FOURIER TRANSFORM

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Very broadly speaking, the Fourier transform is a systematic way to decompose "generic" functions into a superposition of "symmetric" functions. These symmetric functions are usually quite explicit (such as a trigonometric function $\sin(nx)$ or $\cos(nx)$), and are often associated with physical concepts such as frequency or energy. What "symmetric" means here will be left vague, but it will usually be associated with some sort of group G, which is usually (though not always) abelian. Indeed, the Fourier transform is a fundamental tool in the study of groups (and more precisely in the *representation theory* of groups, which roughly speaking describes how a group can define a notion of symmetry). The Fourier transform is also related to topics in linear algebra, such as the representation of a vector as linear combinations of an orthonormal basis, or as linear combinations of eigenvectors of a matrix (or a linear operator).

To give a very simple prototype of the Fourier transform, consider a real-valued function $f : \mathbf{R} \to \mathbf{R}$. Recall that such a function $f(x)$ is *even* if $f(-x) = f(x)$ for all $x \in \mathbf{R}$, and is *odd* if $f(-x) = -f(x)$ for all $x \in \mathbf{R}$. A typical function f, such as $f(x) = x^3 + 3x^2 + 3x + 1$, will be neither even nor odd. However, one can always write f as the *superposition* $f = f_e + f_o$ of an even function f_e and an odd function f_o by the formulae

$$f_e(x) := \frac{f(x) + f(-x)}{2}; \quad f_o(x) := \frac{f(x) - f(-x)}{2}.$$

For instance, if $f(x) = x^3 + 3x^2 + 3x + 1$, then $f_e(x) = 3x^2 + 1$ and $f_o(x) = x^3 + 3x$. Note also that this decomposition is unique; there are no other even functions $\tilde{f}_e$ and odd functions $\tilde{f}_o$ such that $f = \tilde{f}_e + \tilde{f}_o$. This rudimentary Fourier transform is associated with the two-element multiplicative group $\{-1, +1\}$, with the identity element $+1$ associated to the identity map $x \mapsto x$ on the real line, and the other element -1 associated to the reflection map $x \mapsto -x$.

For a more complicated example, let $n \geq 1$ be an integer and consider a complex-valued function $f : \mathbf{C} \to \mathbf{C}$. If $0 \leq j \leq n - 1$ is an integer, let us say that such a function $f(z)$ is a *harmonic of order* j if we have $f(e^{2\pi i/n} z) = e^{2\pi ij/n} f(z)$ for all $z \in \mathbf{C}$; note that the notions of even function and odd function correspond to the cases $j = 0, n = 2$ and $j = 1, n = 2$ respectively. As another example, the functions z^j, z^{j+n}, z^{j+2n}, etc. are harmonics of order j. Then we can split any function f uniquely as a superposition $f = \sum_{j=0}^{n-1} f_j$ of harmonics of order j, by means of the formula

$$f_j(x) := \frac{1}{n} \sum_{k=0}^{n-1} f(e^{2\pi ik/n} x) e^{-2\pi ijk/n};$$

note that the previous decomposition into even and odd functions was simply the $n = 2$ special case of this formula. The group associated to this Fourier transform is the n^{th} roots of unity $\{e^{2\pi ik/n} : 0 \leq k \leq n-1\}$, with each root of unity $e^{2\pi ik/n}$ associated with the rotation $z \mapsto e^{2\pi ik/n} z$ on the complex plane.

Moving now to the case of infinite groups, consider a function $f : \mathbf{T} \to \mathbf{C}$ defined on the unit circle $\mathbf{T} := \{z \in \mathbf{C} : |z| = 1\}$; to avoid technical issues let us assume that f is smooth (i.e. infinitely differentiable). Observe that if f is a monomial function $f(z) = c_n z^n$ for some integer n, then f will obey the rotational symmetry of order n $f(e^{i\theta} z) = e^{in\theta} f(z)$ for all complex numbers z and all phases θ. It should now be no surprise that an arbitrary smooth function f can be expressed as a superposition of such rotationally symmetric functions:

$$f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n) z^n, \text{ where } \hat{f}(n) := \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} \, d\theta.$$

This formula can be thought of as the limiting case $n \to \infty$ of the previous decomposition, restricted to the unit circle. It also generalizes the Taylor series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \text{ where } a_n = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z^{n+1}} \, dz$$

from complex analysis, when f is a complex analytic function on the closed unit disk $\{z \in \mathbf{C} : |z| \leq 1\}$; indeed there are very strong links between Fourier analysis and complex analysis. The complex numbers $\hat{f}(n)$ are known as the *Fourier coefficients* of f at given *frequencies* or *modes* n. When f is smooth, then these coefficients decay very quickly and there is no problem in establishing convergence of the Fourier series $\sum_{n=-\infty}^{\infty} \hat{f}(n) z^n$. The issue becomes more subtle if f is not smooth (for instance, if it is merely continuous), and one has to specify the nature of this convergence; in fact a significant portion of harmonic analysis is devoted to these types of questions, and in developing tools (and estimates) that can address them. The group associated with this Fourier analysis is the circle group $\mathbf{T}$. But there is now also a second group which is important here, which is the integer group $\mathbf{Z}$; this group indexes the type of symmetries available on the original group $\mathbf{T}$ (for each integer $n \in \mathbf{Z}$, one has a notion of a rotationally symmetric function of order n on $\mathbf{T}$), and is known as the *Pontryagin dual* to $\mathbf{T}$. (In the previous examples, the underlying group and its Pontryagin dual were the same.)

In the theory of partial differential equations and in related areas of harmonic analysis, the most important Fourier transform is that on a Euclidean space $\mathbf{R}^d$. Among all functions $f : \mathbf{R}^d \to \mathbf{C}$, there are the *plane waves* $f(x) = c_\xi e^{2\pi i x \cdot \xi}$, where $\xi \in \mathbf{R}^d$ is a vector (known as the *frequency* of the plane wave), $x \cdot \xi$ is the dot product between the position x and the frequency ξ, and c_ξ is a complex number (whose magnitude is the *amplitude* of the plane wave). It turns out that if a function f is sufficiently "nice" (e.g. smooth and rapidly decreasing), then it can be represented uniquely as the superposition of plane waves, where a "superposition" is interpreted as an integral now rather than a summation. More precisely, we have the formulae[1]

$$f(x) = \int_{\mathbf{R}^d} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi, \text{ where } \hat{f}(\xi) = \int_{\mathbf{R}^d} f(x) e^{-2\pi i x\cdot\xi}\, dx.$$

The function $\hat{f}(\xi)$ is known as the *Fourier transform* of f, thus the above two formulas show how to determine the Fourier transformed function from the original function or vice versa. One can view the quantity $\hat{f}(\xi)$ as the extent to which the function f contains a component which oscillates at frequency ξ. As it turns out, there is no difficulty justifying the convergence of these integrals when f is sufficiently nice, though the issue again becomes more subtle for f which are somewhat rough or slowly decaying. In this case, the underlying group is the Euclidean group $\mathbf{R}^d$ (which can also be thought of as the group of d-dimensional translations); note that both the position variable x and the frequency variable ξ are contained in $\mathbf{R}^d$, so $\mathbf{R}^d$ is also the Pontryagin dual group in this setting[2].

One major application of the Fourier transform lies in understanding various linear operations on functions, for instance the *Laplacian* on $\mathbf{R}^d$,

$$f \mapsto \Delta f = \sum_{j=1}^{d} \frac{\partial^2 f}{\partial x_j^2},$$

where we write the position variable $x \in \mathbf{R}^d$ in co-ordinates as $x = (x_1, \dots, x_d)$. To avoid technicalities let us only consider functions that are smooth enough for the Laplacian to make sense without any difficulty. In general, there is no obvious relationship between a function f and its Laplacian Δf. But when f is a plane wave such as $f(x) = e^{2\pi i x\cdot\xi}$, then there is a very simple relationship:

$$\Delta e^{2\pi i x\cdot\xi} = -4\pi^2 |\xi|^2 e^{2\pi i x\cdot\xi}.$$

In other words, the plane wave is an[3] *eigenfunction* for the Laplacian Δ, with *eigenvalue* $-4\pi^2|\xi|^2$. (More generally, plane waves will be eigenfunctions for any linear operation which commutes with translations.) Since the Fourier transform lets one write an arbitrary function as a superposition of plane waves, and since the Laplacian is a linear operator, we thus have a formula for the Laplacian of a general function:

$$\begin{aligned}\Delta f(x) &= \Delta \int_{\mathbf{R}^d} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi \\ &= \int_{\mathbf{R}^d} \hat{f}(\xi) \Delta e^{2\pi i x\cdot\xi}\, d\xi \\ &= \int_{\mathbf{R}^d} (-4\pi^2|\xi|^2) \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi.\end{aligned}$$

[1]In some texts, the Fourier transform is defined slightly differently, with factors such as 2π and -1 being moved to other places. These notational differences have some minor benefits and drawbacks, but they are all equivalent to each other.

[2]This is because of our reliance on the dot product; if one did not want to use this dot product, the Pontryagin dual would instead be $(\mathbf{R}^d)^*$, the dual vector space to $\mathbf{R}^d$. But this subtlety is not too important in most applications.

[3]Strictly speaking, this is a *generalized* eigenfunction, as plane waves are not square-integrable on $\mathbf{R}^d$.

Here we have interchanged the Laplacian Δ with an integral; this can be rigourously justified for suitably nice f, but we omit the details. Since Δf has a unique representation $\int_{\mathbf{R}^d} \widehat{\Delta f}(\xi) e^{2\pi i x \cdot \xi}\, d\xi$, we conclude that

$$\widehat{\Delta f}(\xi) = (-4\pi |\xi|^2) \hat{f}(\xi);$$

this identity can also be derived directly from the definition of the Fourier transform and from integration by parts. This identity shows that the Fourier transform *diagonalizes* the Laplacian; the operation of taking the Laplacian, when viewed using the Fourier transform, is nothing more than a multiplication operator by an explicit *multiplier*, in this case the function $-4\pi|\xi|^2$; this quantity can also be interpreted as the *energy level* associated[4] to the frequency ξ. In other words, the Laplacian can be viewed as a *Fourier multiplier*. This viewpoint allows one to manipulate the Laplacian very easily. For instance, we can iterate the above formula to compute higher powers of the Laplacian:

$$\widehat{\Delta^n f}(\xi) = (-4\pi|\xi|^2)^n \hat{f}(\xi) \text{ for } n = 0, 1, 2, \ldots$$

Indeed, this now suggests a way to develop more general functions of the Laplacian, for instance a square root:

$$\widehat{\sqrt{-\Delta} f}(\xi) = \sqrt{4\pi|\xi|^2}\hat{f}(\xi).$$

This leads to the theory of fractional differential operators (which are in turn a special case of *pseudodifferential operators*), as well as the more general theory of *functional calculus*, in which one starts with a given operator (such as the Laplacian) and then studies various functions of that operator, such as square roots, exponentials, inverses, and so forth.

As the above discussion shows, the Fourier transform can be used to develop a number of interesting operations, which have particular importance in the theory of differential equations. To analyze these operations effectively one needs various estimates on the Fourier transform, for instance knowing how the size of a function f (in some norm) relates to the size of its Fourier transform (perhaps in a different norm). One particularly important and striking estimate of this type is the *Plancherel identity*

$$\int_{\mathbf{R}^d} |f(x)|^2\, dx = \int_{\mathbf{R}^d} |\hat{f}(\xi)|^2\, d\xi$$

which shows, among other things, that the Fourier transform is in fact a unitary operation, and so one can view the frequency space representation of a function as being in some sense a "rotation" of the physical space representation. Developing further estimates related to the Fourier transform and associated operators is a major component of harmonic analysis. A variant of the Plancherel identity is the *convolution formula*

$$\int_{\mathbf{R}^d} f(y) g(x-y)\, dy = \int_{\mathbf{R}^d} \hat{f}(\xi)\hat{g}(\xi) e^{2\pi i x\cdot\xi}\, d\xi.$$

This formula allows one to analyze the convolution $f * g(x) := \int_{\mathbf{R}^d} f(y) g(x-y)\, dy$ of two functions f, g in terms of their Fourier transform; in particular, if f or g have small Fourier coefficients then we expect their convolution $f * g$ to also be small.

[4]When taking this perspective, it is customary to replace Δ by $-\Delta$ in order to make the energies positive.

This relationship means that the Fourier transform controls certain *correlations* of a function with itself or with other functions, which makes the Fourier transform an important tool in understanding the randomness and uniform distribution properties of various objects in probability theory, harmonic analysis, and number theory. For instance, one can pursue the above ideas to establish the *central limit theorem* (also known as the *law of large numbers*), which asserts that the sum of many independent random variables will eventually resemble a Gaussian distribution; one can even use such methods to establish *Vinogradov's theorem*, that every sufficiently large odd number is the sum of three primes.

There are many directions in which to generalize the above set of ideas. For instance, one can replace the Laplacian by a more general operator, and the plane waves by (generalized) eigenfunctions of that operator, which leads to the subject of spectral theory and functional calculus; one can also study the algebra of Fourier multipliers (and of convolution) more abstractly, which leads to the theory of C^*-algebras. One can also go beyond the theory of linear operators and study bilinear, multilinear, or even fully nonlinear operations, leading in particular to the theory of *paraproducts*, which are generalizations of the pointwise product operation $(f(x), g(x)) \mapsto fg(x)$ that are of importance in differential equations. In another direction, one can replace Euclidean space $\mathbf{R}^d$ by a more general group, in which case the notion of a plane wave is replaced by the notion of a *character* (if the group is abelian) or a *representation* (if the group is non-abelian). There are other variants of the Fourier transform, such as the Laplace transform or the Mellin transform, which are very similar algebraically to the Fourier transform and play similar roles (for instance, the Laplace transform is also useful in analyzing differential equations). We have already seen that Fourier transforms are connected to Taylor series; there is also a connection to some other important series expansions, notably Dirichlet series, as well as expansions of functions in terms of special polynomials such as orthogonal polynomials or spherical harmonics.

The Fourier transform decomposes a function exactly into many components, each of which has a precise frequency. In some applications it is more useful to adopt a "fuzzier" approach, in which a function is decomposed into fewer components, and each component has a range of frequencies rather than consisting purely of a single frequency. Such decompositions can have the advantage of being less constrained by the *uncertainty principle*, which asserts that a strong localization in the frequency variable must be accompanied by a strong delocalization in the position variable; this leads to some variants of the Fourier transform, such as *wavelet transforms*, which are better suited for a number of problems in applied and computational mathematics, and also in certain questions in harmonic analysis and differential equations. The uncertainty principle, being fundamental to quantum mechanics, also connects the Fourier transform to mathematical physics, and in particular to the connections between classical and quantum physics, which can be studied rigourously using the methods of geometric quantization and microlocal analysis.

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