PERIODIC PERMUTATIONS AND THE ROBINSON–SCHENSTED CORRESPONDENCE

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Abstract. We introduce a group of periodic permutations, a new version of the infinite symmetric group. We then generalize and study the Robinson–Schensted correspondence for such permutations.

Introduction

In the last few decades the study of combinatorics and representation theory of the symmetric group has exploded and really became a field of its own, with hundreds of papers and several books dedicated solely to the subject. We refer the reader to [F,M,Sa,St] for the introduction to the field, connections to other areas, and an overview of extensions too numerous to be listed here.

By an 'infinite symmetric group' most authors think of the group Σ_1 of permutations of $\mathbb{N}$ with bounded support. The study of Σ_1 led to remarkable successes with connections and applications to a number of fields from Representation Theory to Probability to Mathematical Physics (see e.g. [AD,BO,VK]). In this paper we consider a family of groups Σ_m which become periodic with period m after a certain point. These groups generalize Σ_1, and have a number of interesting algebraic and combinatorial properties which we explore in this paper.

Our main tool is the *Robinson–Schensted correspondence*, one of the most important bijections in the combinatorics of Young tableaux. The correspondence is a classical special case of the celebrated Robinson–Schensted–Knuth (RSK) correspondence, which has countless reformulations and generalizations in the literature (see references in [F,St]). In the classical case, it maps a permutation $\sigma \in S_n$ into a pair of standard Young tableaux of the same shape λ, and can be viewed as a combinatorial way to represent Burnside's identity for the symmetric group S_n. What makes this bijection important is its robustness and various 'bonus' properties, such as the length of the longest increasing subsequence in σ being equal to the first part of λ.

Key words and phrases. Infinite symmetric group, affine Weyl group, Robinson–Schensted correspondence.

To analyze periodic permutations, we consider a *stable* version of the Robinson–Schensted correspondence, defined by applying the correspondence to longer and longer subpermutations and taking a limit. The resulting *stable Robinson–Schensted map* is well defined, but is no longer one-to-one. We are mostly concerned with the *inverse problem:* given a resulting pair of tableaux, can one reconstruct the original permutation? Reminiscent to the "can you hear the shape of a drum?" question, this problem also has a negative solution. On the other hand, it has enough structure to make it interesting and deserving further investigation.

To give a more technical overview of the results, let us start with a formal definition of our main object of study. Let Σ_m be the group of *m-periodic permutations* $\sigma : \mathbb{N} \to \mathbb{N}$ such that: $\sigma(i+m) = \sigma(i) + m$ for all i large enough. Observe that Σ_1 is indeed a group of permutations with bounded support. Perhaps surprisingly, and in contrast with Σ_1, groups Σ_m are finitely generated for $m \geq 2$. We show that group Σ_m is mapped onto a Weyl group W_m of affine root systems $\widehat{A}_{m-1}$, and this homomorphism proves to be important in the inverse problem.

Roughly speaking, when formulated for all one-to-one maps from $\mathbb{N}$ into itself, the inverse problem has little structure, as some pairs of tableaux have no preimages of the stable Robinson–Schensted map, while others have infinitely many. When restricted to m-periodic permutations we discover that the resulting pairs of tableaux are also periodic in certain precise sense, and have additional properties, including a connection to W_n

The paper has the following outline. In the first section we study algebraic properties of Σ_m. The stable Robinson–Schensted map is defined in Section 2, and the inverse problem is introduced in Section 3. Then, in Section 4 we study these for m-periodic permutation. We conclude with questions, open problems and final remarks.

1. Periodic permutations

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ denote the set of positive integers, and let $\sigma : \mathbb{N} \to \mathbb{N}$ be a one-to-one correspondence. Define a multiplication on all such σ by taking compositions, and let S_∞ denote the resulting group of permutations.

We say that $\sigma \in S_\infty$ is *m-periodic* if there exists $N > 0$ such that $\sigma(i+m) = \sigma(i) + m$ for all $i \geq N$. Let Σ_m be the set of all m-periodic permutations, and let $\Sigma = \cup_m \Sigma_m$. We refer to elements $\sigma \in \Sigma$ as *periodic permutations.*

Note that 1-periodic permutations are simply all permutations $\sigma \in S_\infty$ with finite support: $Supp(\sigma) = \{i \mid \sigma(i) \neq i\}$.

Proposition 1.1 *$\Sigma_1 \subsetneq \Sigma_2 \subsetneq \Sigma_3 \subsetneq \dots \subsetneq \Sigma \subsetneq S_\infty$ is a proper chain of subgroups.*

Proof. First, let us check that Σ_m and Σ are closed under multiplication. Indeed, suppose $\sigma_1, \sigma_2 \in \Sigma_m$. Then $[\sigma_1\sigma_2](i+m) = \sigma_1\big(\sigma_2(i+m)\big) = \sigma_1\big(\sigma_2(i)+m\big) = [\sigma_1\sigma_2](i) + m \in \Sigma_m$. Similarly, if $\sigma \in \Sigma_m$, then $\sigma^{-1} \in \Sigma_m$, which proves that Σ_m is a subgroup of S_∞. Also, if $\sigma_1 \in \Sigma_m$, $\sigma_2 \in \Sigma_n$, then $\sigma_1, \sigma_2 \in \Sigma_{mn} \subset \Sigma$, which implies that $\Sigma = \cup_m \Sigma_m$ is a subgroup of S_∞.

To show the inclusion $\Sigma_{m-1} \subset \Sigma_m$, consider a subgroup $H \subset \Sigma_m$ of m-periodic permutations with $\sigma(n) = n$ for all $m|n$. It is easy to see that the map $\beta : \Sigma_{m-1} \to H$ defined by $[\beta(\sigma)]((m-1)r+i) = mr+1+i$, $0 \leq i < m-1$, is a group isomorphism.

Finally, a permutation $\tau = (1,2)(m+1,m+2)(2m+1,2m+2)\cdots \in \Sigma_m$, while clearly $\tau \notin \Sigma_n$ for all $n < m$, which completes the proof. $\square$

Proposition 1.2 *Groups Σ_1, Σ, and S_∞ are not finitely generated.*

This shows that despite attractive appearance, the most natural versions of the 'infinite symmetric groups' are not finitely generated, in contrast with the group Σ_m for $m \geq 2$ (see below). This, of course, is well known and easy to prove. We present here an elementary proof for completeness.

Proof. For the first part, suppose $H = \langle \sigma_1, \ldots, \sigma_k \rangle$, with $Supp(\sigma_i) \subset A$. Then $Supp(\sigma) \subset A$ for all $\sigma \in H$, and thus $H \subsetneq \Sigma_1$. For the second part, consider $H = \langle \sigma_1, \ldots, \sigma_k \rangle$, for any $\sigma_1 \in \Sigma_{m_1}, \ldots, \sigma_k \in \Sigma_{m_k}$. Then $H \subset \Sigma_M \subsetneq \Sigma$, where $M = m_1 \cdots m_k$. The third part is obvious as S_∞ is uncountable. $\square$

Example 1.3 Consider the first nontrivial example of a group of 2-periodic permutations Σ_2. Define

$$\alpha = (1,2), \quad \tau_1 = (1,2)(3,4)(5,6)\cdots, \quad \tau_2 = (2,3)(4,5)(6,7)\cdots.$$

Note that $\alpha, \tau_1, \tau_2 \in \Sigma_2$ and $\alpha^2 = \tau_1^2 = \tau_2^2 = 1$. We claim that $\Sigma_2 = \langle \alpha, \tau_1, \tau_2 \rangle$. By definition, for every $\sigma \in \Sigma_2$ there exists an N such that $\sigma(n+2) - \sigma(n) = 2$ for all $n \geq N$. Define $a = a(\sigma) := \sigma(2n) - 2n$, $b = b(\sigma) := \sigma(2n+1) - (2n+1)$, which are independent of n, for $n \geq N$. Since σ is one-to-one, we must have the in-flow $|\{j \mid \sigma(j) \leq n\}|$ equal to out-flow $|\{i \mid \sigma(i) \geq n\}|$. Taking n large enough this implies that $a = -b$.

Now, observe that $\langle \tau_1, \tau_2 \rangle \simeq \mathbb{Z}_2 * \mathbb{Z}_2 \simeq D_\infty$ is an infinite dihedral group with elements $\rho_k = (\tau_1\tau_2)^k$ satisfying $a(\rho_k) = k$ and $b(\rho_k) = -k$, for all $k \in \mathbb{Z}$. Therefore, if $\sigma \in \Sigma_2$, then $\sigma\rho_k \in \Sigma_1$ for $k = -a(\sigma)$.

Finally, since $\rho_1 = \tau_1\tau_2$ is an infinite cycle on $\mathbb{N}$:

$$\ldots 8 \to 6 \to 4 \to 2 \to 1 \to 3 \to 5 \to 7 \to \ldots$$

Let $\rho = \alpha\rho_1$. Then $\langle \alpha, \rho \rangle \supset \langle \rho^{-k}\alpha\rho^k, \forall k \in \mathbb{Z} \rangle = \langle (1,r), \forall r > 1 \rangle = \Sigma_1$. Therefore $\Sigma_2 = \langle \tau_1, \tau_2 \rangle \cdot \langle \alpha, \rho \rangle = \langle \alpha, \tau_1, \tau_2 \rangle$, and thus Σ_2 is finitely generated. Furthermore, $\Sigma_2 \simeq \Sigma_1 \rtimes \mathbb{Z}$, where $\mathbb{Z} = \langle \rho_1 \rangle$ acts on Σ_1 by conjugation.

Note that Σ_2 contains a lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$, and thus has an exponential growth. Clearly, Σ_2 does not have Kazhdan's property T since it is mapped ont $\mathbb{Z}$. Moreover, Σ_2 is amenable as can be shown by the following calculation. Consider finite sets $B_n = \{(S_n, k), -n/2 < k < n/2\}$ of vertices in the Cayley graph of $\Sigma_1 \rtimes \mathbb{Z}$ with generating set corresponding to $(1,2) \in \Sigma_1$ and $\pm 1 \in \mathbb{Z}$. Then the relative size of the boundary $|\partial B_n|/|B_n| = O(1/n) \to 0$ as $n \to \infty$, which implies that $\Sigma_2 \simeq \Sigma_1 \rtimes \mathbb{Z}$ is amenable [dlH].

For the rest of this section we consider only a group of m-periodic permutations Σ_m with $m \geq 3$. Let $\alpha = (1,2)$, and

$$\tau_i = (i, i+1)(m+i, m+i+1)(2m+i, 2m+i+1)\cdots, \quad \text{for} \quad i = 1, \ldots, m.$$

Clearly, $\alpha, \tau_i \in \Sigma_m$. Denote this set of m-periodic permutations by R_m. The main result of this section is the following theorem.

Theorem 1.4 *Group of m-periodic permutations Σ_m with $m \geq 3$ is finitely generated by R_m. In other words, $\Sigma_m = \langle \alpha, \tau_1, \dots, \tau_m \rangle$.*

Before we prove the theorem, let us recall the structure of the Coxeter group

$$W_n = \langle s_1, \dots, s_n \rangle / \big(s_k^2 = (s_i s_j)^2 = (s_r s_{r+1 \mod m})^3 = 1, \ \ \forall\, k,\, r,\, i-j \neq \pm 1 \big).$$

Group W_n is a Weyl group of affine root system $\widehat{A}_{n-1}$ and is well understood [B,H]. It can be viewed as an automorphism group of a lattice $L = \{(a_1, \dots, a_n) \in \mathbb{Z}^n \mid a_1 + \ldots + a_n = 0\}$, with the action of s_i on L given by reflections:

$$\begin{aligned} s_i &: (\dots, a_i, a_{i+1}, \dots) \to (\dots, a_{i+1}+1, a_i - 1, \dots), \quad i = 1, \dots, n-1, \\ s_n &: (a_1, \dots, a_n) \to (a_n - 1, \dots, a_1 + 1). \end{aligned}$$

Now consider a group Υ_m of all one-to-one correspondences $\omega : \mathbb{Z} \to \mathbb{Z}$ such that $\omega(i+m) = \omega(i) + m$ for all $i \in \mathbb{Z}$, and $\omega(0) + \omega(1) + \ldots + \omega(m-1) = \binom{m}{2}$. One can view Υ_m as m-periodic doubly infinite permutations. Define a natural map $\gamma : \Sigma_m \to \Upsilon_m$ by setting $\gamma(\sigma) = \omega$ if $\sigma(i) = \omega(i)$, for all i large enough.

Lemma 1.5 *1. Groups W_n and Υ_n are isomorphic for all $n \geq 2$.*
2. Map $\gamma : \Sigma_m \to \Upsilon_m$ defined above is a group homomorphism.

Proof. For $\omega \in \Upsilon_n$, define $a_i(\omega) = \omega(i) - i$, $i = 1, \dots, n$. Clearly, ω is uniquely determined by the values $a_i(\omega)$. Also, from $\omega(0) + \ldots + \omega(n-1) = \binom{n}{2}$ we have:

$$a_1(\omega) + \ldots + a_n(\omega) = \big(\omega(1) - 1\big) + \ldots + \big(\omega(n-1) - (n-1)\big) + \big(\omega(0) - 0\big) = 0.$$

Define $\pi_i := \gamma(\tau_i) = \cdots (i-m, i-m+1)(i, i+1)(i+m, i+m+1) \cdots \in \Upsilon_n$, for $i = 1, \dots, n$. Consider a group homomorphism $\psi : W_n \to \Upsilon_n$ defined by $\psi(s_i) = \pi_i$. It is easy to see that ψ is an isomorphism, which implies part *1.* Proof of part *2* is straightforward. $\square$

Proof of Theorem 1.4 From the first part of Lemma 1.5, Υ_m is finitely generated: $\Upsilon_m = \langle \pi_1, \dots, \pi_m \rangle$. The second part implies that for every $\sigma \in \Sigma_m$ there exists $\rho \in H := \langle \tau_1, \dots, \tau_m \rangle$ such that $\sigma\rho \in \Sigma_1$. Consider an element $h = (\tau_1 \tau_2 \cdots \tau_m) \in \Sigma_m$, and check that h is an infinite cycle on $\mathbb{N}$. Thus $\Sigma_1 = \langle \alpha, \rho \rangle$, and $\sigma \in H \cdot \langle \alpha, \rho \rangle \subset \langle R_m \rangle$. $\square$

We conclude this section with two side results of independent interest.

Proposition 1.6 *Groups Σ_m, $m \geq 2$, are amenable, non-solvable, and have exponential growth.*

Proof. Since $S_5 \subset \Sigma_1 \subset \Sigma_m$, for all $m \geq 2$, this implies that Σ_m are not solvable. The exponential growth of Σ_m follows from the exponential growth of $\Sigma_2 \subset \Sigma_m$. Amenability can be shown by a direct calculation similar to that in Example 1.3. We omit the details. $\square$

Proposition 1.7 *Group Σ contains W_n as a subgroup, for all $n \geq 1$.*

Proof. It suffice to show that the group $\Sigma_{2n} \subset \Sigma$, contains W_n as a subgroup, for all $n \geq 2$. Consider a map $\phi : W_n \to \Sigma_{2n}$ defined on generators s_i, $1 \leq i < n$, and s_n as follows:

$$\begin{aligned}
\phi(s_i) &= (2i, 2i+2)(2i+2n, 2i+2+2n)\cdots \\
&\quad \cdot (2n-2i-1, 2n-2i+1)(4n-2i-1, 4n-2i+1)\cdots \\
\phi(s_n) &= (1,2)\,(2n, 2n+2)(4n, 4n+2)\cdots \\
&\quad \cdot (2n-1, 2n+1)(4n-1, 4n+1)\cdots
\end{aligned}$$

One can view $\phi(W_n)$ as a group of periodic doubly infinite permutations of the set $\mathbb{N}$ ordered as $\{\dots, 9, 7, 5, 3, 1, 2, 4, 6, 8, \dots\}$. Such permutations are obviously $2n$-periodic (as permutations in S_∞). A straightforward check shows that the map ϕ is a group homomorphism. $\square$

2. The stable Robinson–Schensted correspondence

Recall that a *Young tableau* A of shape $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$ is a function $A : [\lambda] \to \mathbb{N}$ on Young diagram $[\lambda]$ corresponding to λ with nonnegative integers increasing in rows and columns. We will think of values A as being written in squares of $[\lambda]$ (cf. [F,M,St]). The *weight* of A, denotes $w(A)$, is a set of integers in A. Tableau A is called *standard* if $w(A) = \{1, \dots, n\}$, where $n = |\lambda| = \lambda_1 + \lambda_2 + \dots$ Let A, B be two Young tableaux of the same shape λ, $|\lambda| = n$, such that B is standard. We call these RS-pairs.

An *insertion* of an integer c into RS-pair (A, B) is a RS-pair (A', B'), obtained by the following procedure. Start with the first row of A. If c is larger than all integers in the first row, add a new square with c at the end. Otherwise, find the smallest integer $d > c$, replace d with c, and 'bump' d to insert in the next row. Repeat until a new Young tableau A' is obtained of shape μ, $|\mu| = n+1$. For B, add a new square with $(n+1)$, to obtain B' of the same shape μ.

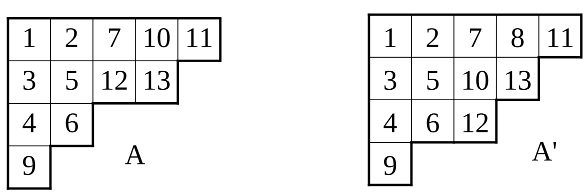

Figure 1. Insertion of 8 into tableau A gives a tableau A'. Here 8 'bumps' 10, which in turn 'bumps' 12.

For any permutation $(c_1, c_2, \dots, c_n, \dots) \in S_\infty$, define a sequence of RS-pairs $\{(A_n, B_n)\}$ obtained by successive insertion of c_1, c_2, … , starting with an empty RS-pair. The correspondence $\varphi_n : (c_1, \dots, c_n) \to (A_n, B_n)$ is called *Robinson–Schensted correspondence.*

Note that integers in B_n are never erased, while in A_n they may change after future insertions. We say that $\{A_n\}$ *stabilizes* if for every $i \in \mathbb{N}$ there exist $N = N(i)$ such that i is never erased after the first N insertions.

Proposition 2.1 *The sequence $\{A_n\}$ stabilizes for all $(c_1, c_2, \dots) \in S_\infty$.*

Proof. The only way (i) is erased if the inserted integer c 'bumps' i, which can happen only if $c < i$. Thus it suffices to take the smallest integer $N = N(i)$, such that $i < c_{N+1}, c_{N+2}, \dots$ $\square$

Thus we can speak of pairs of *infinite Young tableaux* (A_∞, B_∞) obtained as stable limits of (A_n, B_n). Denote by $\varphi : \sigma \to (A_\infty, B_\infty)$ the *stable Robinson–Schensted correspondence* defined as above. The shape of A_∞ and B_∞ is going to be the same (possibly infinite) order ideal in $\mathbb{Z}^2_{\geq 0}$, which can denoted by an infinite sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$, where $\lambda_i \in \mathbb{N} \cup \{0, \infty\}$. We refer to λ as a *stable shape* of σ. Note that the weight of A and B is $\mathbb{N}$. When speaking of infinite Young tableaux, we will always assume that, unless explicitly stated otherwise.

Example 2.2 Consider a 2-periodic permutation

$$\sigma = (2,3)(4,5)(6,7)\cdots = (1,3,2,5,4,7,6,\dots) \in \Sigma_2$$

Tableaux $(A_\infty, B_\infty) = \varphi(\sigma)$ of stable shape (∞, ∞) are given in Figure 2. Note that $A_\infty = B_\infty$ and the fact that σ is an involution: $\sigma^2 = 1$.

1	2	4	6	8	
3	5	7	9	11	

A = B

FIGURE 2. Stable Robinson-Schensted bijection $\varphi(\sigma) = \varphi(\omega) = (A, B)$.

Similarly, for another 2-periodic permutation

$$\omega = (1,3)\,(2,5)(4,7)(6,9)(8,11) = (3,5,1,7,2,9,4,11,6,\dots) \in \Sigma_2$$

we have $\omega^2 = 1$ and $(A_\infty, B_\infty) = \varphi(\omega)$. This shows that the stable Robinson–Schensted correspondence is not one-to-one. On the other hand, an exhaustive search shows that σ and ω are the only preimages of (A_∞, B_∞).

Example 2.3 Let $\pi = (1,3,2,6,5,4,10,9,8,7,15,14,13,12,11,\dots) \in S_\infty$ be another involution, and $(A_\infty, B_\infty) = \varphi(\pi)$ (see Figure 3). Here tableaux $A_\infty = B_\infty$ have stable shape $(\infty, \infty, \infty, \dots)$. In the next section we show that there exist infinitely many permutations that are mapped into (A_∞, B_∞).

1	2	4	7	11	
3	5	8	12		
6	9	13			
10	14				
15					

A = B

FIGURE 3. Stable Young tableaux $(A, B) = \varphi(\pi)$.

The following result extends the *duality* property of the Robinson–Schensted correspondence. As a corollary it implies that involutions are mapped into equal tableaux.

Proposition 2.4 *If $\varphi(\sigma) = (A_\infty, B_\infty)$ for $\sigma \in S_\infty$, then $\varphi(\sigma^{-1}) = (B_\infty, A_\infty)$.*

Proof. Let $(A_n, B_n) = \varphi(\sigma(1), \dots, \sigma(n))$, $(A'_n, B'_n) = \varphi(\sigma^{-1}(1), \dots, \sigma^{-1}(n))$, and $\varphi(\sigma^{-1}) = (A'_\infty, B'_\infty)$. As in the proof above, let $N = N(i)$ be smallest integers such that $\sigma(k) > i$ and $\sigma^{-1}(k) > i$ for all $k \geq N$. Therefore, integers $1, \dots, i$ are never erased in A_N and A'_N. By duality of the Robinson–Schensted correspondence (see e.g. [St]), subtableaux of integers smaller than 1 in (A_N, B_N) coincide with that in (B'_N, A'_N). This implies the result. □

3. The inverse problem

We already know that the stable Robinson–Schensted correspondence is not one-to-one (see Example 2.3). The main question we ask is the following problem.

Inverse Problem: *Find all preimages $\sigma = \varphi^{-1}(A, B)$ of the stable Robinson–Schensted correspondence for any given pair of infinite Young tableaux (A, B) of the same stable shape.*

Below we prove both negative and positive results, showing that certain pairs of tableaux have no preimages, while others have infinitely many. In the next section we return to periodic permutations for which the problem is more structured.

Proposition 3.1 *If $\varphi(\sigma) = (A, B)$, then the stable shape λ of A, B cannot end with an infinite tail of nonzero integers.*

Proof. Suppose $\lambda_k = \lambda_{k+1} = \ldots = \ell > 1$ be the tail of λ. If $\varphi(\sigma) = (A, B)$, there exist an integer N so that the value $A(k, \ell)$ is never erased. The only way we can have $B(m, \ell) > N$ with $m > k$, is by inserting an integer smaller than $A(k, \ell)$ (and bumping all the way to m-th row). Note that there is only a finite number of integers smaller than $\sigma(N)$ while there exist infinitely many values $B(m, \ell) > N$ with $m > k$. The result now follows by contradiction. □

Define the *height* $h(\lambda) \in \mathbb{N} \cup \{\infty\}$ to be the number of positive parts in a partition λ. The following proposition proves existence of preimages in the inverse problem for pairs of tableaux of the same stable shape with a finite height.

Theorem 3.2 *Every pair of infinite Young tableaux (A, B) of stable shape λ with $h(\lambda) < \infty$, has a preimage $\sigma \in S_\infty : \varphi(\sigma) = (A, B)$.*

Proof. The result follows by induction on the height $h(\lambda) = k$. First, consider the case $\ell = \lambda_k < \infty$. This case is similar to the classical inverse of the Robinson–Schensted correspondence (see [F,Sa,St]). Use induction on ℓ. The base of induction is straightforward as for $k = 1$ there exist a unique Young tableau corresponding to an identity permutation.

For $k \geq 2$, let $a_k = A(k, \ell)$, $b = B(k, \ell)$. Find the largest integer a_{k-1} in $(k-1)$-th row such that $a_{k-1} < a_k$, then find the largest integer a_{k-2} in $(k-2)$-th row such that $a_{k-2} < a_{k-1}$, etc. Now remove b from B, replace each a_i with a_{i+1} to obtain tableaux A' and B' of stable shape μ. Set $\sigma(b) = a_1$ and use induction assumption to obtain the rest of the permutation.

Now suppose $\lambda_k = \infty$. Set $\sigma(B(k,i)) = A(1,i)$ for all $i = 1, 2, \dots$ Remove the first row in A, the last (k-th) row in B, to obtain two Young tableaux of the same stable shape μ, $h(\mu) = k-1$. Finally, use the inductive assumption to obtain the rest of the permutation. $\square$

Let us note that the method in the proof above breaks completely for $\lambda = (\infty, \infty, \dots)$. Basically, as the height k grows the resulting permutation never stabilizes. Intuitively, this suggests that in this case of the inverse problem there are no preimages. As Example 2.3 shows, this is not true. In fact, our rather limited set of experiments suggest the opposite:

Conjecture 3.4 *Every pair of infinite tableaux of stable shape $\lambda = (\infty, \infty, \dots)$ has at least one preimage of the stable Robinson–Schensted map.*

The following result suggests that the inverse problem is infeasible in certain cases.

Proposition 3.5 *The number of preimages is unbounded for tableaux with stable shape of finite height.*

First proof. We use an explicit construction. Consider a partition $\lambda = (\infty, \dots, \infty)$ (k times). Consider two equal tableaux $(A^k, B^k) = \varphi(\sigma_k)$, where

$$\sigma_k = (1, 3, 2, 6, 5, 4, \dots, r+k, r+k-1, \dots, r+1, r+2k, r+2k-1, \dots, r+k+1, \dots)$$

and $r = \binom{k}{2}$. See Figures 2, 4 for $k = 2, 3$ examples. We prove by induction that a pair of tableaux (A^k, B^k) defined above has at least k distinct preimages. This immediately implies the result.

1	2	4	7	10	13
3	5	8	11	14	
6	9	12	15		

FIGURE 4. Stable Young tableau $A^3 = B^3$.

The base of induction, the case $k = 2$, is given in Example 2.2. For the step of induction, use the construction given in the second part of the proof of Theorem 3.2. Set $\sigma(B(k,i)) = A(1,i)$ for all $i \in \mathbb{N}$. As before, remove the first row in A^k, the last row in B^k. Observe that we obtain two tableaux with the numbers in the same relative order as in A^{k-1} and B^{k-1}. By induction assumption, there are now $(k-1)$ ways to extend σ to a permutation. The k-th permutation is given by σ_k defined as above. It is straightforward to check that all these permutations are different. $\square$

Second proof. Let $A^k = B^k$ be two infinite tableaux of stable shape (∞, ∞) with integers $(1, 2, \dots, k, k+2, k+4, \dots)$ in the first row, and integers $(k+1, k+3, k+5, \dots)$ in the second row. For every $i \in \{1, \dots, k\}$, define

$$\begin{aligned}\sigma_{k,i} = {} & (1, 2, \dots, i-1, k+1, k+3, \dots, m-2, m, i, m+2, i+1, m+4, \\ & i+2, m+6, \dots, r-2, k-1, r, k, r+2, k+2, r+4, k+4, \dots),\end{aligned}$$

where $m = 3k-2i+1$, $r = 5k-4i+1$. Routine check shows that $\varphi(\sigma_{k,i}) = (A^k, B^k)$, and that $\sigma_{k,i} \neq \sigma_{k,j}$ for $1 \le i < j \le k$. $\square$

Example 3.6 When $k = 2$, the constructions in two proofs coincide with those in Example 2.2. When $k = 3$, the first construction gives the following 3-periodic permutations:

$$(6, 9, 3, 12, 5, 1, 15, 8, 2, 18, 11, 4, 21, 14, 7, \dots),$$
$$(3, 6, 1, 9, 5, 2, 12, 8, 4, 15, 11, 7, 18, 14, 10, \dots),$$
$$(1, 3, 2, 6, 5, 4, 9, 8, 7, 12, 11, 10, 15, 14, 13, \dots).$$

The second construction gives the following 2-periodic permutations, all mapped into (A^3, B^3) as in Figure 4.

$$\sigma_{3,1} = (4, 6, 8, 1, 10, 2, 12, 3, 14, 5, 16, 7, 18, 9, \dots),$$
$$\sigma_{3,2} = (1, 4, 6, 2, 8, 3, 10, 5, 12, 7, 14, 9, 16, 11, \dots),$$
$$\sigma_{3,3} = (1, 2, 4, 3, 6, 5, 8, 7, 10, 9, 12, 11, 14, 13, \dots).$$

Both constructions can be generalized in several, very different directions. The idea behind them is used to obtain the following result.

Theorem 3.7 *There exists a pair of infinite Young tableaux with uncountably many preimages.*

Proof. Let $A = B = \varphi(\pi)$ be as in Example 2.3. Let A' be obtained by removing the first row in A. Note relabelling integers (preserving the order) give tableau A again. Consider a sequence $c = (3, 6, 5, 10, 9, 8, 15, 14, 13, 12, 21, 20, 19, 18, 17, \dots)$ obtained by permuting integers in A'. From above, $\varphi(c) = (A', B)$ is well defined, although $c \notin S_\infty$. Substitute a subsequence $3, 5, 8, 12, 17, \dots$ in c of integers that appear in the first row in A' (and the second row in A) by a sequence variables $a = (a_1, a_2, a_3, \dots)$ This gives a sequence

$$u = (a_1, 6, a_2, 10, 9, a_3, 15, 14, 13, a_4, 21, 20, 19, 18, a_5, \dots)$$

Define an infinite permutation $\sigma \in S_\infty$ by the following substitution of a_r in u. Observe that in order for $\sigma \in S_\infty$ we need a sequence a to be a permutation of $I \cup J$, where $I = \{i_1, i_2, \dots\} = \{1, 2, 4, 7, 11, 16, \dots\}$, and $J = \{j_1, j_2, \dots\} = \{3, 5, 8, 12, 17, \dots\}$. Check that $\varphi(\sigma) = (A, B)$ only if every a_i is smaller than all integers in u which appear before a_i, with possible exception of elements of I that *not* in the same row of A. Fix $a_1 = 3$, $a_2 = 5$, and let (a_3, a_4), (a_5, a_6), (a_7, a_8), etc. be a permutation of $\{1, 8\}$, $\{2, 12\}$, $\{4, 17\}$, etc. In general, take (a_{2r-1}, a_{2r}) to be any permutation of (i_{r-1}, j_{r+1}), for all $r > 1$. This substitution satisfies the conditions above and a straightforward check shows that the resulting permutation σ is mapped into (A, B). Clearly, all $2^{\mathbb{N}}$ choices of substitutions produce distinct permutations. $\square$

Note that a permutation π in Example 2.3 is not included in the uncountable set of permutations σ produced in the proof above. Let us mention here a connection to the first proof of Theorem 3.5 which underscores the importance of tableaux A'.

In conclusion, let us state the following conjecture which contrasts Theorem 3.7 above:

Conjecture 3.8 *Every pair of Young tableaux of the same stable shape with finite height has a finite number of preimages of the stable Robinson–Schensted map.*

This suggests that the inverse problem may have a positive resolution for all partitions with finite height.

4. Periodic tableaux

In this section we introduce a new class of periodic (infinite) Young tableaux and connect them to periodic permutations by means of the stable Robinson–Schensted correspondence. Everywhere below we assume that partitions have finite height.

We say that an infinite Young tableau A is *m-periodic* if there for some $N \in \mathbb{N}$ we have: if an integer k is in i-th row, then so is $k+m$. Clearly, this is equivalent to saying that large enough integers are place in rows of A according to $i \mod m$.

Suppose $\lambda = (\lambda_1, \lambda_2, \dots)$ is the shape of m-periodic tableau A, $\lambda_i \in \mathbb{N} \cup \{\infty\}$, and let $\ell = h(\lambda)$. Clearly, $\ell(\lambda) < \infty$ since $\lambda_i = \infty$ for at most m rows. Let $\mu_r(A)$ denotes the number of integers i placed in r-th row of A, and such that $N \le i < N+m$. We call the sequence $\mu = (\mu_1, \mu_2, \dots)$ the *periodic shape* of a tableaux A.

Lemma 4.1 *The periodic shape of a m-periodic tableau A is a partition of m.*

Proof. Indeed, if $a_r(n)$ is the number of integers $\le n$ in r-th row, then $a_r(n) \ge a_{r+1}(n)$ by definition of a Young tableau. Observe that $\mu_r = \lim_{n\to\infty} a_r(n)/m$ for all $r < \ell$, which implies the result. □

The following result is the main connection between periodic permutations and periodic tableaux:

Theorem 4.2 *If $\sigma \in \Sigma_m$ is a m-periodic permutation, then Young tableaux $(A, B) = \varphi(\sigma)$ produced by the stable Robinson–Schensted correspondence are m-periodic with the same stable shape λ and the same periodic shape μ.*

Proof. We need the following well known property of the classical Robinson–Schensted correspondence (see e.g. [St]). Let $\sigma \in S_n$, and $(A_n, B_n) = \varphi_n(\sigma)$. Then the first row in A_n consists of all elements $a_i = \sigma(k_i)$ such that $\sigma(j) > a_i$ for all $j > k_i$, $1 \le j \le n$. The second row of A_n can be obtained in a similar way after elements $a_1, a_2, \dots$ are removed from σ. Similarly we obtain the third row, etc.

Let us show that the above description works verbatim as we let $n \to \infty$. Indeed, for any permutation $\sigma \in S_\infty$, consider a sequence $P_1 = (a_1, a_2, \dots)$, such that $a_i = \sigma(k_i)$ and $\sigma(j) > a_i$ for all $j > k_i$. From the above construction, the element a_i is always in the first row of A_n for all $n \ge k_i$, and thus are in the first row of $A_\infty = A$. Same for the sequence of integers P_2 in the second row, etc. Note that every integer $a = \sigma(k)$ must belong to one of the P_i for $i \le k$ since nothing can possibly precede it after the $P_1, \dots, P_{k-1}$ are removed.

Now suppose $\sigma \in \Sigma_m$ and N is an integer such that $\sigma(i+m) = \sigma(i)$ for all $i \ge N$. By definition, if $a \in P_1$, then $(a+m) \in P_1$, and thus P_1 is m-periodic. Use induction to show that the same holds for all P_i. This proves that tableaux A is periodic. By proposition 2.4, we have $\varphi(\sigma^{-1}) = (B, A)$. Since $\sigma^{-1} \in \Sigma_m$, we conclude that B is m-periodic as well.

Recall that Proposition 2.1 implies that A and B have the same stable shape. For the periodic shape, let $s_r(n)$ be the length of the r-th row of A_n or, equivalently, of B_n. Define $M = \max\{\sigma(i)-i, i \in \mathbb{N}\} < \infty$ for m-periodic permutations. Let $\mu = (\mu_1, \mu_2, \dots)$ be the periodic shapes of A. Finally, define $p_r(n) = |P_r \cap \{1, \dots, n\}|$. From above, $p_r(n) = a_r(n)$. Observe that for all $n \geq N$ we have $s_r(n) - p_r(n) < N + mM$ since an integer cannot be bumped more than m times, each coming after at most M steps from the previous. Therefore, as in Lemma 4.1, we have

$$\mu_r \,=\, \lim_{n\to\infty} a_r(n)/m \,=\, \lim_{n\to\infty} p_r(n)/m \,=\, \lim_{n\to\infty} s_r(n)/m\,.$$

Now symmetry in the definition of $s_r(n)$ implies the result. $\square$

Example 4.3 Let us show that the inverse of Theorem 4.2 is not true. Consider 3-periodic tableaux $A = B$ with the stable shape (∞, ∞) with the first row $P_1 = (1, 2, 4, 5, 7, 8, 10, 11, \dots)$ and the second row $P_2 = (3, 6, 9, 12, \dots)$. The periodic shape is a partition $\mu = (2, 1)$. By construction as in the proof of Theorem 3.2 we have $\varphi(\sigma) = (A, B)$ for $\sigma = (3, 6, 1, 9, 12, 2, 15, 18, 4, 21, 24, 5, \dots)$. Since $\sigma \notin \Sigma$, we conclude that not every preimage of an m-periodic pair of Young tableaux (A, B) with the same stable and periodic shape, is m-periodic.

In view of Theorem 4.2 and Example 4.3 we can consider the a restriction on the Inverse Problem to m-periodic permutations and tableaux. The following result show what we know about the number of preimages of the Robinson–Schensted map. Note that it still does not resolve the inverse problem in this case.

Theorem 4.4 *The is a finite number permutations $\sigma \in \Sigma_m$ with $\sigma(i+m) = \sigma(i) + m$ for all $i \geq N$, and which are preimages $\sigma = \varphi^{-1}(A, B)$ of any given two m-periodic Young tableaux (A, B) of the same stable shape.*

Proof. Let N_1 be the smallest integers such that i and $i+m$ are in the same row in A respectively, for all $i \geq N_1$. Define N_2 similarly, for the tableau B. We shall find a bound on the smallest M such that the $J = \{N_1, \dots, N_1 + m - 1\}$ are never erased after the first M steps of the stable Robinson–Schensted correspondence.

By construction, after N_2 steps the sequence of rows in which the new elements appear in B is periodic. Let d denote any element of J that lies in the largest (r-th) row of A. Since it can't be 'bumped', after at most $t = N_2 + (N_1 + m - 1)$ steps we the element d is never erased. Next, the elements of J in the previous row of A can be 'bumped' only to the last row, and the number of times that can happen is at most $\mu_r \leq \mu_{r-1}$ times. Thus these elements are never erased after $t + m$ steps. Repeat this to obtain a bound $M \leq N_1 + N_2 + m^2$.

To summarize, we obtain $\sigma(j) < M$ for all $N_1 \leq j < N_1 + m$. Since $\sigma \in \Sigma_m$, these values determine $\sigma(i) - i$ for all $i \geq N$. There is a finite number of possibilities: at most m^M, so we can try them all. In each possible case, we obtain $N-1$ elements of $\mathbb{N}$ which are unaccounted. These are the elements in $I = \{\sigma(1), \dots, \sigma(N-1)\}$. Trying all $(N-1)!$ permutations of I exhaust all possibilities. $\square$

Now recall the group homomorphism $\gamma : \Sigma_m \to \Upsilon_m$ defined in section 1. By Lemma 1.5, group Υ_m has a natural isomorphism with the Weyl group W_n of root system $\widehat{A}_{n-1}$.

Denote by $\eta : \Sigma_m \to W_n$ the homomorphism $\eta(\tau_i) = s_i$ with $\tau_i \in \Sigma_m$, $s_i \in W_n$ defined as in section 1. We obtain:

Corollary 4.5 *Let $Q = \varphi^{-1}(A, B) \cap \Sigma_m$ be the set of preimages of the stable Robinson–Schensted correspondence for a given pair of m-periodic Young tableaux (A, B) of the same stable and periodic shape. Then $\eta(Q) \subset W_n$ is finite.*

Proof. Indeed, from the proof of Theorem 4.4 there exists a finite number of values $\sigma(i) - i$ for i large enough. From the proof of Lemma 1.5 this is exactly what we need. $\square$

One can view Corollary 4.5 as a stable analogue of the Robinson–Schensted correspondence. What is says, basically, is that for m-periodic pairs of tableaux the "right" correspondence is not with m-periodic permutations, but with elements of W_n instead. As our very first Example 1.2 shows, this "correspondence" is not unique. Still, our finiteness result shows that this connection deserves further investigation.

5. Final remarks

1) Let us speculate on some probabilistic models for generation of infinite permutations. First, consider a probability distribution $P = \{p_i, i \in \mathbb{N}\}$ and think of an urn with balls labeled $i \in \mathbb{N}$ and having probability p_i of being selected. If we now start removing the balls from the urn without replacement. The result is a permutation $\sigma \in S_\infty$ with properties which depend on P. For example, when $p_n = (1 - \frac{1}{2^n})/2^{\binom{n}{2}}$, an easy calculation shows that $\sigma \in \Sigma_1$ with probability $\varepsilon > 0$.

Another way to obtain a random infinite permutation is by taking infinitely many samples $\{a_n\}$ from a continuous distribution P on $\mathbb{R}_+$ and then record the relative order of a sequence $\{n + a_n\}$. Note that without adding a growing term the infinite permutation does not stabilize. One can study the rows of tableaux and their dependence of P. For example, if P is exponential, the rows of A grow linearly, while columns at least exponentially fast.

2) It would be nice to find more about the algebraic structure of groups Σ_m. For example, we do not know whether these groups are finitely presented. In a different direction, can one extend Dixon's theorem on probability of generating S_n by two random permutations?

3) One is tempted to generalize the stable Robinson–Schensted map in direction of the classical Robinson–Schensted–Knuth (RSK) correspondence [F,St]. Simply consider all infinite matrices $X = (x_{i,j})$ such that every row and column contains only a finite number of nonzero entries. We say that X is m-periodic if for some integer N we have $x_{i+m,j+m} = x_{i,j}$ for all $i, j \geq N$. From that point in one can follow the standard construction, or the "continuous" version given in [P].

One can show an RSK–analogues of Proposition 2.1, Proposition 2.4, and Theorem 4.2. It would be interesting to find the analogue of the group W_n in this case. We leave the exploration of this subject to the reader.

4) Even if Conjecture 3.8 is false, it may still be true when restricted to m-periodic permutations. One can, perhaps find efficient bounds on the number of preimages in this case. Can one describe the pairs of m-periodic tableaux with

exactly one preimage? We believe in the case $m = 2$ the inverse problem can be completely resolved. Now, what about involutions, when both tableaux are identical? Can one completely resolve the inverse problem in this case?

5) Finally, in recent years there has been much work done on combinatorics of the wight multiplicities, generalizing Young tableaux, for affine root systems including $\widehat{A}_n$ (see [L,vL]). It would be interesting to see if our stable map construction and the periodic shape appear in this context.

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