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SOME HOMOTOPY PROPERTIES OF SPACES OF FINITE SUBSETS OF TOPOLOGICAL SPACES

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Abstract. For X a non-empty topological space and k a positive integer, we denote by $Sub(X,k)$ the set of non-empty subsets of X having cardinality $\leq k$, suitably topologized. The $Sub(\,\cdot\,,k)$ are homotopy functors and their properties are studied. We prove that if X is Hausdorff and path-connected, then for all $k \geq 1$ and $n \geq 0$, the maps $\pi_n\big(Sub(X,k)\big) \to \pi_n\big(Sub(X,2k+1)\big)$ induced by the inclusion are the 0-maps. In the direction of non-triviality, we prove that if X is a non-empty closed manifold of dimension ≥ 2, then for each $k \geq 1$, $Sub(X,k)$ is homologically non-trivial.

1. Introduction

Let X be a non-empty topological space and k a positive integer. We denote by $Sub(X,k)$ the set of non-empty subsets of X having cardinality $\leq k$. As a set, $Sub(X,k)$ contains the configuration spaces $C(X,i)$ for $1 \leq i \leq k$ where $C(X,i)$ is the space of unordered i-tuples of distinct points of X. The $C(X,i)$ have proved important in homotopy theory (e.g. [1], [4]) and certain geometric applications (e.g. [3], [5], [7], [8], [9], [10], [11]). Our topologization of $Sub(X,k)$ will be such that for $1 \leq i \leq k$, $C(X,i)$ with its standard topology will be a subspace of $Sub(X,k)$, and will take into account the fact that finite subsets of different cardinalities may nevertheless be close. In contrast with the $C(\,\cdot\,,k)$, the $Sub(\,\cdot\,,k)$ are functors (in fact, homotopy functors).

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Our first main result is that if X is a non-empty path-connected Hausdorff space, then for each $k \geq 1$ and $n \geq 0$ the map

$$\pi_n\big(Sub(X,k)\big) \to \pi_n\big(Sub(X,2k+1)\big)$$

induced by the inclusion is the 0-map. In contrast with this, our second main result is that if X is a non-empty closed manifold of dimension ≥ 2, then $Sub(X,k)$ is homologically non-trivial for all $k \geq 1$.

In §2 we topologize the $Sub(X,k)$ and establish some general topological properties. In §3 we establish some homotopy properties of the $Sub(X,k)$, and our main results are proved in §4.

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2. General Topology of $Sub(X,k)$

Let X be a non-empty Hausdorff space and k a positive integer. Write X^k for the k-fold Cartesian product $X \times \cdots \times X$ and let $q_k^X : X^k \to Sub(X,k)$ be given by $q_k^X(x_1,\ldots,x_k) = \{x_1,\ldots,x_k\}$. Thus, for example, if x, $y \in X$, then $q_3^X(x,x,y) = q_3^X(x,y,y) = \{x,y\}$. We give $Sub(X,k)$ the quotient topology relative to the surjection q_k^X, and will henceforth call this topology the *standard topology* on $Sub(X,k)$. Note that q_k^X factors through the k^{th} symmetric product $Sp^k(X)$. We sometimes abbreviate q_k^X by leaving off the subscript k and/or the superscript X when there is no danger of confusion. Trivially, the quotient map $q : X^1 \to Sub(X,1)$ is a homeomorphism.

If k is a positive integer, let $\underline{k}$ denote $\{1,\ldots,k\}$. For positive integers k,l and any function $\alpha : \underline{k} \to \underline{l}$ we obtain, for any topological space X, a continuous map $\alpha_X : X^l \to X^k$ given by $\alpha_X(x_1,\ldots,x_l) = \big(x_{\alpha(1)},\ldots,x_{\alpha(k)}\big)$. Let $\mathcal{N}$ denote the full sub-category of the category of sets whose objects are the $\underline{k}$, $k \geq 1$. The following Lemma is immediate:

Lemma 2.1. *For each fixed topological space X, the assignments $\underline{k} \longmapsto X^k$ and $\alpha \longmapsto \alpha_X$ constitute a contravariant functor from $\mathcal{N}$ to the category of topological spaces. Furthermore, if $\alpha : \underline{k} \to \underline{l}$, then for each non-empty topological space X, the image of α_X is* $\{(x_1,\ldots,x_k) \mid x_i = x_j \text{ whenever } \alpha(i) = \alpha(j)\}$. □

Lemma 2.2. *Let $\alpha : \underline{k} \to \underline{l}$ and suppose X is a non-empty Hausdorff space. Then the image of α_X is closed in X^k.*

Proof. Suppose $x = (x_1,\ldots,x_k) \in X^k - \operatorname{Im}\alpha_X$. Then there exist i, $j \in \underline{k}$ such that $\alpha(i) = \alpha(j)$ but $x_i \neq x_j$. Choose disjoint neighborhoods U, V in X of x_i,

x_j, respectively. Then $\{(y_1, \ldots, y_k) \mid y_i \in U,\ y_j \in V\}$ is a neighborhood of x in X^k which is disjoint from $\operatorname{Im} \alpha_X$. $\square$

For $k \geq l \geq 1$ let $\operatorname{surj}(k, l)$ denote the set of all surjections $\underline{k} \to \underline{l}$.

Lemma 2.3. *Let X be a non-empty Hausdorff space. Then for integers $k \geq l \geq 1$ and $\alpha \in \operatorname{surj}(k, l)$, $\alpha_X : X^l \to X^k$ is a homeomorphism onto a closed subspace of X^k.*

Proof. Since α is surjective, we can choose a function $\beta : \underline{l} \to \underline{k}$ such that $\alpha\beta$ is the identity on $\underline{l}$. It follows that $\beta_X \alpha_X$ is the identity on X^l and so α_X is a homeomorphism onto its image. The latter is closed in X^k by Lemma 2.2. $\square$

We have set inclusions

$$Sub(X, 1) \subset Sub(X, 2) \subset Sub(X, 3) \subset \cdots .$$

The question arises as to whether the standard topology on $Sub(X, k)$ agrees with the subspace topology derived from the standard topology on $Sub(X, k + 1)$. Fortunately the two topologies agree:

Proposition 2.4. *Let X be a non-empty Hausdorff space and k a positive integer. Then the standard topology on $Sub(X, k)$ agrees with the subspace topology derived from the standard topology on $Sub(X, k + 1)$. Moreover, $Sub(X, k)$ is closed in $Sub(X, k + 1)$.*

Proof. For each $\alpha \in \operatorname{surj}(k + 1, k)$, it follows from Lemma 2.3 that $\alpha_X : X^k \to X^{k+1}$ is a closed map. For each such α the diagram

$$\begin{array}{ccc} X^k & \xrightarrow{\ \alpha_X\ } & X^{k+1} \\ \big\downarrow{\scriptstyle q_k} & & \big\downarrow{\scriptstyle q_{k+1}} \\ Sub(X, k) & \xrightarrow[i]{} & Sub(X, k + 1) \end{array}$$

commutes where i is the inclusion map. Thus i is continuous with respect to the standard topologies on $Sub(X, k)$ and $Sub(X, k + 1)$. Thus all assertions will follow if we show that i is a closed map with respect to the standard topologies.

Let A be closed in $Sub(X, k)$. Then $q_k^{-1}(A)$ is closed in X^k and hence each $\alpha_X\left(q_k^{-1}(A)\right)$ is closed in X^{k+1}. We have

$$q_{k+1}^{-1}\left(i(A)\right) = \bigcup_{\alpha \in \operatorname{surj}(k+1, k)} \alpha_X\left(q_k^{-1}(A)\right).$$

Since $\operatorname{surj}(k+1,k)$ is finite, $q_{k+1}^{-1}\big(i(A)\big)$ is closed in X^{k+1}, and hence $i(A)$ is closed in $Sub(X,k+1)$. □

Proposition 2.5. *Let X be a non-empty Hausdorff space and k a positive integer. Then the quotient map $q_k : X^k \to Sub(X,k)$ is a closed map.*

PROOF. We proceed by induction on k, the result being trivial for $k=1$. Suppose $k>1$ and that $q_{k-1} : X^{k-1} \to Sub(X,k-1)$ is a closed map. Note that for any subset A of X^k,

$$q_k^{-1}q_k(A) = \left(\bigcup_{\alpha \in \operatorname{surj}(k,k-1)} \alpha_X\, q_{k-1}^{-1}\Big(q_k(A) \cap Sub(X,k-1)\Big)\right) \cup \bigcup_{\beta \in \operatorname{surj}(k,k)} \beta_X(A).$$

Since, by Lemma 2.3, the α_X and β_X are all closed maps, it remains only to show that if A is closed in X^k, then $q_k(A) \cap Sub(X,k-1)$ is closed in $Sub(X,k-1)$. This follows immediately from the fact that

$$q_k(A) \cap Sub(X,k-1) = \bigcup_{\alpha \in \operatorname{surj}(k,k-1)} q_{k-1}{\alpha_X}^{-1}(A)$$

and the inductive hypothesis that q_{k-1} is a closed map. □

In general, q_k^X need not be an open map. For example, if $X = \boldsymbol{R}$ and $U = (0,2)\times(0,2)\times(2,4)$, then $(1,3,3) \in q_3^{-1}q_3(U)$ but $(1,3,3)$ is not an interior point of $q_3^{-1}q_3(U)$. However, we do have the following:

Lemma 2.6. *Let X be a non-empty Hausdorff space, k a positive integer, and suppose U is open in X. Then $q(U \times X^{k-1})$ and $q(U^k)$ are open in $Sub(X,k)$.*

PROOF. Each $\sigma \in \operatorname{surj}(k,k) = \Sigma_k$ yields a self-homeomorphism $\sigma_X : X^k \to X^k$. Note that

$$q^{-1}q(U \times X^{k-1}) = \bigcup_{\sigma \in \Sigma_k} \sigma_X(U \times X^{k-1}),$$

a union of open sets, proving the openness of $q(U \times X^{k-1})$. Since $q^{-1}q(U^k) = U^k$, the openness of $q(U^k)$ follows. □

Proposition 2.7. *If X is a non-empty Hausdorff space and k a positive integer, then $Sub(X,k)$ is Hausdorff.*

PROOF. Let S and T be distinct points in $Sub(X,k)$. We can suppose that there exists an $x \in S$ such that $x \notin T$. Since X is Hausdorff we can choose disjoint open subsets U and V of X such that $x \in U$ and $T \subset V$. By Lemma 2.6, $q(U \times X^{k-1})$ and $q(V^k)$ are open in $Sub(X,k)$. Note that they are disjoint and that $S \in q(U \times X^{k-1})$, $T \in q(V^k)$. □

Suppose X is a pointed Hausdorff space with basepoint x_0. For any positive integer k, let $Sub_0(X,k)$ denote the subspace of $Sub(X,k)$ consisting of those subsets which contain x_0. Then $Sub_0(X,k)$ is a pointed Hausdorff space with basepoint $\{x_0\}$.

Proposition 2.8. *Let X be a pointed Hausdorff space and k a positive integer. Then $Sub_0(X,k)$ is a closed subspace of $Sub(X,k)$.*

Proof. $q^{-1}\big(Sub_0(X,k)\big)$ consists of all k-tuples of points of X with at least one coordinate equal to x_0, and this is closed in X^k. □

Suppose X and Y are non-empty Hausdorff spaces and $f : X \to Y$ continuous. For $k \geq 1$ define $Sub(f,k) : Sub(X,k) \to Sub(Y,k)$ by $Sub(f,k)(S) = f(S)$ for each $S \in Sub(X,k)$. If X and Y are pointed and f is a pointed map, define $Sub_0(f,k) : Sub_0(X,k) \to Sub_0(Y,k)$ to be the restriction of $Sub(f,k)$.

Proposition 2.9. *For each $k \geq 1$, $Sub(\,\cdot\,,k)$ is a covariant functor from the category of non-empty Hausdorff spaces to itself. If $f : X \to Y$ is a continuous map of non-empty Hausdorff spaces, the diagram*

$$\begin{array}{ccc} Sub(X,k) & \xrightarrow{Sub(f,k)} & Sub(Y,k) \\ \downarrow & & \downarrow \\ Sub(X,k+1) & \xrightarrow[Sub(f,k+1)]{} & Sub(Y,k+1) \end{array}$$

commutes, where the vertical maps are the inclusions.

Proof. The only issue is continuity of $Sub(f,k)$ when $f : X \to Y$ is continuous. This is immediate from commutativity of the diagram

$$\begin{array}{ccc} X^k & \xrightarrow{f^k} & Y^k \\ {\scriptstyle q^X}\downarrow & & \downarrow{\scriptstyle q^Y} \\ Sub(X,k) & \xrightarrow[Sub(f,k)]{} & Sub(Y,k)\ , \end{array}$$

the continuity of the top and two vertical maps, and the fact that q^X is a quotient map. □

By restriction we obtain:

Proposition 2.10. *For each $k \geq 1$, $Sub_0(\,\cdot\,, k)$ is a covariant functor from the category of pointed Hausdorff spaces to itself. If $f : X \to Y$ is a pointed continuous map of pointed Hausdorff spaces, the diagram*

$$\begin{array}{ccc} Sub_0(X,k) & \xrightarrow{Sub_0(f,k)} & Sub_0(Y,k) \\ \downarrow & & \downarrow \\ Sub_0(X,k+1) & \xrightarrow[Sub_0(f,k+1)]{} & Sub_0(Y,k+1) \end{array}$$

commutes, where the vertical maps are the inclusions. □

We next describe a base for the standard topology on $Sub(X, k)$. Suppose $U_1, \dots, U_r$ are pairwise-disjoint, non-empty open subsets of X with $r \leq k$. Write $(U_1, \dots, U_r)_k^X = \{A \in Sub(X, k) \mid A \cap U_i \neq \emptyset \text{ for } 1 \leq i \leq r \text{ and } A \subset U_1 \cup \cdots \cup U_r\}$. Let $\mathcal{B}$ be any base for the topology on X and let

$$\mathcal{B}_k = \big\{(U_1, \dots, U_r)_k^X \mid U_i \in \mathcal{B} \text{ for all } i\big\}.$$

Proposition 2.11. *Let X be a non-empty Hausdorff space, $\mathcal{B}$ a base for the topology on X, and k a positive integer. Then $\mathcal{B}_k$ is a base for the standard topology on $Sub(X, k)$. Moreover, if V is open in $Sub(X, k)$ and $A = \{x_1, \dots, x_r\} \in V$ where the x_i are distinct, there exist pairwise-disjoint neighborhoods $U_1, \dots, U_r$ of $x_1, \dots, x_r$, respectively, in $\mathcal{B}$ such that $A \in (U_1, \dots, U_r)_k^X \subset V$.*

Proof. Let $(U_1, \dots, U_r)_k^X \in \mathcal{B}_k$. We have

$$q^{-1}\big((U_1, \dots, U_r)_k^X\big) = \bigcup_{\alpha \in \mathrm{surj}(k,r)} U_{\alpha(1)} \times \cdots \times U_{\alpha(k)},$$

a union of open rectangles and hence open in X^k, whence $\mathcal{B}_k$ consists of open sets in $Sub(X, k)$.

Now let V be any open subset of $Sub(X, k)$ and suppose $A = \{x_1, \dots, x_r\} \in V$ where the x_i are distinct. Then $q^{-1}(V)$ is open in X^k and $q^{-1}(A) \subset q^{-1}(V)$. Note that $q^{-1}(A) = \big\{(x_{\alpha(1)}, \dots, x_{\alpha(k)}) \mid \alpha \in \mathrm{surj}(k, r)\big\}$. For each $\alpha \in \mathrm{surj}(k, r)$ we can choose pairwise-disjoint open neighborhoods $U_1^\alpha, \dots, U_r^\alpha$ of $x_1, \dots, x_r$, respectively, such that $U_{\alpha(1)}^\alpha \times \cdots \times U_{\alpha(k)}^\alpha \subset q^{-1}(V)$. For $1 \leq i \leq r$ we can choose $U_i \in \mathcal{B}$ such that

$$x_i \in U_i \subset \bigcap_{\alpha \in \mathrm{surj}(k,r)} U_i^\alpha.$$

Then $A \in (U_1, \dots, U_r)_k^X \subset V$. □

Corollary 2.12. *If X is a non-empty second-countable Hausdorff space, then so are the $Sub(X,k)$ for all $k \geq 1$.* □

Let X be a non-empty Hausdorff space and A a non-empty subspace of X. Then for each $k \geq 1$, $Sub(A,k)$ is a subset of $Sub(X,k)$. If $i : A \to X$ denotes the inclusion map, then $Sub(i,k) : Sub(A,k) \to Sub(X,k)$ is the inclusion map. The question arises as to whether or not the subspace topology on $Sub(A,k)$ derived from the standard topology on $Sub(X,k)$ coincides with the standard topology on $Sub(A,k)$. Fortunately, the two are the same:

Proposition 2.13. *Let A be a non-empty subspace of the Hausdorff space X and $k \geq 1$. Then:*
(a) *The subspace topology on $Sub(A,k)$ derived from the standard topology on $Sub(X,k)$ coincides with the standard topology on $Sub(A,k)$.*
(b) *If A is open (respectively closed) in X, then $Sub(A,k)$ is open (respectively closed) in $Sub(X,k)$.*

Proof of (a). Since the inclusion map $Sub(A,k) \to Sub(X,k)$ is $Sub(i,k)$ where $i : A \to X$ is the inclusion, it follows that the inclusion of $Sub(A,k)$ into $Sub(X,k)$ is continuous with respect to the standard topologies. Thus it remains only to show that each subset U of $Sub(A,k)$ which is open in the standard topology on $Sub(A,k)$ is also open in the subspace topology derived from the standard topology on $Sub(X,k)$. It suffices to show that whenever $S = \{x_1, \ldots, x_r\} \in U \subset Sub(A,k)$ where the x_i are distinct and U is open in the standard topology on $Sub(A,k)$, then there exist pairwise-disjoint open neighborhoods $V_1, \ldots, V_r$ in X of $x_1, \ldots, x_r$, respectively, such that $(V_1, \ldots, V_r)_k^X \cap Sub(A,k) \subset U$. We can choose pairwise-disjoint open neighborhoods $U_1, \ldots, U_r$ in A of $x_1, \ldots, x_r$, respectively, such that $(U_1, \ldots, U_r)_k^A \subset U$. Since A has the subspace topology derived from X, there exist open subsets $T_1, \ldots, T_r$ of X such that $U_i = T_i \cap A$ for each i. Using the Hausdorffness of X, we can choose pairwise-disjoint open sets $V_1, \ldots, V_r$ in X such that $x_i \in V_i \subset T_i$ for each i. It is immediate that

$$(V_1, \ldots, V_r)_k^X \cap Sub(A,k) \subset (U_1, \ldots, U_r)_k^A \subset U. \qquad \square$$

Proof of (b). Let A be open in X and write $\mathcal{T}(A)$, $\mathcal{T}(X)$ for the topologies on A and X, respectively. Then the inclusion of bases $\mathcal{T}(A)_k \subset \mathcal{T}(X)_k$ yields that $Sub(A,k)$ is open in $Sub(X,k)$.

Suppose A is closed in X and $S \in Sub(X,k) - Sub(A,k)$. Say $S = \{x_1, \ldots, x_r\}$ where the x_i are distinct and $x_1 \notin A$. We can choose pairwise-disjoint open neighborhoods $U_1, \ldots, U_r$ in X of $x_1, \ldots, x_r$, respectively, with $U_1 \subset X - A$. Then $S \in (U_1, \ldots, U_r)_k^X \subset Sub(X,k) - Sub(A,k)$. $\square$

Proposition 2.14. *Let X be a non-empty Hausdorff space and k, l positive integers. Then the union map*

$$\mu : Sub(X,k) \times Sub(X,l) \to Sub(X,k+l)$$

given by $\mu(S,T) = S \cup T$ is continuous.

Proof. Let $\mathcal{T}$ denote the topology on X. Suppose $V = (U_1, \ldots, U_r)_{k+l}^X \in \mathcal{T}_{k+l}$ and that $(S,T) \in \mu^{-1}(V)$. Suppose $U_{m_1}, \ldots, U_{m_s}$ are the distinct U_i which meet S, and $U_{n_1}, \ldots, U_{n_t}$ the distinct U_i which meet T. Then $S \in (U_{m_1}, \ldots, U_{m_s})_k^X \in \mathcal{T}_k$, $T \in (U_{n_1}, \ldots, U_{n_t})_l^X \in \mathcal{T}_l$, and each U_i occurs either among the U_{m_i} or U_{n_j} (possibly both). It is immediate that

$$(S,T) \in (U_{m_1}, \ldots, U_{m_s})_k^X \times (U_{n_1}, \ldots, U_{n_t})_l^X \subset \mu^{-1}(V),$$

establishing the openness of $\mu^{-1}(V)$. $\square$

Proposition 2.15. *Suppose X is a non-empty, locally compact, Hausdorff space. Then for each $k \geq 1$, $Sub(X,k)$ is locally compact.*

Proof. Let $S = \{x_1, \ldots, x_r\} \in Sub(X,k)$ where the x_i are distinct. Since X is locally compact and Hausdorff we can find pairwise-disjoint open neighborhoods $U_1, \ldots, U_r$ of $x_1, \ldots, x_r$, respectively, whose closures $\overline{U}_i$ are all compact. Then

$$S \in (U_1, \ldots, U_r)_k^X \subset \bigcup_{\alpha \in \mathrm{surj}(k,r)} q\big(\overline{U}_{\alpha(1)} \times \cdots \times \overline{U}_{\alpha(k)}\big).$$

The latter union is a finite union of compact spaces and hence compact, providing a compact neighborhood of S in $Sub(X,k)$. $\square$

Proposition 2.16. *Let X and Y be non-empty Hausdorff spaces. Let k and l be positive integers. Suppose either X and Y are both locally compact, or that Y is locally compact and $l = 1$. Then the cartesian product map*

$$cp : Sub(X,k) \times Sub(Y,l) \to Sub(X \times Y, kl)$$

given by $cp(S,T) = S \times T$ is continuous.

PROOF. We have the commutative diagram

$$\begin{array}{ccc} X^k \times Y^l & \xrightarrow{\quad f \quad} & (X \times Y)^{kl} \\ {\scriptstyle q_k^X \times q_l^Y}\downarrow & & \downarrow{\scriptstyle q_{kl}^{X \times Y}} \\ Sub(X,k) \times Sub(Y,l) & \xrightarrow[cp]{} & Sub(X \times Y, kl) \end{array}$$

where, regarding $(X \times Y)^{kl}$ as the space of $k \times l$ matrices with entries in $X \times Y$, f is given by

$$f(x_1, \ldots, x_k, y_1, \ldots, y_l)_{ij} = (x_i, y_j).$$

Under the hypotheses, either all the spaces involved are locally compact and Hausdorff, or q_l^Y is the identity map on a locally compact Hausdorff space. Under either hypothesis, $q_k^X \times q_l^Y$ is a quotient map. The continuity of *cp* now follows from the continuity of the other maps in the above diagram. □

Proposition 2.17. *Suppose X is a non-empty regular space. Then for each $k \geq 1$, $Sub(X, k)$ is regular.*

PROOF. Let $S = \{x_1, \ldots, x_r\} \in Sub(X, k)$ where the x_i are distinct. By Proposition 2.11, it suffices to show that if $U_1, \ldots, U_r$ are pairwise-disjoint open neighborhoods of $x_1, \ldots, x_r$, respectively, in X, then there exists an open neighborhood V of S in $Sub(X, k)$ such that $\overline{V} \subset (U_1, \ldots, U_r)_k^X$. By regularity of X there exist open neighborhoods $V_1, \ldots, V_r$ in X of $x_1, \ldots, x_r$, respectively, such that for each i, $\overline{V}_i \subset U_i$. Let

$$A = \bigcup_{\alpha \in \mathrm{surj}(k,r)} \overline{V}_{\alpha(1)} \times \cdots \times \overline{V}_{\alpha(k)}.$$

Then A is closed in X^k and so by Proposition 2.5, $q(A)$ is closed in $Sub(X, k)$. Taking $V = (V_1, \ldots, V_r)_k^X$ we have

$$S \in V \subset q(A) \subset (U_1, \ldots, U_r)_k^X.$$ □

Corollary 2.12 and Proposition 2.17, together with the Urysohn Metrization Theorem, yield:

Theorem 2.18. *Let X be a non-empty second-countable metric space. Then for all $k \geq 1$, $Sub(X, k)$ is metrizable.* □

Proposition 2.19. *Let X and $Y_1, \ldots, Y_k$ be non-empty topological spaces with X Hausdorff. Suppose $f_i : Y_i \to X$ are continuous, $1 \leq i \leq k$. Then the map $f : Y_1 \times \cdots \times Y_k \to Sub(X, k)$ given by $f(y_1, \ldots, y_k) = \{f_1(y_1), \ldots, f_k(y_k)\}$ is continuous.*

PROOF. f is the composition

$$Y_1 \times \cdots \times Y_k \xrightarrow{f_1 \times \cdots \times f_k} X \times \cdots \times X \xrightarrow{q} Sub(X, k). \qquad \square$$

Proposition 2.20. *Let X and Y be non-empty topological spaces with X Hausdorff. Suppose $f_1, \ldots, f_k : Y \to X$ are continuous. Then $g : Y \to Sub(X, k)$ given by $g(y) = \{f_1(y), \ldots, f_k(y)\}$ is continuous.*

PROOF. g is the composition

$$Y \xrightarrow{\Delta} Y^k \xrightarrow{f} Sub(X, k)$$

where Δ is the k-fold diagonal map and f is as in Proposition 2.19. $\square$

Suppose we are given a topological group G, a non-empty Hausdorff space X, and a continuous group action $\alpha : G \times X \to X$. For each $k \geq 1$, α induces a group action $\alpha_k : G \times Sub(X, k) \to Sub(X, k)$ in the evident way.

Proposition 2.21. *Let $\alpha : G \times X \to X$ be a continuous group action where X is a non-empty Hausdorff space and G a locally compact Hausdorff topological group. Then for each $k \geq 1$, $\alpha_k : G \times Sub(X, k) \to Sub(X, k)$ is a continuous group action.*

PROOF. The only issue is continuity of α_k. We have the commutative diagram

$$\begin{array}{ccc} G \times X^k & \xrightarrow{f} & X^k \\ {\scriptstyle 1_G \times q_k}\downarrow & & \downarrow{\scriptstyle q_k} \\ G \times Sub(X, k) & \xrightarrow[\alpha_k]{} & Sub(X, k) \end{array}$$

where $f(g, x_1, \ldots, x_k) = \big(gx_1, \ldots, gx_k\big)$. $1_G \times q_k$ is a quotient map since G is locally compact and Hausdorff. Since f and q_k are continuous, continuity of α_k follows. $\square$

Let X be a non-empty Hausdorff space. Recall that $C(X, k)$, the configuration space of unordered k-tuples of distinct points of X, is the quotient space $F(X, k)/\Sigma_k$ where

$$F(X, k) = \{(x_1, \ldots, x_k) \in X^k \mid x_i \neq x_j \text{ if } i \neq j\}$$

and the symmetric group Σ_k acts by permutating coordinates. As a set, $C(X,k) \subset Sub(X,k)$. Note that $F(X,k)$ is open in X^k, the diagram

$$\begin{array}{ccc} F(X,k) & \xrightarrow{\subset} & X^k \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} \\ C(X,k) & \xrightarrow[\subset]{} & Sub(X,k) \end{array}$$

is commutative where q and q' are the respective quotient maps, and $F(X,k) = q^{-1}\big(C(X,k)\big)$. Thus:

Proposition 2.22. *Let X be a non-empty Hausdorff space and $k \geq 1$. Then the topology on $C(X,k)$ as a quotient of $F(X,k)$ coincides with the subspace topology derived from the standard topology on $Sub(X,k)$. Moreover, $C(X,k)$ is open in $Sub(X,k)$.* □

For X a locally compact Hausdorff space, let X^+ denote the one-point compactification of X. We follow the convention that if X is already compact, then X^+ is the union of X with a new isolated point.

For any non-empty Hausdorff space X and any $k \geq 2$, the composition

$$C(X,k) \overset{\subset}{\to} Sub(X,k) \overset{p}{\to} Sub(X,k)/Sub(X,k-1)$$

where p is the collapsing map, is a continuous injection onto a subspace whose complement consists of a single point $*$.

Proposition 2.23. *Let X be a non-empty regular space and $k \geq 2$. Then:*
(a) *The injection $C(X,k) \to Sub(X,k)/Sub(X,k-1)$ is a homeomorphism of $C(X,k)$ onto an open subspace of $Sub(X,k)/Sub(X,k-1)$.*
(b) *If, additionally, X is compact, then $Sub(X,k)/Sub(X,k-1)$ is $C(X,k)^+$, the one-point compactification of $C(X,k)$.*

Proof. $Sub(X,k)$ is regular by Proposition 2.17, and $Sub(X,k-1)$ is closed in $Sub(X,k)$ by Proposition 2.4. Thus $Sub(X,k)/Sub(X,k-1)$ is Hausdorff. Write $i : C(X,k) \to Sub(X,k)/Sub(X,k-1)$ for the above injection. It follows easily from the openness of $C(X,k)$ in $Sub(X,k)$ (Proposition 2.22) that i is an open map. Part (a) now follows, and we henceforth identify $C(X,k)$ with an open subspace of $Sub(X,k)/Sub(X,k-1)$.

Suppose, additionally, X is compact. Then $C(X,k)$ is locally compact and Hausdorff, and so $C(X,k)^+$ is defined. Since

$$Sub(X,k)/Sub(X,k-1) = C(X,k) \cup \{*\}$$

and the former is compact Hausdorff, part (b) follows. □

Theorem 2.24. *For each $n \geq 1$ and $k \geq 1$, $Sub(\boldsymbol{R}^n, k)$ is topologically embeddable in some finite-dimensional Euclidean space.*

Proof. We proceed by induction on k, the result being immediate for $k = 1$. Suppose $k > 1$ and that $f : Sub(\boldsymbol{R}^n, k-1) \to \boldsymbol{R}^a$ is a topological embedding for some positive integer a. By Theorem 2.18, $Sub(\boldsymbol{R}^n, k)$ is normal and so, by Proposition 2.4 and the Tietze Extension Theorem, f extends to a continuous map $g : Sub(\boldsymbol{R}^n, k) \to \boldsymbol{R}^a$. Again, using Theorem 2.18 and Proposition 2.4, there exists a non-negative-valued continuous map $\alpha : Sub(\boldsymbol{R}^n, k) \to \boldsymbol{R}$ such that $\alpha^{-1}(0) = Sub(\boldsymbol{R}^n, k-1)$. Since $C(\boldsymbol{R}^n, k)$ is a smooth manifold, there exists a topological embedding $h : C(\boldsymbol{R}^n, k) \to S^b$ for some positive integer b. Define $i : Sub(\boldsymbol{R}^n, k) \to \boldsymbol{R}^{b+1}$ by

$$i(x) = \begin{cases} \alpha(x)h(x) & \text{if } x \in C(\boldsymbol{R}^n, k), \\ 0 & \text{if } x \in Sub(\boldsymbol{R}^n, k-1). \end{cases}$$

Continuity of i follows easily from the facts that h and i are continuous, h is bounded, and α vanishes on $Sub(\boldsymbol{R}^n, k-1)$. Define $j : Sub(\boldsymbol{R}^n, k) \to \boldsymbol{R}^{a+b+2}$ by $j(x) = \big(g(x), \alpha(x), i(x)\big)$. Then j is continuous. Since g distinguishes different points of $Sub(\boldsymbol{R}^n, k-1)$, α distinguishes points of $Sub(\boldsymbol{R}^n, k-1)$ from points of $C(\boldsymbol{R}^n, k)$, and i distinguishes different points of $C(\boldsymbol{R}^n, k)$, j is injective. Thus, writing D^n for the closed unit disk in $\boldsymbol{R}^n$, compactness of $Sub(D^n, k)$ implies that the restriction of j to $Sub(D^n, k)$ is a topological embedding. Since the interior of D^n is homeomorphic to $\boldsymbol{R}^n$, the result now follows from the functoriality of $Sub(\,\cdot\,, k)$ and Proposition 2.13. □

Corollary 2.25. *Suppose X is homeomorphic to a non-empty subspace of some finite-dimensional Euclidean space. Then for each $k \geq 1$, $Sub(X, k)$ is topologically embeddable in some finite-dimensional Euclidean space.* □

3. Homotopy Properties of $Sub(X, k)$

Proposition 3.1. *Let (X, x_0) be a path-connected pointed Hausdorff space. Then for all $k \geq 1$, $Sub(X, k)$ and $Sub_0(X, k)$ are path-connected.*

Proof. Since X^k and $\{x_0\} \times X^{k-1}$ are path-connected, so are their images under the quotient map q. □

Proposition 3.2. *Let $h : X \times I \to Y$ be a homotopy from f to g where X and Y are non-empty Hausdorff spaces. Then:*

(a) *$h_k : Sub(X,k) \times I \to Sub(Y,k)$ given by*

$$h_k\Big(\{x_1,\ldots,x_r\},t\Big) = \big\{h(x_1,t),\ldots,h(x_r,t)\big\}$$

is a homotopy from $Sub(f,k)$ to $Sub(g,k)$. Moreover, the diagram

$$\begin{array}{ccc} Sub(X,k)\times I & \xrightarrow{h_k} & Sub(Y,k) \\ \downarrow & & \downarrow \\ Sub(X,k+1)\times I & \xrightarrow[h_{k+1}]{} & Sub(Y,k+1) \end{array}$$

commutes, where the vertical maps are the inclusions.
(b) *If A is a non-empty subset of X and h is a homotopy rel A (i.e. $h(a,t)$ is independent of t for each $a \in A$), then h_k is a homotopy rel $Sub(A,k)$.*

Proof. The only issue is the continuity of h_k. We have the commutative diagram

$$\begin{array}{ccccccc} X^k\times I & \xrightarrow{1_X\times\Delta} & X^k\times I^k & \xrightarrow{\iota} & (X\times I)^k & \xrightarrow{h^k} & Y^k \\ \downarrow{\scriptstyle q_k^X\times 1_I} & & & & & & \downarrow{\scriptstyle q_k^Y} \\ Sub(X,k)\times I & & & \xrightarrow[h_k]{\qquad\qquad} & & & Sub(Y,k) \end{array}$$

where $\Delta : I \to I^k$ is the k-fold diagonal map, and ι is the permutation map which interleaves the coordinates of X^k with those of I^k. Since I is locally compact and Hausdorff, $q_k^X \times 1_I$ is a quotient map. The continuity of h_k now follows. □

Corollary 3.3. *Let $h : X \times I \to Y$ be a pointed homotopy from f to g where X and Y are pointed Hausdorff spaces. Then $h_k : Sub_0(X,k) \times I \to Sub_0(Y,k)$ given by*

$$h_k\Big(\{x_1,\ldots,x_r\},t\Big) = \big\{h(x_1,t),\ldots,h(x_r,t)\big\}$$

is a pointed homotopy from $Sub_0(f,k)$ to $Sub_0(g,k)$. □

Corollary 3.4. *For non-empty Hausdorff spaces X and $k \geq 2$, the homotopy type of $Sub(X,k)/Sub(X,k-1)$ depends only on the homotopy type of X.* □

In general, the homotopy type of $C(X,k)$ is *not* determined by the homotopy type of X (see [4]). However, by Proposition 2.23 and Corollary 3.4, for non-empty compact Hausdorff spaces X, the homotopy type of $C(X,k)^+$ depends only on the homotopy type of X.

Proposition 3.5. *Suppose $i : A \overset{\subset}{\to} X$ is a cofibration where X is Hausdorff, and A non-empty. Then for each $k \geq 1$, $Sub(i,k) : Sub(A,k) \to Sub(X,k)$ is a cofibration. If X is pointed with basepoint in A, then $Sub_0(i,k) : Sub_0(A,k) \to Sub_0(X,k)$ is a cofibration.*

Proof. Let $r : X \times I \to X \times I$ be a retraction of $X \times I$ onto $(X \times \{0\}) \cup (A \times I)$. Let $\pi_1 : X \times I \to X$ and $\pi_2 : X \times I \to I$ denote the respective projections. Let $f : Sub(X,k) \times I \to Sub(X,k) \times I$ be the composition

$$\begin{array}{ccc} Sub(X,k) \times I = Sub(X,k) \times Sub(I,1) & \xrightarrow{cp} & Sub(X \times I, k) \\ & & \downarrow{\scriptstyle \left(Sub(\pi_1 r,\, k),\, Sub(\pi_2 r,\, k)\right)} \\ & & Sub(X,k) \times Sub(I,k) \\ & & \downarrow{\scriptstyle 1_{Sub(X,\,k)} \times min} \\ & & Sub(X,k) \times I \end{array}.$$

The cartesian product map *cp* is continuous by Proposition 2.16. The composition $min \circ q_k^I$ is clearly continuous, and so min is continuous. Hence f is continuous. It is easily checked that f is a retraction onto $\bigl(Sub(X,k) \times \{0\}\bigr) \cup \bigl(Sub(A,k) \times I\bigr)$. The required retraction in the pointed case is obtained by restriction of this f. $\square$

Proposition 3.6. *Suppose X is a non-empty locally contractible Hausdorff space. Then for each $k \geq 1$, $Sub(X,k)$ is locally contractible.*

Proof. It suffices to show that whenever $U_1, \ldots, U_r$ are mutually disjoint open subsets of X with $h_i : U_i \times I \to U_i$ a strong deformation retraction to a one-point space $\{x_i\}$, $1 \leq i \leq r \leq k$, then $\bigl\{\{x_1, \ldots, x_r\}\bigr\}$ is a strong deformation retract of $(U_1, \ldots, U_r)_k^X$. For each $\alpha \in \mathrm{surj}(k,r)$ let

$$h^\alpha : U_{\alpha(1)} \times \cdots \times U_{\alpha(k)} \times I \to U_{\alpha(1)} \times \cdots \times U_{\alpha(k)}$$

be given by

$$h^\alpha\bigl((u_1, \ldots, u_k), t\bigr) = \bigl(h_{\alpha(1)}(u_1, t), \ldots, h_{\alpha(k)}(u_k, t)\bigr)$$

and let

$$h : \left(\bigcup_{\alpha \in \mathrm{surj}(k,r)} U_{\alpha(1)} \times \cdots \times U_{\alpha(k)} \right) \times I \to \bigcup_{\alpha \in \mathrm{surj}(k,r)} U_{\alpha(1)} \times \cdots \times U_{\alpha(k)}$$

be the disjoint union of the h^α. Passage to quotients yields a continuous

$$\bar{h} : (U_1, \ldots, U_r)_k^X \times I \to (U_1, \ldots, U_r)_k^X$$

which is the desired strong deformation retraction. □

Lemma 3.7. *Let X be a non-empty compact locally contractible space which is topologically embeddable in some finite-dimensional Euclidean space. Then:*
(a) *For each $k \geq 1$, $Sub(X, k)$ is an* ANR.
(b) *For each $k \geq 2$, $Sub(X, k)/Sub(X, k-1)$ is an* ANR.

Proof. By Corollary 2.25 and Proposition 3.6, $Sub(X, k)$ is a compact, locally contractible space which is topologically embeddable in some finite-dimensional Euclidean space. Part (a) now follows from [2, p. 240].

Applying [12, Theorem 8.2] to the case $X_1 = Sub(X, k)$, $A_1 = Sub(X, k-1)$ and $X_2 =$ a one-point space, part (b) follows. □

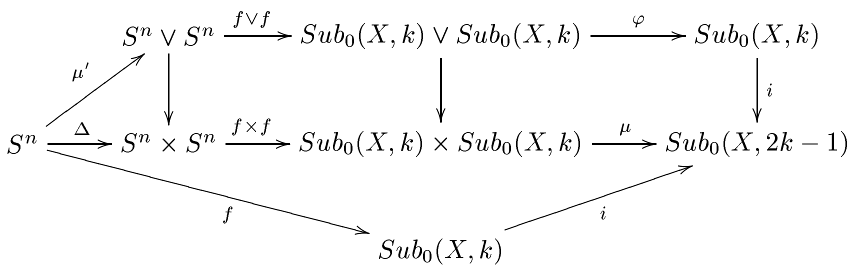
Theorem 3.8. *Let X be a non-empty compact locally contractible space which is topologically embeddable in some finite-dimensional Euclidean space. Then for all $k > 1$, the inclusion $Sub(X, k-1) \to Sub(X, k)$ is a cofibration.*

Proof. By Lemma 3.7(a), the spaces $Sub(X, i)$ are locally compact, separable metric ANRs, and hence ENRs. The assertion is now a consequence of [6, p. 84, Problem 3*]. □

4. Main Theorems

Theorem 4.1. *Let X be a path-connected pointed Hausdorff space. Then for each $k \geq 1$ and $n \geq 0$, the map $\pi_n\big(Sub_0(X, k)\big) \to \pi_n\big(Sub_0(X, 2k-1)\big)$ induced by the inclusion is the 0-map.*

Proof. The result is immediate for $n = 0$ by Proposition 3.1. Let $n \geq 1$. Thus π_n is group-valued. We use additive notation even though the group operation might be non-commutative in case $n = 1$. Let $f : S^n \to Sub_0(X, k)$ be a pointed map. We have the homotopy-commutative diagram



(in fact, all regions are strictly commutative except for the triangle involving the diagonal and comultiplication on S^n). In this diagram, φ is the folding map, μ

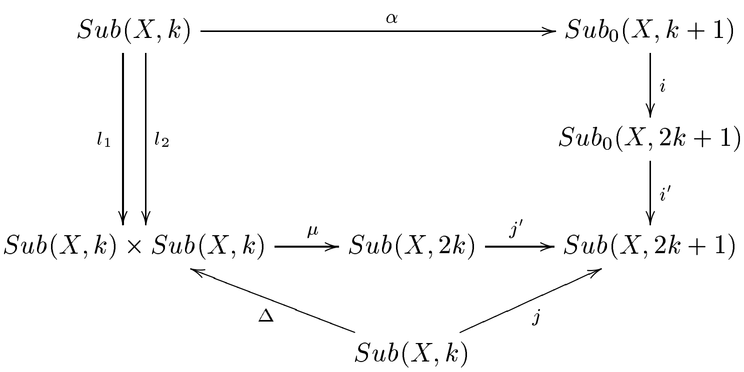
the restriction of the union map, and i the inclusion. Thus, writing ι for the identity map on S^n,

$$i_*[f] = i_* f_*[\iota] = i_* \varphi_* (f \vee f)_* \mu'_*[\iota] = i_*([f] + [f])$$

and so $i_*[f] = 0$. □

Theorem 4.2. *Let (X, x_0) be a path-connected pointed Hausdorff space. Then for each $k \geq 1$ and $n \geq 0$, the map $\pi_n\big(Sub(X,k)\big) \to \pi_n\big(Sub(X, 2k+1)\big)$ induced by the inclusion is the 0 map.*

Proof. The case $n = 0$ is immediate from Proposition 3.1. Let $n \geq 1$. Then π_n is group-valued and as in the proof of Theorem 4.1 we use additive notation. We have the commutative diagrams



where α adjoins x_0 to each set, Δ is the diagonal map, l_1 and l_2 are the axial inclusions, μ the union map, and the other maps are inclusions.

Let $f : S^n \to Sub(X,k)$ be a pointed map. Then from general homotopy theory, $\Delta_*[f] = l_{1*}[f] + l_{2*}[f]$. Thus

$$j_*[f] = j'_* \mu_* \Delta_*[f] = j'_* \mu_* \big(l_{1*}[f] + l_{2*}[f]\big) = 2 i'_* i_* \alpha_*[f] = 0$$

since $i_* = 0$ by Theorem 4.1. □

For any non-empty Hausdorff space X, let $Sub\,(X) = \bigcup_{k \geq 1} Sub(X,k)$ with the weak topology. Thus $Sub\,(X)$ is the space of all non-empty finite subsets of X. From Theorem 4.2 we have:

Corollary 4.3. *Let X be a non-empty path-connected Hausdorff space. Then $Sub\,(X)$ is weakly contractible.* □

Theorem 4.4. *Let M be a non-empty compact connected n-dimensional manifold without boundary, $n \geq 2$. Then for each $k \geq 1$, the mod 2 singular cohomology group $H^{nk}(Sub(M,k); \mathbf{Z}/2)$ is isomorphic to $\mathbf{Z}/2$, and $H^i(Sub(M,k); \mathbf{Z}/2) = 0$ for $i > nk$.*

Proof. All homology and cohomology groups below are with $\mathbf{Z}/2$ coefficients, and for brevity we write M_k for $Sub(M,k)$. We proceed by induction on k, the result being immediate for $k = 1$. Suppose $k > 1$ and, inductively, that

$$H^i(M_{k-1}) = 0 \quad \text{for} \quad i > n(k-1). \tag{1}$$

Since $M_k - M_{k-1}$ is the configuration space $C(M,k)$, an nk-dimensional manifold, the pair (M_k, M_{k-1}) is a compact relative nk-manifold and so by Lefschetz duality (see, e.g. [13, p. 297, Theorem 19]), we have isomorphisms

$$\bar{H}^j(M_k, M_{k-1}) \cong H_{nk-j}(C(M,k))$$

for all j, where $\bar{H}$ denotes Alexander cohomology. Since M_k and M_{k-1} are compact ANR's by Lemma 3.7, it follows from [13, p. 290, Theorem 10] that the above Alexander cohomology groups are isomorphic to the corresponding singular cohomology groups. Thus

$$H^j(M_k, M_{k-1}) = 0 \quad \text{for} \quad j > nk \tag{2}$$

and

$$H^{nk}(M_k, M_{k-1}) \cong H_0(C(M,k)) \cong \mathbf{Z}/2. \tag{3}$$

Let $i > nk$. Then exactness of

$$H^i(M_k, M_{k-1}) \to H^i(M_k) \to H^i(M_{k-1})$$

and the vanishing of the extreme groups by (2) and (1), we have $H^i(M_k) = 0$.

From exactness of

$$H^{nk-1}(M_{k-1}) \to H^{nk}(M_k, M_{k-1}) \to H^{nk}(M_k) \to H^{nk}(M_{k-1}),$$

it follows from (1) that the extreme groups vanish, and so by (3), $H^{nk}(M_k) \cong \mathbf{Z}/2$, completing the proof. □

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