

THE LAW OF COSINES

A mathematical vignette

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A key difficulty with any syllabus is having it come across to students as a collection of isolated facts and techniques. It is desirable to have topics that are central to the syllabus but are capable of making connections across it. A wonderful example of this is based on the law of cosines and its use in connecting trigonometry, the theory of the quadratic and the ambiguous case in Euclidean geometry. Recall that the Cosine Law states that, if a, b, c are the sides of a triangle and A is the angle opposite side a , then

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

This formula is often used to determine the third side when two sides and the contained angle is given, or to determine an angle of the triangle when the three sides are given. However, it is of more than passing interest when the third side to be determine is not opposite the given angle.

Thus, suppose that we are given sides a and b , along with an angle A opposite to one of the given sides and are asked to find the third side. Then, according to the Cosine Law, we are being asked to solve the quadratic equation

$$x^2 - (2b \cos A)x + (b^2 - a^2) = 0.$$

The discriminant of this quadratic equation is

$$4(a^2 - b^2 \sin^2 A).$$

The quadratic will have real solutions if and only if $a \geq b \sin A$. What does this correspond to geometrically? In a triangle ABC , $b \sin A$ is the length of the perpendicular dropped from C to the side AB , and for a viable triangle, this cannot be greater than the side opposite the angle A .

If $a = b \sin A$, then the triangle must be right with hypotenuse b . In this case the quadratic has a unique solution $x = b \cos A$ and this is the length of one arm of the right triangle.

Henceforth, we will assume that $a < b \sin A$. Suppose to begin with that the angle A is acute, *i.e.*, $0 < A < 90^\circ$. Then the coefficient of the linear term is negative, so that the sum of the roots of the quadratic is positive.

When $a < b$, the constant term (which is the product of the roots) is positive, so the quadratic has two positive real roots. When we look at the situation geometrically, and specify the vertices A and C and the angle at A , we see that there are two possible ways of locating B to obtain a triangle ABC with angle A interior to the triangle and $a = |BC|$.

When $a = b$, then we still have two nonnegative roots, one of which is 0. The positive root corresponds to the configuration of an isosceles triangle with $b = a$ and the root 0 to the degenerate triangle for which the vertices A and B coincide.

When $a > b$, then the constant term of the quadratic is negative and so the roots have opposite signs. In this case, there is only one option. When we look at the situation geometrically, we see that there is only one position for B that will have the given angle at A inside the triangle CAB . However, is there any significance to the negative root? This corresponds to the situation where the point B lies on the other side of A , so that the triangle CAB contains the supplement of the given angle A . In this case, we can think of the negative root as corresponding to the situation where we measure the distance from A to B in a direction that takes us outside the desired triangle.

When $A = 90^\circ$, then the quadratic equation becomes

$$x^2 - (a^2 - b^2) = 0,$$

which has real roots only if $a \geq b$. When $a = b$, we have a degenerate triangle in which A and B coincide. Otherwise, we have one positive and one negative root, and exactly one solution that corresponds to a right triangle with hypotenuse CB .

Finally, sticking with the condition $a \geq b \sin A$, suppose that A is obtuse. Then the linear term of the quadratic is positive, so that there is at least one negative root. In the case that $a < b$, the product of the roots is positive, so that quadratic has two negative roots and so there is no positive value of x to satisfy the equation. When we check this with the geometry, we see that this corresponds to the situation that both positions for the side BC must lie outside the angle. However, when $a > b$, then the product of the roots is negative, so there is one positive and one negative values of x . In this case, there is one possible location for the side BC that lies across the angle A .

There are three standard triangle congruence theorems, where two sides and the contained angle are given; where three sides are given; and where one side and two adjacent angles are given. However, there is a third theorem that deals with to what extent a triangle can be determined by specifying two sides and a non-contained angle. In the last case, we are given the lengths BA and BC and angle A . When the length of BC exceeds the length of BA , the triangle is uniquely determined by these specifications. However, when the length of BC is less than that of BA , but not too small, there are two noncongruent triangles that fit the given data; this is the so-called *ambiguous case*. We can see how the algebraic formulation and our geometric representations fit together.

It is important that the syllabus contain instances that point out the consistency of mathematics, in that one can approach a situation from many different directions, each lending its own insight, the conclusions that we can draw are the same.