Lecture 5

CLASSIFICATION OF SURFACES

In this lecture, we present the topological classification of surfaces. This will be done by a combinatorial argument imitating Morse theory and will make use of the Euler characteristic.

5.1. Main definitions

In this course, by a *surface* we mean a connected compact topological space M such that that any point $x \in M$ possesses an open neighborhood $U \ni x$ whose closure is a 2-dimensional disk. By a *surface-with-holes* (поверхность с краем in Russian) we mean a connected compact topological space M such that any point $x \in M$ possesses either an open neighborhood $U \ni x$ whose closure is a 2-dimensional disk, or a whose closure is the open half disk

$$C = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0, x^2 + y^2 < 1\}.$$

(A synonym of "surface" is "two-dimensional compact connected manifold", but we will use the shorter term.) In the previous lecture, we presented several examples of surfaces and surfaces-with-holes.

It easily follows from the definitions that the set of all points of a surface-with-holes that have half-disk neighborhoods is a finite family of topological circles. We call each such circle the *boundary* of a hole. For example, the Möbius strip has one hole, pants have three holes.

5.2. Triangulating surfaces

In the previous lecture, we gave examples of triangulated surfaces (see Fig. 4.6). Actually, it can be proved that *any* surface (or any surface-with holes) can be triangulated, but the known proofs are difficult, rather ugly, and based on the Jordan Curve Theorem (whose known proofs are also difficult). So we will accept this as a fact without proof.

Fact 1. *Any surface and any surface-with-holes can be triangulated.*

To state the next fact about triangulated surfaces, we need some definitions. Recall that a (continuous) map $f : M \to N$ of triangulated surfaces is called *simplicial* if it sends each simplex of M onto a simplex of N (not necessarily of the same dimension) linearly. Any bijective simplicial map map $f : M \to N$ is said to be an *isomorphism*, and then M and N are called *isomorphic*.

Suppose M is a triangulated surface, σ^2 is a face of M and w is an interior point of σ^2. Then the new triangulation of M obtained by joining w to the three vertices of σ^2 is called a *face subdivision* of M at σ (Fig. 5.1(a)); the *barycentric subdivision* of a 2-simplex is shown in Fig. 5.1(c); the *barycentric subdivision of M* is obtained by barycentrically subdividing all its 2-simplices. If σ^1 is an edge (1-simplex) of M, then the *edge subdivision* of M at σ^2 is shown on Fig. 5.1(b). If a triangulated surface M' is obtained from M by subdividing some simplices of M in some way, we say that M' is a *subdivision of M*.

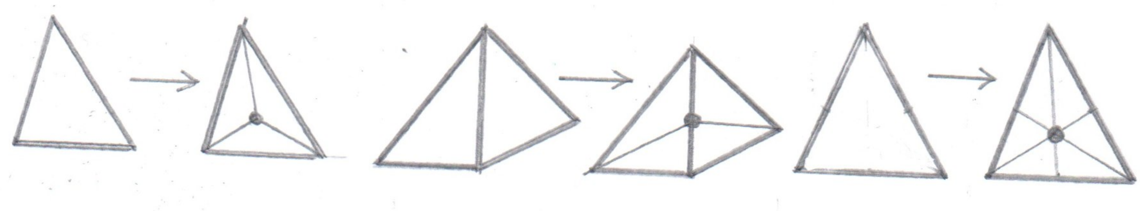

FIGURE 5.1. Face, edge, and barycentric subdivisions

A map $f : M \to N$ is called a PL-*map* if there exist subdivisions of M', N' of M, N such that f is a simplicial map of M' to N'. A bijective PL-map $f : M \to N$ is said to be a PL-*equivalence*, and then M and N are called PL-*equivalent*. The following statement, known as the *hauptvermutung for surfaces*, will be stated without proof.

Fact 2. *Two surfaces are homeomorphic if and only if they are PL-equivalent. Homeomorphic triangulated surfaces have isomorphic triangulations.*

If x, y are vertices of M, then the *star* of x, $\mathrm{St}(x)$, is defined as the union of all simplices for which x is a vertex, and the *link* of y, $\mathrm{Lk}(y)$, is the union of all 1-simplices opposite to the vertex y of the 2-simplices forming $\mathrm{St}(x)$. It is easy to show that $\mathrm{St}(x)$ is, topologically, a 2-disk, and $\mathrm{Lk}(y)$, a circle (see Figure 5.2).

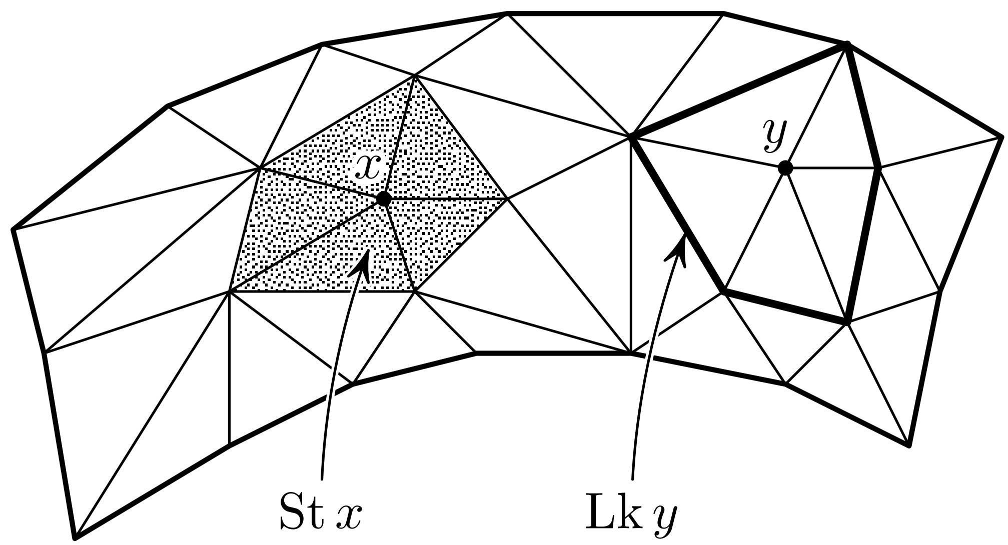

FIGURE 5.2. Star and link of points on a surface

In the previous lecture, orientable surfaces were defined as surfaces not containing a Möbius strip. Now we give another (equivalent) definition of orientability for triangulated surfaces. A simplex $\sigma^2 = [0, 1, 2]$ is called *oriented* if a cyclic order of its vertices is chosen. Adjacent oriented simplices are *coherently oriented* if their common edge acquires opposite orientations induced by the two oriented simplices. Thus if the two simplices $\sigma_1^2 = [0, 1, 2]$ and $\sigma_2^2 = [0, 1, 3]$ are coherently oriented if the cyclic orders chosen in the two simplices are $(0, 1, 2)$ and $(1, 0, 3)$, respectively. A triangulated surface is called *orientable* if all its 2-simplices can be coherently oriented.

It is easy to prove that *a surface is orientable if and only if it does not contain a Möbius strip.*

5.3. Classification of orientable surfaces

The main result of this section is the following theorem.

Theorem 5.3. [Classification of orientable surfaces] *Any orientable surface is homeomorphic to one of the surfaces in the following list*

$$\mathbb{S}^2,\ \mathbb{S}^1 \times \mathbb{S}^1 \ (\textit{torus}),\ (\mathbb{S}^1 \times \mathbb{S}^1)\#(\mathbb{S}^1 \times \mathbb{S}^1) \ (\textit{sphere with 2 handles}), \dots$$

$$\dots, (\mathbb{S}^1 \times \mathbb{S}^1)\#(\mathbb{S}^1 \times \mathbb{S}^1)\# \dots \#(\mathbb{S}^1 \times \mathbb{S}^1) \ (\textit{sphere with } k \textit{ handles}), \dots$$

Any two (different) surfaces in the list are not homeomorphic.

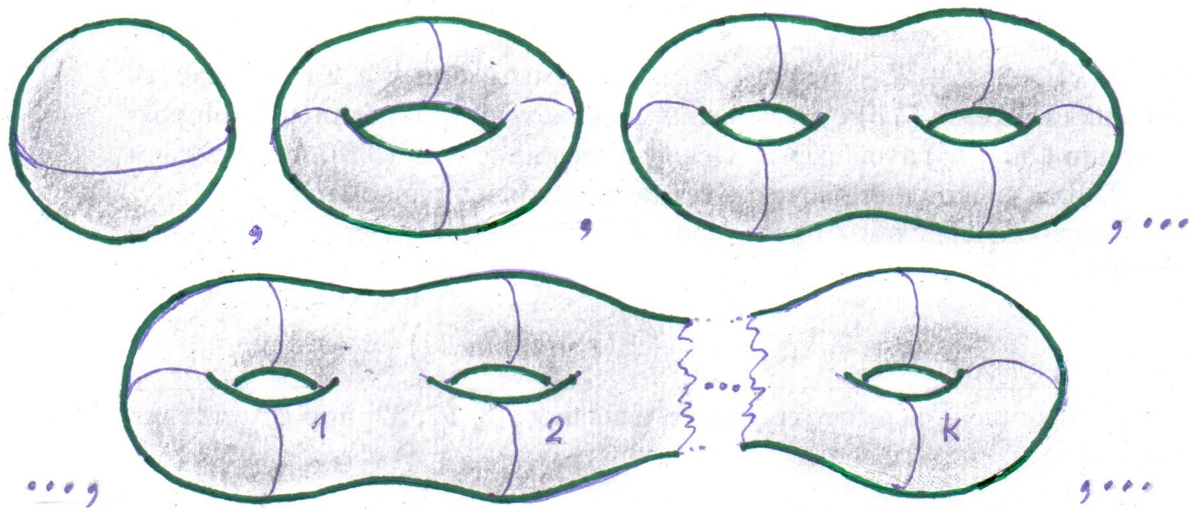

FIGURE 5.3. The orientable surfaces

Proof. In view of Fact 1, we can assume that M is triangulated and take the double baricentric subdivision M'' of M. In this triangulation, the star of a vertex of M'' is called a *cap*, the union of all faces of M'' intersecting an edge of M but not contained in the caps is called a *strip*, and the connected components of the union of the remaining faces of M'' are called *patches*.

Consider the union of all the edges of M; this union is a graph (denoted G). Let G_0 be a maximal tree of G. Denote by M_0 the union of all caps and strips surrounding G_0. Clearly M_0 is homeomorphic to the 2-disk (why?). If we successively add the strips and patches from $M - M_0$ to M_0, obtaining an increasing sequence

$$M_0 \subset M_1 \subset M_2 \subset \dots \subset M_p = M,$$

we shall recover M.

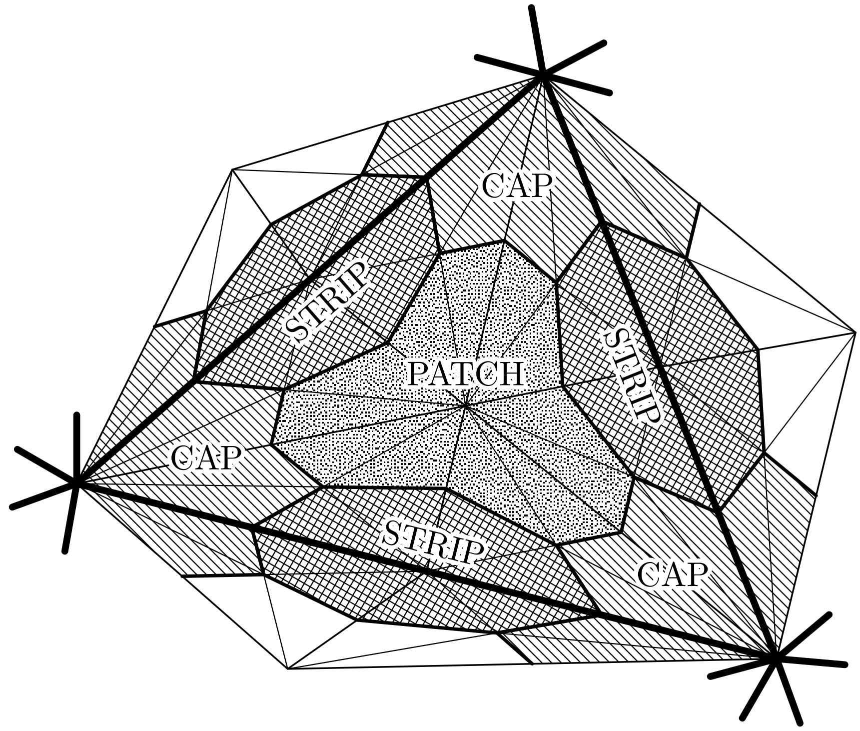

FIGURE 5.4. Caps, strips, and patches

Let us see what happens when we go from M_0 to M_1.

If there are no strips left [1] , then there must be a patch (topologically, a disk), which is attached along its boundary to the boundary circle Σ_0 of M_0; the result is a 2-sphere and the theorem is proved.

Suppose there are strips left. At least one of them, say S, is attached along one end to Σ_0 (because M is connected) and its other end is also attached to Σ_0 (otherwise S would have been part of M_0). Denote by K_0 the closed *collar neighborhood* of Σ_0 in M_0 (i.e., the union of all simplices having at least one vertex on Σ_0). The collar K_0 is homeomorphic to the annulus (and not to the Möbius strip) because M is orientable. Attaching S to M_0 is the same as attaching another copy of $K_0 \cup S$ to M_0. But $K \cup S$ is homeomorphic to the disk with two holes (what we have called “pants"), because attaching S cannot make create a Möbius strip in M because M is orientable (for that reason the twisting of the strip shown in Figure 5.5 (a) cannot occur). Thus M_1 is obtained from M_0 by attaching the pants $K \cup S$ by the waist, and M_1 has two boundary circles (Figure 5.5 (b)).

Now let us see what happens when we pass from M_1 to M_2. If there are no strips left, there are two patches that must be attached to the two boundary circles of M_1, and we get the 2-sphere again.

[1]Actually, this case cannot occur, but it is more complicated to prove this than to prove that the theorem holds in this case.

Suppose there are patches left. Pick one, say S, which is attached at one end to one of the boundary circles, say Σ_1 of M_1. Two cases are possible: either

(i) the second end of S is attached to Σ_2, or
(ii) the second end of S is attached to Σ_1.

Consider the first case. Take collar neighborhoods K_1 and K_2 of Σ_1 and Σ_2; both are homoeomorphic to the annulus (because M is orientable). Attaching S to M_1 is the same as attaching another copy of $K_1 \cup K_2 \cup S$ to M_1 (because the copy of $K_1 \cup K_2$ can be homeomorphically pushed into the collars K_1 and K_2). But $K_1 \cup K_2 \cup S$ is obviously homeomorphic to the disk with two holes. Thus, in the case considered, M_2 is obtained from M_1 by attaching pants to M_1 along the legs, thus decreasing the number of boundary circles by one.

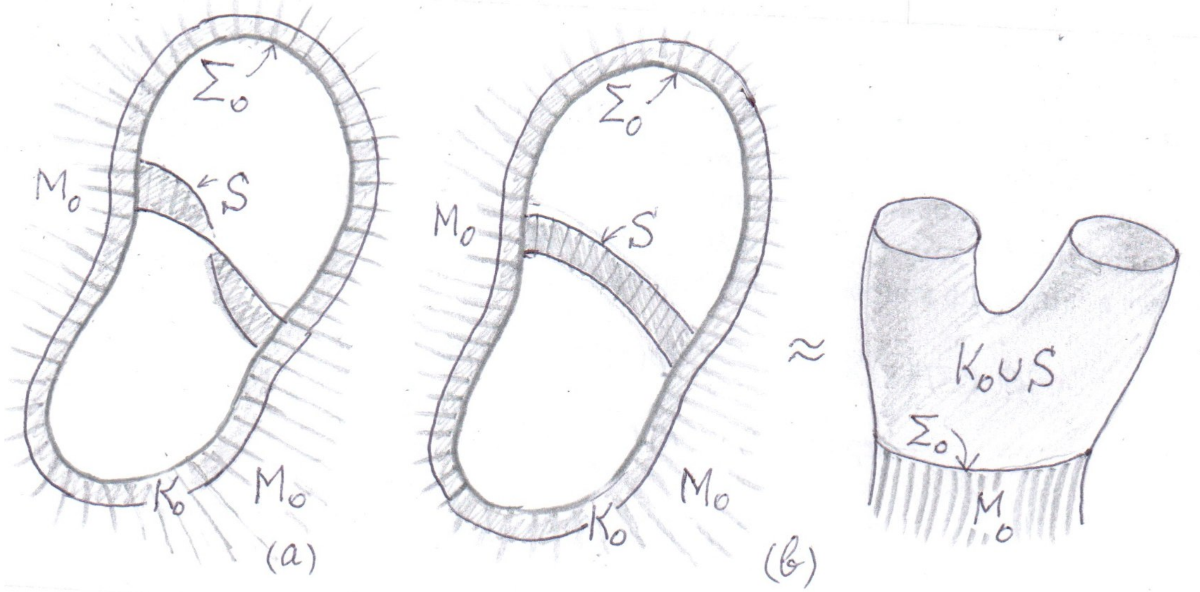

FIGURE 5.5. Adding pants along the legs

The second case is quite similar to adding a strip to M_0 (see above), and results in attaching pants to M_1 along the waist, increasing the number of boundary circles by one.

What happens when we add a strip at the ith step? As we have seen above, two cases are possible: either the number of boundary circles of M_{i-1} increases by one or it decreases by one. We have seen that in the first case “inverted pants” are attached to M_{i-1} and in the second case “upright pants” are added to M_{i-1}.

After we have added all the strips, what will happen when we add the patches? The addition of each patch will “close” a pair of pants either at the “legs” or at the “waist”. As the result, we obtain a surface. Let us prove that *this surface is a sphere with m handles, $m \geq 0$.*

We will prove this by induction over the number k of attached pants.

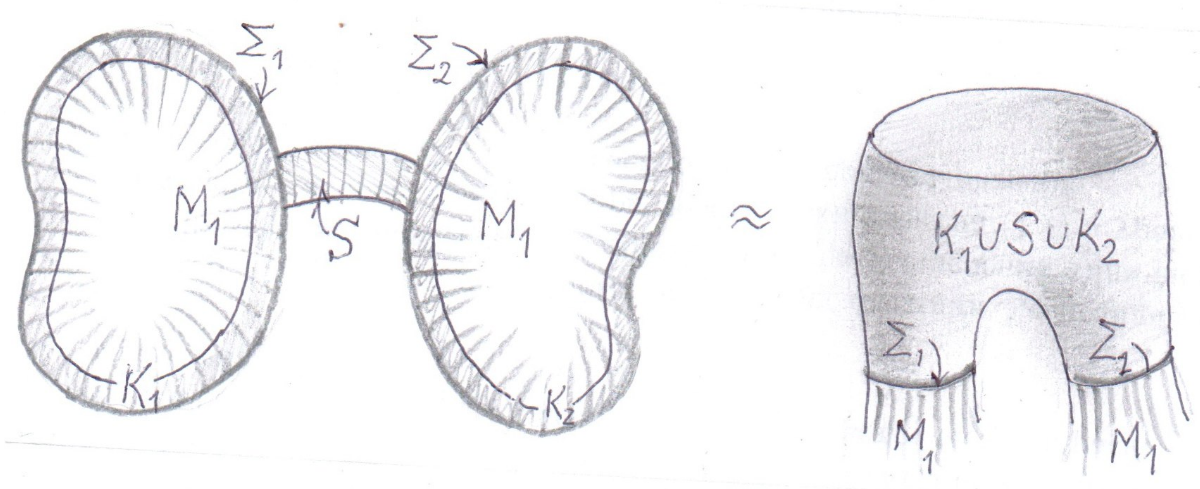

FIGURE 5.5. Adding pants along the waist

The base of induction ($k = 0$) was established above. Assume that by attaching $k-1$ pants by the waist and by the legs and patching up (attaching disks to the free boundaries) we always obtain a sphere with some number (≥ 0) of handles. Let us prove that this will be true for k pants. We will consider two cases.

Case 1: The last pants were attached by the waist (and then the legs were patched up). Removing the pants (together with the two patches) from our surface M and patching up the waist W, we obtain a surface M_1 constructed from $k-1$ pants. By the induction hypothesis, M_1 is a sphere with $m_1 \geq 0$ handles. But M is obtained from M_1 by removing the patch of W and attaching pants by the waist and patching up. But then M is obviously a sphere with the same (m_1) number of handles.

Case 2: The last pants were attached by the legs (and then the waist was patched up). Removing the pants (together with the two patches) from our surface M and patching up the waist W, we obtain a surface M_1 constructed from $k-1$ pants. By the induction hypothesis, M_1 is a sphere with $m_2 \geq 0$ handles. But M is obtained from M_1 by removing the patch of W and attaching pants by the waist and patching up. But then M is obviously a sphere with ($m_1 + 1$) handles.

The first part of the theorem is proved.

To prove the second part, it suffices to show that

(1) *homeomorphic surfaces have the same Euler characteristic;*

(2) *all the surfaces in the list have different Euler characteristics* (*namely* $2, 0, -2, -4, \dots,$ *respectively*).

The first statement follows from Fact 2. Indeed, if two surfaces are homeomorphic, then they have isomorphic subdivisions. It is easy to verify that *the Euler characteristic does not change under subdivision.* To do that, it suffices to check that the Euler characteristic does not change under face, edge, barycentric subdivision, which is obvious. This proves (1).

The second statement is proved by simple computations using the formula for the Euler characteristic of a connected sum (Theorem 4.2).

The theorem is proved.

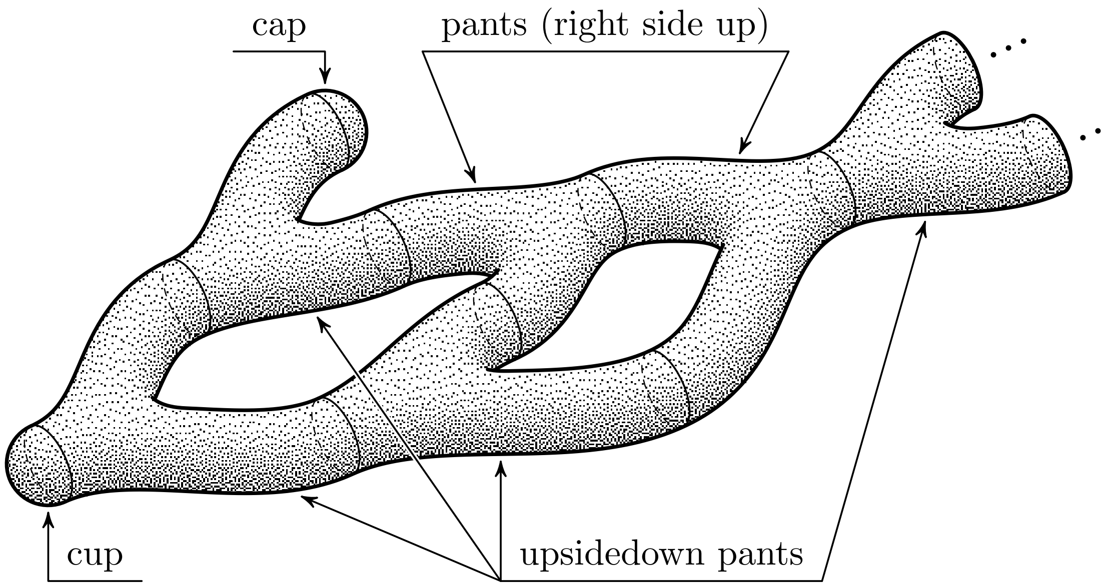

FIGURE 5.6. Constructing an orientable surface

The *genus* g of an orientable surface can be defined as the number of its handles and can be expressed in terms of the Euler characteristic in the following way:

$$g(M) = \frac{1}{2}\Big(2 - \chi(M)\Big).$$

In fact, this has already been established in the above computation of the Euler characteristic of orientable surfaces.

5.4. Classifying nonorientable surfaces and surfaces-with-holes

Theorem 5.4. *Any nonorientable surface is contained in the following list:*

$$\mathbb{R}P^2,\ \mathbb{R}P^2\#\mathbb{R}P^2, \dots, \mathbb{R}P^2\#\mathbb{R}P^2\#\dots\#\mathbb{R}P^2 (n \text{ summands}), \dots$$

Two different surfaces in the list are not homeomorphic.

The proof is similar to the proof of Theorem 5.3, but slightly more complicated. We omit it.

Actually, the assertion of Theorem 5.4 is equivalent to saying that any nonorientable surface is obtained from the sphere by attaching a *Möbius cap*, i.e., deleting an open disk and attaching a Möbius strip along the boundary circle, and then attaching $g \geq 0$ handles.

The nonnegative integer g is called the *genus* of the nonorientable surface. It can easily be expressed in terms of the Euler characteristic. Namely,

$$g(M) = \frac{1}{2}\Big(1 - \chi(M)\Big).$$

We leave the statement of the general classification theorem of all surfaces-with-holes to the reader. We only note that a sphere with h handles, m Möbius caps, and d deleted open disks has Euler characteristic

$$\chi(M) = 2 - 2h - m - d.$$

5.1. Exercises

5.1. Prove that $\chi(m\mathbb{T}^2) = 2 - 2m$ and $\chi(n\mathbb{R}P^2) = 2 - n$. (Here the notation nM stands for the connected sum of n copies of M.)

5.2. Prove that an orientable surface is not homeomorphic to a nonorientable surface.

5.3. (a) Prove that any graph has a maximal subtree. (b) Prove that a simplicial neighborhood of a tree in a surface is homeomorphic to the disk.

5.4. Find the Euler characteristic of the Klein bottle.

5.5. Consider the quotient space $(S^1 \times S^1)/\big((x, y) \sim (y, x)\big)$. This space is a surface. Which one?

5.6. Show that the standard circle can be spanned by a Möbius band, i.e., the Möbius band can be homeomorphically deformed in 3-space so that its boundary becomes a circle lying in some plane.

5.7. Prove that the boundary of $\mathrm{Mb}^2 \times [0, 1]$ is the Klein bottle.

5.8. Prove that on the sphere with g handles, the maximal number of nonintersecting closed curves not dividing this surface is equal to g.

5.9. Can $K_{3,3}$ be embedded (a) in the sphere; (b) in the torus; (c) in the Klein bottle; (d) in the Möbius strip?

5.10*. Prove that the Klein bottle cannot be embedded in $\mathbb{R}^3$.