+1} = 6a_n - a_{n-1}$ and $b_{n+1} = 6b_n - b_{n-1}$.) We start with $(x,y,z) = (2,4,2c-6)$ where $k = c^2 - 8$. In the table, we indicate values of k for which the triple is congenial along with the squares involved.

k	c	(x, y, z)	(a, b, c)	$z/(a+b)$	Congenial k	(a, b, c)
1	-3	(2, 4, 0)	$(1, -1, -3)$		1	$(1, -1, -3)$
		(2, 4, 12))	$(7, 5, -3)$	1	1	(7, 5, 3)
		(2, 4, 420)	$(41, 29, -3)$	6	42841	211, 209, 207)
		(2, 4, 14280)	$(239, 169, -3)$	35	50936761	(7141, 7139, 7137)
		(2, 4, 485112)	$(1393, 985, 3)$	204		
-4	-2	(2, 4, 2)	$(2, 0, -2)$	1	-4	$(2, 0, -2)$
		(2, 4, 10)	$(6, 4, -2)$	1	-4	(6, 4, 2)
		(2, 4, 290)	$(34, 24, -2)$	5		
-7	-1	(2, 4, 4)	$(3, 1, -1)$ 1	-7	$(3, 1, -1)$	
		(2, 4, 8)	$(5, 3, -1)$	1	-7	(5, 3, 1)
		(2, 4, 44)	$(13, 9-1)$	2	353	(23, 21, 19)
		(2, 4, 184)	$(27, 19, -1)$	4	7913	(93, 91, 89)
		(2, 4, 1408)	$(73, 53, -1)$	11	491391	(705, 703, 701)
-8	0	(2, 4, 6)	(4, 2, 0)	1	-8	(4, 2, 0)
		(2, 4, 102)	(20, 14, 0)	3	2296	(52, 50, 48)
		(2, 4, 3366)	(116, 82, 0)	17	2822392	(1684, 1682, 1680)

The set of $(-1)-$triples for which $(x, y) = (1, 5)$ is noteworthy. While the table can be generated from the solutions of $a^2 - 5b^2 = 4$, there is a convenient formula for its entries. Only the first triple on the table is congenial.

(x, y, z)	(a, b, c)	sequence extension
$(1,\ 5,\ f_{2n}^2 + 1)$	$(f_{2n-1} + t_{2n+1}), f_n, 2)$	
(1, 5, 10)	(7, 3, 2)	$(\dots, 1, 2, 1, 5, 10, 29, 73, \dots)$
(1, 5, 65)	(18, 8, 2)	$(\dots, -53, 1, 5, 65, 139, 403, \dots)$
(1, 5, 442)	(47, 21, 2)	$(\dots, -430, 1, 5, 442, 893, 2665, \dots)$
(1, 5, 3026)	(123, 55, 2)	

For $yz - 1$, we have the computation

$$\begin{aligned}5(f_{2n}^2 + 1) - 1 - (f_{2n-1} + f_{2n+1})^2 &= 5f_{2n-1}f_{2n+1} - 1 - (f_{2n-1} + f_{2n+1})^2 \\ &= f_{2n-1}(f_{2n+1} - f_{2n-1}) + (f_{2n-1} - f - 2n + 1)f_{2n+1} + f_{2n-1}f_{2n+1} - 1 \\ &= -f_{2n}(f_{2n+1} - f_{2n-1}) + f_{2n-1}f_{2n+1} - 1 = -f_{2n^2} + f_{2n-1}f_{2n+1} - 1 = 0.\end{aligned}$$

An alternative approach is through Pell's equation. The equations $z - 1 = b^2$ and $5z - 1 = a^2$ leads to $a^2 - 5b^2 = 4$ which has fundamental solutions $(a, b) = (2, 0), (3, 1), (7, 3)$. These lead to the foregoing possibilities.

5. Constructing triples from the related squares

We can construct $k-$triples by starting with the squares involved. Let a, b, c be three arbitrary integers; we will use factorizations of the differences of their squares to construct $k-$triple. For example, if $b^2 - c^2 = x(z - y)$, we can select different possibilities for the factors x and $z - y$.

Thus, $z-y$ will be among the divisors of b^2-c^2, $y-x$ among the divisors of a^2-b^2, and $z-x$ among the factors of a^2-c^2. However, the choice of divisors from the three differences of squares will be constrained by the fact that

$$z-x=(z-y)+(y_x).$$

From these choices for $z-x$, $z-y$, $y-x$, we can get x, y, z from the coefactors of the square differences and check that the values are consistent with their differences. When this is applied to $(a,b,c)=(11,7,3)$, there are a great many choices to try, but the only ones that work are

$$(z-x,z-y,y-x)=(8,4,4),(14,4,10).$$

For example, when $(a,b,c)=(11,7,3)$, $(z-x,z-y,y-x)=(14,4,10)$,

$$\begin{aligned}14y&=(z-x)y=11^2-3^2=112;\\ 4z&=(y-x)z=11^2-7^2=72;\\ 10x&=(z-y)x=7^2-3^2=40;\end{aligned}$$

which lead to $(x,y,z)=(4,8,18)$, which is consistent with the values of the differences. Indeed, $(4,8,18)$ is a $-23-$triple, with $11^2=8\times18-23$, $7^2=4\times18-23$, $3^2=4\times8-23$. We observe that when we factor the difference of squares using the formula, we get $112=8\times14$, $72=4\times8$, $40=4\times10$. This suggests that when we search for $k-$triples, we begin with the difference of squares factorizations.

Suppose first that a,b,c are all distinct.

1. Let $x=b-c$, $y=a-c$, $z=a+b$. Then $a-b=y-x$, $a+c=(a+b)-(b-c)=z-x$, $b+c=(a+b)-(a-c)=z-y$. Then $a^2-b^2=(y-x)z=yz-xz$, $b^2-c^2=x(z-y)=xz-xy$, $a^2-c^2=y(z-x)=yz-yx$.

In fact,

$$\begin{aligned}xy&=c^2-(a+b)c+ab;\\ xz&=b^2-(a+b)c+ab;\\ yz&=a^2-(a+b)c+ab;\end{aligned}$$

so that (x,y,z) is a $k-$triple with $k=(a+b)c-ab$.

We can extend to get a sequence of $k-$triples. Let

$$w=2(y+z)-x=2(2a+b-c)-(b-c)=4a+(b-c).$$

Then

$$yw+k=(a-c)(4a+b-c)+(a+b)c-ab=(2a-c)^2$$

and

$$zw+k=(a+b)(4a+b-c)+(a+b)c-ab=(2a+b)^2.$$

The triple (y, z, w) corresponds to the squares $((2a+b)^2, (2a-c)^2, c^2)$. Analogous to the relationship between (x, y, z) and (a, b, c), we find that

$$y = (2a - c) - a;$$
$$z = (2a + b) - a;$$
$$w = (2a + b) + (2a - c).$$

2. Let $x = a + b$, $y = a + c$, $z = a + b$, so that $y - x = a - b$, $z - x = a - c$, $z - y = b - c$. Then, when $k = -(ab + bc + ca)$,

$$xy + k = c^2; \qquad yz + k = a^2; \qquad zx + k = b^2.$$

Let

$$w = 2(y + z) - x = 2(2a + b + c) - (b + c) = 4a + b + c.$$

Then

$$yw + k = (2a + c)^2; \qquad zw + k = (2a + b)^2.$$

If we specialize to $(a, b, c) = (r - 1, r, r + 1)$, then $(x, y, z) = (2r - 1, 2r, 2r + 1)$ and $k = -(3r^2 - 1)$.

3. Let $x = b - c$, $y = a - c$, $z = a - b$. Then $y - x = a + b$, $z - x = a + c$, $z - y = b + c$. A necessary condition for this to hold is that $b = 0$. In this case, we get the $ac-$triple $(-c, a - c, a)$. This can be extended to the sequence yielding sucessive $ac-$triples:

$$\ldots, a - 4c, -c, a - c, 4a - c, \ldots$$

4. Let $x = b - c$, $y = a + c$, $z = a + b$. Then $z - y = b + c$, $z - x = a - c$, $y - x = a - b$. Then

$$a - c = z - x = (z - y) + (y - x) = (b + c) + (a - b) = a + c,$$

so that $c = 0$. In this case, we get the sequence yielding $-ab-$triples.

$$\ldots, a + 4b, a + b, b, a, a + b, 4a + b, 9a + 4b, \ldots$$

Now we consider the case that $b^2 = c^2$, so that $xy + k = xz + k = b^2$ and $yz + k = a^2$, In this case $x(z - y) = 0$, so that either $x = 0$ or $y = z$.

5. When $x = 0$, we obtain the $3-$triple $(0, a-b, a+b)$ with $k = b^2$. The extended sequence is:

$$\ldots, a - 5b, a - 3b, 0, a - b, a + b, 4a, 9a + 3b, \ldots.$$

However, only $(0, a - b, a + b)$ and $(a - b, a + b, 4a)$ are b^2-triples.

6. When $y = z$, we get the equations $xy + k = b^2$ and $y^2 + k = a^2$. Then $(x - y)y = b^2 - a^2$. If $y = b + a$, we get the $-(b^2 + 2ab)-$triple $(2b, b + a, b + a)$.

When $y = b - a$, we get the $-(b^2 - 2b)-$triple $(2b, b - a, b - a)$.

We consider the particular case where $(a,b,c) = (4,5,5)$. By the foregoing cases, we get the $(-65)-$sequence

$$\ldots, 69, 29, 10, 9, 9, 26, 61, \ldots$$

and the $15-$sequence

$$\ldots, -35, -11, -6, 1, 1, 10, 21, 61, \ldots$$

However, the factorization 3×3 of $b^2 - a^2 = 16$, which is not of the type considered so far, gives the $7-$triple, $(6,3,3)$ which is not embedded in a sequence of triples. However, it is also a $-9-$triple which can be embedded in a sequence of triples. This is because it is a multiple of the $-1-$triple $(2,1,1)$.

6. Open questions

1. For each nonzero integer k, what is the maximum number of entries in a nontrivial set S of integers for which $xy + k$ is a square for pair (x,y) of distince elements of S? (By nontrivial, we insist that the numbers be distinct and nonzero.) In particular, is it always possible to find a set of 4 elements with this property? Are there any values of k for which the answer is 3?

2. For each integer k we form a graph whose vertices are equivalent classes of $k-$triples. Two $k-$ triples are equivalent if the terms of one are the negative of the terms of the other, the terms of one are a permutation of those of the other, or a composite of these conditions. The vertices are the equivalent classes of $k-$triples and two vertices are connected by an edge if and only if a representative triple of one is an associate of a representative triple of the other. Is the graph formed by the equivalence classes of congenial $k-$triples connected?

3. Can a triples (x,y,z) be a congenial $k-$triple for more than one integer k.

4. Are there any $k-$triples (x,y,z) for which none of x,y,z is equal to 0 or 1 and $xyz + k$ is also a square?

5. Let k be an integer. Suppose that for some triple, the product $(xy+k)(yz+k)(zx+k)$ is square. Under what circumstances does this imply that each of the three factors is square?

This has been investigated for $k = 1$ in the paper

Kiran S. Kedlaya, *When is* $(xy+1)(yz+1)(zx+1)$ *a square?* Math. Mag. 71:1 (February, 1998), 61-63 .

6. Does every congenial $k-$sequence have the property that, for every consecutive triple (x,y,z), $z = x + y + 2c$ where $xy + k = c^2$?

7. What are the possible values of the triple (k,m,d) for which there is a $k-$sequence with each term congruent to d modulo m?

For example, if m is a common divisor of r and s, then $(s^2, m, 0)$ is such a triple exemplified by the sequence

$$\dots, r-s, 0, r+s, 4r+8s, 9r+21s, 25r+55s, \dots.$$

The modular pair $(2,0)$ is exemplified by

$$k = 1 : \dots, -2, 0, 0, 2, 4, 12, 30, 44, \dots;$$
$$k = 1 : \dots, -28, 8, 6, 0, 4, 2, 12, 24, 70, \dots;$$
$$k = 1 : \dots, 4, 0, 2, 0, 6, 20, 48, \dots;$$
$$k = 4 : \dots, -6, 0, -2, 2, 0, 6, 10, 32, 78, \dots;$$
$$k = 5 : \dots, -2, -2, 2, 2, 10, 22, 62, 158, \dots.$$

Are there any examples for which $d \neq 0$?

8. Which $k-$triples are arithmetic progressions? geometric progressions? harmonic progressions?

9. Characterize triples (x, y, z) that are not $k-$triples for any value of k?

Further References

For the cases $k = \pm 1$, a few results are given on pages 153-155, 157-159 of the book

Edward J. Barbeau, *Power play.* The Mathematical Association of America, 1997 ISBN 0-88385-523-2

In Mathematics and Informatics Quarterly 6 (1996), 21-26, S.T. Thakar gives one of our parametric examples of a $1-$quadruple, and also for general k, the embedding of a $k-$triples into a congenial sequence.