The Irrationals: a Story of the Numbers You Can't Count On

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What are the real numbers? This is a deep question. It has taken about two millenia to understand and explain their properties, and there is still much to learn. The convoluted history of the real number system is the subject of this interesting book by Julian Havil, his fourth exposition published by Princeton. It is a pleasure to read, with solid mathematical content balanced by historical notes, philosophical discussion and anecdote.

The story begins with the Greeks whose idea of number was tied to counting, so that they had a separate concept of magnitude to deal with the continuum. This dichotomy was forced upon them by the discovery of incommensurable geometric magnitudes whose ratio could not be expressed with whole numbers. As Havil says, "the first significant implication of what we now call irrationality was that mathematical enquiry became geometric enquiry, an approach that was to pervade all European mathematics and last well into the eighteenth century." (p. 35-36) But there were other perspectives. After a generous discussion of Greek mathematics that includes Euclid's treatment of ratio, proportionality and incommensurability, Havil reviews the contributions of Indian, Levantine and early Renaissance European mathematicians. These investigators were comfortable with a number system that embraced surds, which they manipulated stunningly. While the spirit of the these developments is preserved, the mathematics is presented in modern notation.

The next three chapters of the book treat particular irrationals, in particular e, π and $\zeta(3)$. Determining the character of these numbers occupied the attention of many leading mathematicians and involved a variety of tools: continued fractions, integration and infinite series. A full chapter is devoted to the proof of the irrationality of $\zeta(3)$ by the French mathematician, Roger Apéry in 1978. This is a real *tour de force* and the author presents the right amount of detail to convey the essence of the argument without bogging the reader down in technicalities.

The author moves to more general and theoretical considerations, especially the degree of closeness of rational approximations to real numbers. Here among familiar material, I found some that was new and fascinating. The basic question is when a real number α can be approximated by infinitely many rationals p/q with $|\alpha - p/q| < 1/cq^r$ where c and r are given reals exceeding 1. The whole story is far from being wrapped up in the early result that α is irrational if and only if there are infinitely many rationals with $|\alpha - p/q| < 1/q^2$. Havil surveys the theoretical evolution of degree of rational approximation up to the striking result of Klaus Roth in 1955 that, given an algebraic number α and positive ϵ, there are only finitely many rationals p/q for which $|\alpha - p/q| < 1/q^{2+\epsilon}$.

There are many other topics touched upon: straightedge-and-compasses construction, the Gelfond-Schneider theorem, uncountability of the reals (by a nested interval rather than a diagonal argument), approximation using continued fractions, the Lagrange and Markov spectra, "randomness" of decimal digits, rigorous formulation of the real number system. The final chapter, entitled "Does Irrationality Matter¿' contains brief musings on the well-tempered musical scale, the golden ratio, Penrose tiles, the characteristic function $\lim_{m\to\infty} \lim_{n\to\infty} \cos^{2n}(m!\pi x)$ of the rationals, dynamical systems and Samuel Beatty's notable result that, for positive irrationals x and y whose reciprocals add to 1, the sequences $\{\lfloor mx \rfloor\}$ and $\{\lfloor nx \rfloor\}$ together contain each positive integer exactly once. (It is slightly annoying to see Beatty described as an American mathematician, despite being described as being from the University of Toronto.)

On the whole, the author has presented a coherent and compelling account of a sophisticated topic to which he has evidently devoted a lot of time and effort to research and sort out. However, there were several places where I found the treatment heavy-handed. There is an easier way of establishing the incommensurability of side and diagonal of a square than is presented on page 24. On page 79, a modulo 4 argument suffices to show that $a^2 + b^2 = 3c^2$ is not solvable in integers. The argument on page 146 that the least common multiple of the first n natural numbers does not exceed n raised to the number of primes not greater than n is unnecessarily complicated. More seriously, on page 86 appears the assertion that, for a positive continuous function defined on the nonnegative reals and with absolute maximum M_N on $[0, N]$,

$$\int_0^1 f(x)dx = \lim_{N \to \infty} \frac{\sum_{r=0}^{N} f(r)}{M_N(N+1)}$$

(try this on $f(x) = \sin \pi x$); fortunately, this is applied to a situation where it works.

Despite a some minor lapses and rough edges, this is a book worth having. It is accessible to secondary students keen on mathematics and anyone with a first-year university background, although it requires sustained concentration in parts. I particularly recommend it for undergraduate students and secondary teachers.