FUN WITH PYTHAGORAS

A mathematical vignette

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At around age 13, most school children have been introduced to the triple $(3, 4; 5)$. This is called a *Pythagorean triple* because the square of the largest number is equal to the sum of the squares of the two smaller ones. (Pythagoras was an ancient Greek philosopher who lived about 2500 years ago.) There are lots of such triples; another one is $(5, 12; 13)$. It is natural to ask whether we can find other relationships that can fit into some pattern exemplified by $(3, 4; 5)$.

We might notice for example that the difference between the two largest numbers is 1, and ask whether there are other triples besides $(3, 4; 5)$ and $(5, 12; 13)$ for which this is so. Yes. We can go on producing them forever. There are $(7, 24; 25)$, $(9, 40; 41)$ and so on. Just note that the first entry is an odd number and the sum of the last two is its square. Be sure to check that the triples you get work.

We can generalize in another direction. We have the interesting fact that

$$10^2 + 11^2 + 12^2 = 365 = 13^2 + 14^2,$$

which we will indicate by the sequence $(10, 11, 12; 13, 14)$ in which the sum of the squares of the numbers before the colon equals the sum of the squares of the numbers after the colon. The next equation following this pattern is described by $(21, 22, 23, 24; 25, 26, 27)$. Can you get others along the same lines?

However, we get into a luxuriant garden of relationships when we look for Pythagorean triples for which the smaller two numbers differ by 1. The first three are $(0, 1; 1)$, $(3, 4; 5)$ and $(20, 21; 29)$. Your job is to find others.

The two sequences

$$(0, 1, 2, 5, 12, 29, 70, 169, 408, 985, \cdots)$$

and

$$(1, 1, 3, 7, 17, 41, 99, 239, 677, 1593, \cdots)$$

play a role. For each sequence, each term after the second is the sum of twice its predecessor and the previous one. For example, $29 = 2 \times 12 + 5$. I will show you ways in which these sequences are implicated in $(20, 21; 29)$ and leave the rest up to you. We have $29 - 20 = 3^2$; $29 + 20 = 7^2$; $21 = 3 \times 7$; $29 = 2^2 + 5^2$; $7 = 2 + 5$; $20 + 21 = 12 + 29$.

There is a third sequence formed by taking the product of corresponding terms of the foregoing two sequences:

$$(0, 1, 6, 35, 204, 1189, \cdots).$$

I will note simply that $29 = 35 - 6$ and $20 + 21 = 35 + 6$, and leave you to discover its other treasures.

There are also some Pythagorean near missesi that involve consecutive squares: $(8, 9; 12, 1)$ and $(49, 50; 70, 1)$. Can you find others?

Sometimes we see something that might be part of a pattern, but cannot see how to continue. For example, while $3^2 + 4^2 = 5^2$ and $3^3 + 4^3 + 5^3 = 6^3$, it is not clear whether and how this might be continued.

It is of some value to generalize Pythagorean triples with specific characteristics, rather than go whole hog and derive the generic formula for a complete set of Pythagorean triples. We could also ask for triples for which the largest two numbers differ by 2, such as (8, 15, 17) and (12, 35, 37). Students might observe perhaps that the square of the smallest is twice the sum of the largest two. This may lead to a more piecemeal but in the end more satisfying discovery of the general formula.