CONSECUTIVE INTEGER PRODUCTS CLOSE TO SQUARES

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0. Sometime mathematics operates very much like other sciences. People observe things, notice patterns, try to describe them, come up with a theory to explain them, and check the theory by further cases and predictions. In the case of mathematics, these patterns are the basis of conjectures, general statements that we believe are true. Corresponding to scientific theories that explain the patterns, mathematicians construct theorems where the conjectures can be justified by logical argument.

This paper illustrates one of the ways in which we can collect information. The intention is to encourge the reader to make some organized calculations, which can be done on a calculator or by programming a computer. The next stage is to look for patterns, and discover whether they can be described in a way that leads to a theorem. A secondary goal of the paper is to show how school algebra can be used to both describe and justify what we observe.

1. You may be familiar with the fact that the product $n(n+1)$ of two consecutive integers cannot be an integer square. Think how you might justify this. The easiest argument is perhaps to note that $n^2 < n(n+1) < (n+1)^2$, so that $n(n+1)$ lying between two consecutive squares cannot be square. But we could proceed by contradiction argument. Suppose $n(n+1)$ is square. Since n and $n+1$ are coprime (have no common divisor except 1), each of them must be square. But then n and $n+1$ are two positive squares that differ by 1. This cannot happen.

Exercise 1. Explain why, if the product of two positive coprime integers is a square, then each must be a square.

Exercise 2. Find a simple argument to prove that two positive squares cannot differ by 1.

Another contradition argument goes like this. If $n(n+1)$ is a square, then so is $4n(n+1)$. But $4n(n+1) = (2n+1)^2 - 1$, and again we would have two squares differing by 1. These arguments are worth keeping in mind when we look at products of more than two consecutive integers. Is it possible for the product $(n-1)n(n+1)$ of three positive integers to be a square. We can look for the foregoing arguments for inspiration. For example, if $(n-1)n(n+1) = n(n^2-1)$ was square, then both n and n^2-1 would be squares (why?); thus $n = u^2$ and $n^2 - 1 = v^2$ for some positive integers u and v. This would lead to

$$1 = u^4 - v^2 = (u^2 - v)(u^2 + v),$$

whereupon $(u, v) = (1, 0)$ and the product would vanish, contrary to our supposition.

When we come to the product of four consecutive positive integers, which we will write as $(n-1)n(n+1)(n+2)$, something interesting happens. Before reading further, check a few examples on your calculator and see what you notice. The product is always one less than a perfect square:

$$(n-1)n(n+1)(n+2) = [(n-1)(n+2)][n(n+1)] = [n^2+n-2][n^2+n]$$
$$= [(n^2+n-1)-1][(n^2+n-1)+1] = (n^2+n-1)^2 - 1.$$

Exercise 3. Is it possible to express a square as the product of five consecutive positive integers?

Exercise 4. Express the product $(n-3)(n-2)(n-1)n(n+1)(n+2)(n+3)(n+4)$ as the difference of the squares of two polynomials in n, one of which is a quadratic.

2. The case of the product of four consecutive positive integers is worth further investigation.

Exercise 5. Make up a table in which we enter beside each value of the integer n, the value of the product $f_4(n) = (n-1)n(n+1)(n+2)$ and the next four squares that exceed $f_4(n)$; these are $(n^2+n-1+m)^2$ where $0 \leq m \leq 3$. Note in particular the situations in which $(n^2+n-1+m)^2 - f_4(n)$ is a square.

Exercise 6. In Exercise 5, you will notice that $(n^2+n+1)^2 - f_4(n)$ is a square. Verify that it is equal to $(2n+1)^2$.

3. Under some circumstances, we have seen that the product of consecutive integers differ from nearby squares by squares. When we look at products $f_3(n) = (n-1)n(n+1)$ of three consecutive integers, then the situation becomes more textured. Let $g_3(n)$ denote the smallest integer that exceeds the square root of $f_3(n)$; notationally this is written as

$$g_3(n) = \lceil \sqrt{f_3(n)} \rceil.$$

Exercise 7. Make a table that lists values of n, $f_3(n)$, $(g_3(n))^2$ and the next few larger squares. How often is the difference between these squares and $f_3(n)$ itself a square?

The difference between $(g_3(n))^2$ and $f_3(n)$ is square suprisingly often. It occurs when $n = 3, 4, 5, 7, 8, 9, 11, 13, 15, 16, 17, 19, 21, 25$ for example. This suggests that it might be worth looking at the diophantine equation

$$y^2 = x^3 - x + k^2$$

where k is a positive integer parameter, and see what its solutions (x, y) in integer pairs might be.

Exercise 8. With the help of the table you made in Exercise 7, find solutions of $y^2 = x^3 - x + k^2$ for various values of the parameter k. Some of these solutions are generic, in that they exist for all values of k and can be expressed in terms of k, Find them.

Pay particular attention to the case $k = 1$. By looking at small values of x, you should find that $y^2 = x^3 - x + 1$ is satisfied by $(x, y) = (3, 5)$ and $(5, 11)$. There is a useful device that helps you find a third solution when two solutions are known. Using a graphing calculator, obtain a graph of the equation $y^2 = x^3 - x + 1$. This graph will contain these points:

$$(-1, \pm 1), (0, \pm 1), (1, \pm 1), (3, \pm 5), (5, \pm 11).$$

Each pair of these points will determine a line of equation $y = ax + b$. To find all the points where the line intersects the graph, we need to solve a cubic equation $(ax + b)^2 = x^3 - x + 1$, or

$$0 = x^3 - a^2x^2 - (2ab + 1)x - (b^2 - 1).$$

The two points that determine the line provide two roots of the equation, namely the abscissa of the points on both the line and the curve. The sum of the roots is a^2 (the square of the slope of the line). In general, this will be a rational number, so the third root is also rational and we can identify a point of the curve with rational coordinates. However, if we can arrange for slope to be an integer, then we will have found a further solution in integers to the equation $y^2 = x^3 - x + 1$.

Exercise 9. Start with the solutions $(x, y) = (3, 5), (5, 11)$ to determine a new solutions of the diophantine equation. Does this give us anything new? Now start with $(3, -5)$ and $(5, 11)$ and see what you get.

Exercise 10. For other values of $k \geq 2$, see how many solutions apart from the generic ones you can find for $y^2 = x^3 - x + k^2$.

Exercise 11. Determine the lines through pairs of solutions (x, y) to $y^2 = x^3 - x + 25$.

Exercise 12. Determine the lines through pairs of points corresponding to generic solutions of $y^2 = x^3 - x + k^2$.

4. We turn our attention to $y^2 = f_5(x) + k^2$ where $f_5(x) = (x - 2)(x - 1)x(x + 1)(x + 2)$. For each integer x, let $g_5(x) = \lceil \sqrt{f_5(x)} \rceil$. Thus $f_5(3) = 120$, $g_5(3) = 11$, $f_5(4) = 720$, $g_5(4) = 27$.

Exercise 13. Make a table for $3 \leq n \leq 25$ showing $f_5(n)$, $g_5(n)$ and $(g_5(n) = m)^2 - f_5(n)$ for $0 \leq m \leq 3$. is it always true that $g_5(n)^2 - f_5(n)$ is a square?

5. The equation $y^2 = f_6(x) + k^2$ is more interesting, where

$$f_6(x) = (x - 2)(x - 1)x(x + 1)(x + 2)(x + 3) = x^6 + 3x^5 - 5x^4 - 15x^3 + 4x^2 + 12x.$$

Let $g_6(x) = \lceil \sqrt{f_6(x)} \rceil$. It turns out that $(g_6(n) + m)^2 - f(n)$ is square for a string of values of n and for $0 \leq m \leq 2$.

Exercise 14. By looking at the data, try to discern some patterns that might give you some formulas analogous to those in Exercise 6. Since $f_6(x)$ is a polynomial of degree 6, it is reasonable to try to use your data to express it as a difference of squares to two polynomials whose degree do not exceed 3.

Exercise 15. How many solutions (x, y) in positive integers of

$$y^2 = (x-3)(x-2)(x-1)x(x+1)(x+2)(x+3)+1$$

can you find?

Consider the equation $y^2 = f_8(x) + k^2$ where

$$f_8(x) = (x-3)(x-2)(x-1)x(x+1)(x+2)(x+3)(x+4).$$

Let $g_8(x) = \lceil \sqrt{f_8(x)} \rceil$.

Exercise 16. By examining a table showing values of $f_8(n)$ and $(g_8(n)+m)^2 - f_8(n)$ and fitting polynomials, it is possible to find distinct polynomials $u(x)$ and $v(x)$ for which $u(x)^2 - f_8(x)$ and $v(x)^2 - f_8(x)$ are squares of polynomials. Do this.

6. In Exercises 4, 6 and 14, we have seen that $f_4(x)$ and $f_6(x)$ can be written as the difference of squares of two polynomials, each of which takes an integer value when x is an integer. Each polynomial $f(x)$ can, in fact, be written as the difference of squares of two polynomials. One way of doing this is to write

$$f(x) = \left[\frac{1}{2}(f(x)+1)\right]^2 - \left[\frac{1}{2}(f(x)-1)\right]^2.$$

However, if $f(x)$ assume integer values when x in an integer, it may happen that neither of the two polynomials being squared has the same property.

Exercise 17. Suppose that a polynomial $f(x)$ can be written as the product $p(x)q(x)$. Use the factors to construct two polynomials $u(x)$ and $v(x)$ for which $f(x) = u(x)^2 - v(x)^2$.

Exercise 18. Show that $f_2(x) = x(x+1)$ cannot be written as the squares of two polynomials, each of which assume only integer values when the variable is an integer.

One way to check whether a polynomial $f(x)$ is the difference of two integer-valued polynomials is to see if it assumes any numerical values that cannot be expresed as the difference of two squares.

Exercise 19. Show that an integer can be expressed as the difference of squares of two integers if and only if it is not equal to twice an odd number. (Thus 2, 6, 10, 14, *etc.* are not expressible as the difference of integer squares.)

We will finish with a conjecture for you to investigate: the polynomial $f_r(x)$ is equal to $u(x)^2 - v(x)^2$ for some polynomials for which $u(x)$ and $v(x)$ take integer values when x is an integer if and only if r is even.

8. We can extend the investigation to products of consecutive integers of the same parity (all even or all odd).

Exercise 20. Verify that the product of two consecutive integers of the same parity is always 1 less than a perfect square.

Exercise 21. For what integers n does the product $(n-2)n(n+2)$ differ from the next higher square by a square? Does it always happen when $n \geq 3$?

Exercise 22. Check out the situation for the product of four consecutive integers of the same parity.

8. This section contains comments on the exercises.

Exercise 1. Any common divisor of n and $n+1$ must divide 1. Hence the primes that divide n are distinct from the primes that divide $n+1$. If the product is square, any prime dividing it must divide it to an even power, and this prime power divides exactly one of n and $n+1$.

Exercise 2. If $u^2 - v^2 = 1$, then $(u-v)(u+v)$ expresses 1 as the product of two integers, which both can only be 1 or -1.

Exercise 3. Express the product as $(n-2)(n-1)n(n+1)(n+2) = (n^2-4)(n^2-1)n$.

Exercise 4. Write the product as

$$\begin{aligned}
&[(n-3)n(n+1)(n+4)][(n-2)(n-1)(n+2)(n+3)] \\
&\qquad = [n^4 + 2n^3 - 11n^2 - 12n][n^4 + 2n^3 - 7n^2 - 8n + 12] \\
&\qquad = [(n^4 + 2n^3 - 9n^2 - 10n + 6) - (2n^2 + 2n + 6)] \\
&\qquad\qquad [(n^4 + 2n^3 - 9n^2 - 10n + 6) + (2n^2 + 2n + 6)].
\end{aligned}$$

Exercise 8. Some solutions are given by

$$(x, |y|) = (-1, k), (0, k), (1, k), (k^2, k^3), (4k^2 - 1, k(8k^2 - 3)), (4k^2 + 1, k(8k^2 + 3)).$$

Exercise 9. The points $(3, 5)$, $(5, 11)$ lies on the line $y = 3x - 4$. The cubic to be solved is

$$0 = x^3 - 9x^2 + 23x - 15 = (x-1)(x-3)(x-5)$$

yielding the solutions $(1, -1)$, $(3, 5)$,$(5, 11)$.

If we start with $(3, -5)$ and $(5, 11)$, we are led to the line $y = 8x - 29$, the cubic equation

$$0 = x^3 - 64x^2 + 463x - 840 = (x-3)(x-5)(x-56)$$

and the additional solution $(x, y) = (56, 419)$.

Exercise 10. Additional solutions are given by

$$\begin{aligned}(k;x,y) =&(5;3,7),(5;7,19),(5;8,23),(5;13,47),(5;32,181),\\ &(7;5,13),(7;11,37),(7;19,83),(7;40,253),(8;7,20),(8;9,28),\\ &(12;55,408),(13;21,97),(13;31,73),(14;33,190),(15;27,141),\\ &(16;39,244),(17;29,157),(25;51,365),(25;57,431),(31;71,599).\end{aligned}$$

Exercise 11. $[y = 2x+5; (-3,-1), (0,5), (7,19)]$

$$[y = 4x-9; (1,-5), (7,19), (8,23)]$$

$$[y = 2x+7; (-3,1), (-1,5), (8,23)]$$

$$[y = x+4; (-3,1), (1,5), (3,7)]$$

$$[y = 4x-5; (0,-5), (3,7), (13,47)]$$

Exercise 13. $g_5(n)^2 - f(n)$ is square for $3 \le n \le 19$ but not for $n = 20$ and $n = 21$.

Exercise 14. For $3 \le n \le 14$, $g_6(n) = \frac{1}{2}(2n^3 + 3n^2 - 7n - 6)$.

$$g_6(n)^2 - f_6(n) = \left[\frac{1}{2}(n^2 - 3n - 6)\right]^2.$$

$$(g_6(n)+1)^2 - f_6(n) = \left[\frac{1}{2}(n^2 + n + 4)\right]^2.$$

$$(g_6(n)+2)^2 - f_6(n) = \left[\frac{1}{2}(n^2 + 5n - 2)\right]^2.$$

The pattern breaks at $n = 15$. The reason for this is that, if the polynomial $u(n)$ is to yield the smallest square greater than $f_6(x)$ and $d(n) = u(n)^2 - f_6(n)$, then

$$(u(n)-1)^2 < f_6(n) = u(n)^2 - d(n)$$

so that $d(n) < 2u(n) - 1$. This would require the polynomial $d(n)$ to have degree no greater than than of $u(n)$, and this fails to be the case when $d(n) = (n^2 - 3n - 6)^2$. The pattern breaks at $n = 15$, where $g_6(n) = 3656$ but $\frac{1}{2}(2n^3 + 3n^2 - 7n - 6) = 1357$. Note however that $3656^2 - f_6(15) = 16^2$.

Exercise 15. A numerical search turns up the solutions $(x,y) = (4,71), (11,4159)$.

Exercise 16. The situation is not as straightforward as it was for the product of six consecutive integers. By making a judgment in the patterns, we are led to

$$u(4) = g_8(4)+3; u(5) = g_8(5)+1; u(6) = g_8(6)+1; u(n) = g_8(n)$$

and

$$v(4) = g_8(4)+4; v(5) = g_8(5)+3; v(6) = g_8(6)+3; v(n) = g_8(n)+2,$$

for $n \geq 7$. Thus

$$g_8(n) = n^4 + 2n^3 - 9n^2 - 10n + 4;$$
$$u(n) = 8n + 4; \qquad v(n) = 2n^2 + 2n + 6.$$

Exercise 17. Solve the equations $u + v = p; u - v = q$ for u and v.

Exercise 18. There are two ways of factoring $x(x+1)$ as a product of polynomials, giving rise to the representations

$$x(x+1) = \left[\frac{1}{2}(2x+1)\right]^2 - 1^2 = \left[\frac{1}{2}(x^2+x+1)\right]^2 - \left[\frac{1}{2}(x^2+x-1)\right]^2.$$

Exercise 19. If n is odd or divisible by 4, then n is the product uv of two integers of the same parity (both even, or both odd). Then

$$n = \left[\frac{1}{2}(u+v)\right]^2 - \left[\frac{1}{2}(u-v)\right]^2$$

is the desired representation. On the other hand, if n is twice an odd number, it leaves a remainder 2 upon division by 4. Since squares leave remainder 0 or 1 upon division by 4, it is impossible for the difference of two squares to leave remainder 2.