Mathematics in a deck of cards

While the acquisition of skills is important, pupils of mathematics also need an educational regime that authentically conveys to them other aspects of mathematics, in particular the way in which observations can be organized and analyzed. Students should be presented with situations in which structure is visible and can be studied. As mathematicians, we have faith that patterns and phenomena can be understood, and a decent curriculum should provide occasions for demonstrating this.

One vehicle with the young is an ordinary deck of 52 playing cards, with its thirteen ranks and four suits. I will suggest some interactions between a "magician" and his "subject".

1. Three questions for 27 options. The magician deals 27 cards into three 9-card columns and asks the subject to secretly select one of the cards, but tell him which column contains it. Once the magician has this information, he gathers up to three columns, one on top of the next, and then deals the cards across into three 9-card columns. He then ascertains from the subject which column contains the selected card and again deals the cards across into three 9-card columns. Upon being told a third time which column contains the selected card, he is able to identify it.

The trick is based on dealing out the cards so that the first answer narrows the selected card down to one of nine cards, the second answer to one of three cards and the third answer down to a unique card. This trick is known to many youngsters, sometimes in the form of dealing only 7 cards to a column. Often it is set up, so that the named column is gathered up in the middle, so that the selected card turns out to be in the very middle of the deck.

It is possible for many children from the age of 9 to understand and replicate the trick, after the magician walks them through it a couple of times. All that is required is a sufficient level of concentration to keep track of where the nine cards of each selected column go and to make sure that three of them are dealt into each of the three columns the next time around.

The surprise comes from fact that one can isolate one of 27 possibilities with three questions; the cube of 3 is as *big* as 27. The same perspective applied to base ten numeration; it takes only four pieces of information to specify a number less than 10000, namely its four digits. This trick, thus, can possibly alter perceptions, something desirable in a mathematics class.

2. Seven times seven. A trick that illustrates a similar principle to the first involves powers of 7 rather than 3. Take a deck of 49 cards and ask the subject to take one card, look at it and return it to the deck without telling the magician what it is. Then the subject picks a number between 1 and 49 inclusive and tells the magician.

The magician then deals the 49 cards into seven piles all cards face up and asks the subject to remember which pile his card is in. When this is done, the magician then stacks the seven piles and deals the cards consecutively into seven piles again face up. He asks the subject to identify the pile containing his card. The magician again collects up the seven piles and places the pack face down on the table. He asks the subject to count down the number of cards he indicated earlier and the card he originally drew will be that card.

Each choice of pile reduces the number of possibilities to one seventh, so it is just a matter for the magician to pick up the piles in the correct order. We can classify the numbers from 1 to 49 according to which of seven ranges (1-7, 8-14, 15-21, 22-28, 29-35, 36-42, 43-49) they lie in and which remainder they leave upon division by 7 (1, 2, 3, 4, 5, 6, 7). Suppose, for sake of argument, the subject has picked the number 18. Then we want him to find his card after he has counted through two ranges and hit the fourth card in the third range. Accordingly, when he picks up the seven piles from the first round, he makes sure that the pile that contains the drawn card is fourth from the top when the cards are face up.

On the second round, he deals the cards consecutively into seven piles so that in each pile, the drawn card is the fourth card from the bottom in one of the piles. He now picks up piles so that the pile with the drawn card is third from the bottom when the cards are face up. The deck is now turned over and subject counts down to the eighteenth card.

3. The flipover. Select the ten hearts from ace to ten, inclusive, and arrange them in increasing order in a fan. The magician presents the fan, cards face down, to the subject and asks her to pull out two *adjacent* cards, turn them over and reinsert them face up into the spot whence they were taken. Thus, if 4♡ and 5♡ were removed, the 5 will be where the 4 was, face up, and vice versa. He asks the subject to continue performing several times the following: cut the deck and put one end before the other, and pull out two adjacent cards, turn them over and restore them in place (either card chosen can be face down or face up).

Then the magician does something sight unseen by either person and then shows the fan; all the even cards are facing one way and the odd cards the other. What has the magician done, and why does it work?

The key to this is that parity of the cards in the fan alternate, and the actions, in a more general sense, preserve the alternation. Since cutting the deck is like moving the cards around in a ring, we will assume the cards start face down in a ring, ignore the cut, and just focus on the turnover. In each position in the ring, the cards are in one of two states $EU - OD$ (even-up, odd-down) or $ED - OU$ (even-down, odd-up). These states alternate with position, and continue to alternate with each flip. Turning over a single card and restoring it into the same position reverses the state of that card.

To give a hint to the children, one might point out that whatever the magician did at the end should work if no operations at all were carried out.

4. Still complete in the halves. Two packs of 13 cards, one consisting of the 13 spades in order from ace to king and the other consisting of the 13 hearts in reverse order from king to ace are placed face down on the table and subjected to a rough riffle shuffle. This means that they are incorporated into a single pile, with cards incorporated in bunches alternately from the two packs. (For a perfect riffle, the cards are mixed one alternately from each pack.)

The top thirteen cards are taken from the united pack. It turns out that each of the ranks from ace to king appears exactly once among them. The same is true for the pile left behind. Why is that?

Note that in the incorporated pack, the hearts and spades remain in the same order; they are just interspersed. Suppose, for example, that the top thirteen cards contain no six of spades. Then at most five spades made it into the top thirteen, the ace through five. So at least eight hearts must be there, the king through six. Thus, the six of hearts must be present.

5. Picking the correct pair. The magician deals onto the table ten pairs of cards, and asks the subject to select one of the pairs silently. The magician then gathers the pairs up and deals them into four rows of five cards each. Upon being told which rows contain the two cards of the chosen pair, the magician can identify them.

This is easy to explain, as it simply depends on producing a one-one correspondence between the ten pairs and the number of ways of picking two rows out of four, with the possibility of a row being selected twice. The magician picks the cards up keeping the pairs together, and then carefully deals each pair into two particular rows. For example, the ten pairs can be dealt into rows (1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4).

A less transparent way of dealing into rows is possible. Keep in mind the four words ATLAS, BIBLE, GOOSE and THIGH. The words have ten different letters, each occurring exactly twice. Each letter appears in a different pair of the words, and each pair of words has exactly one letter in common (with each word having one letter appearing twice). Cued by these words, you can deal the pairs accordingly.

6. Go to the top! A pack of the thirteen spades is thoroughly shuffled and the cards are laid out from left to right on the table. We adopt the usual convention that $A = 1$, $J = 11$, $Q = 12$ and $K = 13$. If the leftmost card is c, then the cth card from the left is taken from its position and placed in the first position at the left. The order of the remaining cards is left undisturbed. This move is repeated. It is found that, regardless of the original order of the cards, eventually the ace is brought to the left and the process stops. Why is this?

This probably needs to be performed a few times until the students begin to see the dynamic. Basically, the ace either stays in its original position, or gets shoved to the right, until it is suddenly brought to the leftmost position. If the ace starts out in the nth position, then one of the left $n - 1$ cards must have rank n or bigger. One needs to argue that one such card eventually gets "hit", whereupon the ace either comes to the first position or moves one position to the right. This is a nice example for discussion of induction.

A variant of this is to take the c leftmost cards and deal them back in the opposite order, so that once again the cth card goes into the cth condition.

7. Which card comes last? The magician takes 16 cards from the deck and places them upsidedown in a stack on the table. The subject is asked to remove from the top fewer than half of them, leaving a stack of between 9 and 15 cards. The magician then picks this up and shows the subject (but not himself) the cards in the stack. If the subject removed k cards, the subject is asked to remember the value of the kth card from the bottom.

The magician then takes up the stack, cards upsidedown, and deals the cards alternately to the bottom of the stack and face-up onto the table until only one card remains in the stack. This card turns out to be the one identified earlier by the subject.

This is a manifestation of a Josephus situation; a group of people are arranged in a circle, and each rth person is eliminated until only one remains. Here $r = 2$. In the present situation, suppose that n individuals numbered from 1 to n are in a ring, and we start with individual 1 and eliminate every second one as we count around. If $f(n)$ is the last individual to remain, we can see that $f(2^m) = 1$ for every nonnegative integer m and that $f(n + 1) = f(n) + 2$ when $n + 1$ is not a power of 2. Then $f(16 - k) = 17 - 2k = (16 - k) - (k - 1)$ (for $1 \leq k \leq 7$), so that the final card is the kth card from the end of the $16 - k$ cards.

8. Jack the hunter. The magician deals two piles of fifteen cards each face down onto the table. He places a jack of spades face up off to one side. The subject selects a card at random from the remainder of the deck, retains and remembers it but does not tell the magician what it is. Each pack of fifteen cards is cut into two smaller piles; suppose one pack is cut into piles A and B, while the second is cut into piles C and D.

The subject is asked to put his card on top of one of the four piles, say A. The magician then places one of the piles from the other pack of fifteen cards, say C, face down on top of it. He then places the jack of spades face up on the top of pile D and places pile B on top of it. He puts one of the two packs thus obtained on top of the other getting a single deck that now contains 32 cards, including the subject's card and the upturned jack of spades.

The magician deals the 32 cards alternately into two piles, rejects the pile that does not contain the jack of spades, and deals the remaining pile into two smaller piles of 8 cards each. Again he keeps only the pile with the jack of spades and deals it into two piles with 4 cards each. He finally deals the four-card pile with the jack of spades into two piles of two cards each. In the case, subject's card has been hunted down by the jack of spades as the two finally appear together.

The explanation is pretty straightforward. When the two piles are finally incorporated, note that the jack of spades and the subject's card are separated by one of the original packs of fifteen (either C and D, or else A and B, depending on the order of stacking). Since these two cards are separated by an odd number of cards, when the 32 cards are dealt into two piles, the jack and subject's card will be dealt into the same pile and be separated by seven cards. When we deal this pile into two subpiles, we have a similar situation with the two cards separated by an odd number of cards, and so we continue until we get down to two cards.

9. Balancing red and black. Take an ordinary deck of playing cards without the jokers and shuffle well. You are going to make four piles of cards, two face up and two face down, as follows. Turn over the top card and put it face up on the table. Put the next card in the deck in a separate pile face down on the table. Now turn over the third card. If it is the same colour as the first, place it face up on top of the first card and put the fourth card above it face down on top of the first face down card. If the third card is of the opposite colour, start a separate face up pile with it and put the fourth card above it face down in a new pile.

Continue in this way, delivering cards alternately face up and face down, with each face up card going on top of others of the same colour and each face down card going into the corresponding face down pile. When you are done, the red card pile has the same number of cards as the corresponding face down pile; similarly the black card pile has the same number of cards as its corresponding face down pile.

Count the number of black cards in the face down pile corresponding to the black face up pile and the number of red cards in the other face down pile. What do you observe and why?

Let R be the pile of face up red cards and S be its corresponding pile of face down cards. Let B be the pile of face up black cards and C be its corresponding pile of face down cards. Note that R with C constitute half the deck as do B and S.

Suppose that C has n black cards. Then there are n black cards in the pile R and C together. These black cards must displace n red cards, which will be in the other half of the deck. Since they are not in B, they will be in S.

There is a variant of this trick that goes as follows. Separate a standard deck into two piles, one consisting of all the red cards and the other of all black. Remove some number of red cards from its pile and shuffle it thoroughly into the black pile. Now randomly pick the same number from the shuffled black pile and restore it to the red pile, so that each pile has 26 cards. Then the number of black cards in one pile will equal the number of red cards in the other.

This is a version of the water and wine problem: a cup of water is removed from a litre flask of water and stirred into a litre of wine. A cup of the mixture is then poured into the water flask. is there more wine in the water flask that there is water in the wine flask?

10. A little hidden algebra. The magician takes 26 cards from a regular deck and places it face down on the table. He then turns over the cards one by one to show the subject that the deck is randomly mixed, and then restores the 26 cards to the original position; call this the *stock*. Handing the remaining 26 cards to the subject, he instructs the subject to place a card face up on the table. We will use the equivalence $A = 1$, $J = Q = K = 10$. If the card turned up is k, the subject then places on top of it sufficiently many cards face up to count up to ten. The ranks of the additional cards are immaterial, the subject counting k, $k + 1$, $\cdots$, 10 until she reaches 10. Then the subject starts a new pile by placing one of the remaining cards on the table, and performing the same operation. This is repeated as long as there are sufficiently many cards and there are at least three piles. (In the rare case that there are not enough cards to form three piles, the subject can "borrow" from the top of the stock.)

The subject then turns three of the piles over and puts the rest of the cards face down on top of the 26-card stock left by the magician. The subject is then to turn over the top card on each of the three piles, add them and count down that many cards in the stock (the 26-cards augmented by the leftovers). While the subject is doing this, the magician predicts what the terminal card will be.

For example, suppose the subject turns over a 4; then she will place on top of it face up six more cards, counting as she goes 5 - 6 - 7 - 8 - 9 - 10. If the three piles chosen are built on, say, 4, 3 and 8, then the three piles built up on them will have, respectively, 7, 8 and 3 cards. Eight cards will be returned to the stock, which will now have a total of 34 cards. When the subject turns over the three piles and reveals the top cards, these will, of course, be 4, 3 and 8, and the subject will count down 15 cards into the stock. This will go through the eight returned to the stock and end up with the seventh from the top of the original stock of 26.

Remarkably, no matter what cards are turned face up, the count will go down to the seventh card from the top of the 26-card stock, and it is this card that the magician must memorize. I usually convince students that it works in the following way. Suppose that the three cards turned up are all tens. Then twenty three cards are returned to the stock, and we have to count down 30 cards to the seventh from the top of the original stock. For every reduction of one in the sum of the three cards, there is one more card in the three piles and one fewer returned to the stock. At the same time, there is one fewer card to count down, so we will always wind up in the same place.

I am indebted to Peter Taylor of Queen's University for showing me this nice trick.

11. A quick reversion to order. Begin with a new deck of cards in which the suits appear in order, ranked in order. A remarkable fact is that eight perfect inside riffle shuffles (where the top and bottom cards of the deck remain in position) will restore the deck to its original order. If, like me, you cannot perform a perfect riffle shuffle, you can deal them to obtain the inverse effect of a riffle and still get a striking effect. Suppose that the cards are numbered from 0 to 51, inclusive, and are originally in this order from top to bottom. Deal the cards face up alternately into left and right piles, 0 to the left, 1 to the right, and so on. Pick up the piles, putting the right pile on the left one, turn the incorporated deck upside down and repeat. Now 0 goes to the left, 2 to the right, 4 to the left, 6 to the right and so on. Repeat the process.

Each time the process is repeated and the deck incorporated, the value of the card in any given position gets multiplied by 2 modulo 51. Since $2^8 \equiv 1 \pmod{51}$, eight repetitions will bring the cards back to the original order. However, when the cards are dealt face up, students can see how the order changes from one deal to the next and some interesting things occur. Try it!

While one generally cannot go into the number theory involved for most school students, the investigation of how long it takes this shuffle to return a deck to its original order for various numbers of cards is worthwhile.

Pedagogical considerations. Do such card stunts have a place in the curriculum? Most assuredly they do. Apart from the "fun" aspect, there is real mathematics here. None of these involve sleight of hand or any motor skills; they can be carried out by any student. They are all mathematically based, and can be justified through a careful analysis that is accessible, in some cases, even to elementary students. Their value in the curriculum is that they give an authentic view of the analytical side of mathematics that the standard syllabus, with its emphasis on skills, either hardly hints at or obscures with technicalities. In analyzing the arguments for the tricks, one can see that important mathematical ideas, such as pairing, induction, algebraic structure and transformations are adumbrated.

Even though formal proofs might not be appropriate, enough can be said to convince students of what makes the tricks work. The important message is that of the possibility of proof and the adoption of a perspective that helps to see what is going on. The more technical aspects of the construction and presentation of proofs will not come later on in a vacuum.

Because of the difficulty of systematizing and testing such activities, it may be though that they are not suitable in a curriculum. But this is a strong argument for inclusion. Any attempt to formalize or test them would be destructive. It can be argued that some of the most important things we want to convey about the mathematical enterprise are things that cannot and ought not to be tested, but rather insinuated where appropriate into the regular mathematical experiences of the students, so that they become part of the landscape.

Like all attempts to alter the thrust of the curriculum, this will succeed or fail depending on the background and quality of he teaching corps. This is another instance of how we must start with sound policies for the recruitment and formation of teachers before we can contemplate the reforms in mathematical schooling we would all like to see.

A copy of this description can be found at
www.math.utoronto.ca/barbeau/cards.pdf.

Ed Barbeau