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ELEMENTS OF SURGERY THEORY

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Background information

0.1 Homotopy groups

Pointed topological spaces. A *pointed topological space* is a topological space X together with a point $* \in X$ called the *basepoint* of X. A map of pointed topological spaces is required to take the basepoint in the domain to the basepoint in the target. The pointed union $X \vee Y$ of a pointed topological spaces X and Y is the space $X \sqcup Y/_\sim$ in which the basepoints of X and Y are identified.

Example 0.1. Let S^n denote the standard unit sphere centered at origin in the Euclidean space $\mathbb{R}^{n+1}$ with basis $e_1, ..., e_n$. The sphere S^n becomes pointed if we pick its south pole, i.e., the end point of $-e_{n+1}$ to be the basepoint of S^n. The intersection of S^n with the hyperplane perpendicular to e_1 is a meridian S^{n-1} of S^n. The quotient space S^n/S^{n-1} is homeomorphic to the bouquet of spheres $S^n \vee S^n$.

A *homotopy* of a map $f\colon X \to Y$ to a map $g\colon X \to Y$ is a continuous family of maps $F_t\colon X \to Y$ parametrized by $t \in [0,1]$ such that $F_0 = f$ and $F_1 = g$. When the spaces X and Y are pointed, every map F_t in the homotopy is required to be pointed as well. A space X is homotopy equivalent to a space Y if there are maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $g \circ f$ is homotopic to the identity map of X, and $f \circ g$ is homotopy equivalent to the identity map of Y.

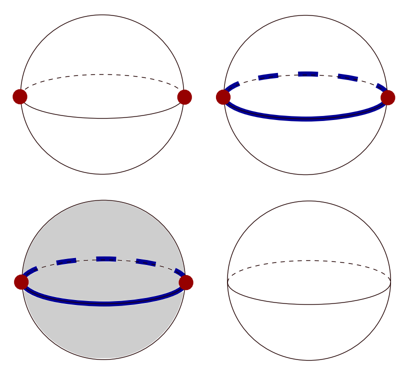


Figure 1: A CW-complex is constructed by induction beginning with a (red) discrete set X^0, by attaching (blue) segments, then (grey) 2-dimensional cells and so on.

CW complexes. Most topological spaces that we will encounter are *CW complexes*. A CW complex X is constructed inductively beginning with a discrete set X^0 called the 0-*skeleton* of X. The n-skeleton X^n is obtained from the $(n-1)$-skeleton X^{n-1} by attaching a collection

$\sqcup D^n_\alpha$ of closed discs of dimension n with respect to attaching maps $\varphi_\alpha\colon \partial D^n_\alpha \to X^{n-1}$. In other words, X^n consists of X^{n-1} and the discs D^n_α with each x in the boundary ∂D^n_α identified with $\varphi_\alpha(x) \in X^{n-1}$. We say that the interior of ∂D^n_α is an *open cell* e^n_α. The so-constructed space is a CW-complex $X = \cup X^n$. A set U in X is declared to be open (respectively, closed) if each intersection $U \cap X^n$ is open (respectively, closed).

Homotopy groups. For $n \geq 0$, the set of homotopy classes $[f]$ of pointed maps $f\colon S^n \to X$ into a pointed topological space is denoted by $\pi_n X$. The set $\pi_n X$ is a group when $n > 0$ with the sum $[f] + [g]$ represented by the composition

$$S^n \longrightarrow S^n/S^{n-1} \approx S^n \vee S^n \xrightarrow{f \vee g} X,$$

where the first map is one collapsing the meridian of S^n into a point. The set $\pi_0 X$ is the set of path components of X. The group $\pi_1 X$ is the *fundamental group*. The *homotopy groups* $\pi_n X$ with $n > 1$ are abelian. A space X is *n-connected* if the groups $\pi_i X$ are trivial in the range $i \leq n$. Thus, a non-empty space is -1-connected, a path connected space is 0-connected, while a simply connected space is 1-connected. A map $f\colon X \to Y$ of topological spaces induces a homomorphism $f_*\colon \pi_i X \to \pi_i Y$ of homotopy groups by $[g] \mapsto [f \circ g]$.

Theorem 0.2 (Whitehead)**.** *Two path-connected CW-complexes X and Y are homotopy equivalent if and only if there exists a map $f\colon X \to Y$ which induces an isomorphism of homotopy groups $f_*\colon \pi_i X \to \pi_i Y$ for all $i \geq 0$.*

Relative homotopy groups. A pair (X, A) of pointed topological spaces consists of a pointed topological space X and a pointed subspace A such that the distinguished point of A is the same as the distinguished point of X. A map of pairs $(X, A) \to (Y, B)$ is a pointed map from X to Y which takes the subspace A to B. Similarly, a homotopy of maps f_0, f_1 of pairs is a continuous family F_t of maps of pairs parametrized by $t \in [0, 1]$ such that $F_0 = f_0$ and $F_1 = f_1$. Let D^n denote the standard disc of dimension n, and S^{n-1} its boundary. Then, the set of homotopy classes of maps from the pair (D^n, S^{n-1}) to a pair (X, A) is denoted by $\pi_n(X, A)$. It is a group if $n \geq 2$, and an abelian group for $n \geq 3$. We note that when the subspace A is a point, the relative homotopy groups reduce to homotopy groups: $\pi_n(X, *) = \pi_n X$. Every map of pairs $f\colon (X, A) \to (Y, B)$ induces a map f_* of relative homotopy groups $\pi_*(X, A) \to \pi_*(Y, B)$. For any pair (X, A), there is a long exact sequence of homotopy groups:

$$\cdots \to \pi_{n+1}(X, A) \xrightarrow{\partial} \pi_n A \xrightarrow{i_*} \pi_n X \xrightarrow{j_*} \pi_n(X, A) \to \cdots,$$

where $i\colon A \to X$ is the inclusion, j is the inclusion $(X,*) \subset (X,A)$ of pairs, while ∂ takes the class $[f]$ of a map $f\colon D^n \to X$ to the class $[f|\partial D^n]$ of the map $f|\partial D^n\colon S^{n-1} \to A$.

A pair (X,A) is said to be n-connected if $\pi_i(X,A) = 0$ for $i \leq n$ and any choice of a base point $* \in A$.

Theorem 0.3. *Let X be a CW complex decomposed as the union of subcomplexes A and B with nonempty connected intersection $C = A \cup B$. If (A,C) is m-connected and (B,C) is n-connected, $m,n \geq 0$, then the map $\pi_i(A,C) \to \pi_i(X,B)$ induced by inclusion is an isomorphism for $i < m+n$ and a surjection for $i = m+n$.*

Theorem 0.4 (Hatcher, Proposition 4.28)**.** *If a CW pair (X,A) is r-connected and A is s-connected, with $r,s \geq 0$, then the map $\pi_i(X,A) \to \pi_i(X/A)$ induced by the quotient map $X \to X/A$ is an isomorphism for $i \leq r+s$ and a surjection for $i = r+s+1$.*

Let A and B be two vector spaces over a field k, or finitely generated free abelian groups (in which case we put define k to be the ring $\mathbb{Z}$). We say that a bilinear function $\psi\colon A \otimes B \to k$ is non-degenerate with respect to A if for any linear function f on A there is a unique vector $b \in B$ such that $f(a) = \psi(a,b)$. Similarly, we say that ψ is non-degenerate with respect to the second factor if for any function f on B there is a vector $a \in A$ such that $f(b) = \psi(a,b)$.

Theorem 0.5. *Suppose that W is an oriented compact manifold of dimension $2q+1 \geq 5$. Suppose that W and ∂W are $q-1$ connected manifolds. Then the intersection pairing*

$$\pi_q W/Tor \otimes \pi_{q+1}(W,\partial W)/Tor \to \mathbb{Z}$$

is non-degenerate with respect to the first factor, while the pairing

$$\pi_{q+1} W/Tor \otimes \pi_q(W,\partial W)/Tor \to \mathbb{Z}$$

is non-degenerate with respect to the second factor.

Proof. Let us prove the non-degeneracy of the first pairing. For the second pairing, the proof is similar.

By the Hurewicz isomorphism, the free part of $\pi_q W$ is isomorphic to the free part of $H_q W$. Since the homology pairing is non-degenerate, for every function f on the free part of $H_q W$ there is an element x_f in the free part of $H_{q+1}(W,\partial W)$ such that $f(y) = x_f \cdot y$ for all elements y

in the free part of $H_{q+1}(W, \partial W)$. By the Hurewicz theorem, the homomorphism $\pi_{q+1}(W/\partial W) \to H_{q+1}(W/\partial W)$ is surjective.[1] Therefore, the element x_f lifts to an element in $\pi_{q+1}(W/\partial W)$. Finally, x_f admits a lift with respect to the map $\pi_{q+1}(W, \partial W) \to H_{q+1}(W/\partial W)$ since such a map is surjective by Theorem 0.4.

[1] See Exercise 23 in section 4.2 of Hatcher's book.

For the second pairing, we have $\pi_q(W, \partial W) \approx \pi_q(W/\partial W)$ is isomorphic to $H_q(W, \partial W)$. Therefore, for any function f on the part of this group, there is a corresponding element x_f in $H_{q+1}(W)$ which can be lifted to an element in $\pi_{q+1} W$. □

0.2 *Bordisms and Cobordisms*

Bordisms. We say that two maps $f_0 : M_0 \to X$ and $f_1 : M_1 \to X$ of closed oriented manifolds into a topological space are *bordant*, if there is a map $f : W \to X$ of a compact manifold W with boundary $\partial W = M_0 \sqcup (-M_1)$ such that $f|M_0 = f_0$ and $f|M_1 = f_1$. The set of bordism classes of maps of manifolds of dimension m into a manifold N forms a group $\Omega_m X$ with operation given by taking the disjoint union of maps.

More generally, given a compact oriented manifold M, a continuous map $f : (M, \partial M) \to (X, A)$ is bordant to zero if there is a compact oriented manifold W with boundary $\partial W = M \sqcup M'$ and a map $F : W \to X$ such that $F|M = f$ and $F(M') \subset A$. Two maps $f_i : (M_i, \partial M_i) \to (X, A)$ are said to be bordant if $f_0 \sqcup f_1 : (M_0 \cup -M_1, \partial M_0 \cup -\partial M_1) \to (X, A)$ is bordant to zero. The set of bordant maps of manifolds of dimension m forms an abelian group $\Omega_m(X, A)$. Clearly, we have $\Omega_m(X, \emptyset) = \Omega_m X$.

Every map $g : (X', A') \to (X, A)$ of pairs induces a homomorphism $g_* : \Omega_*(X', A') \to \Omega_*(X, A)$ by $[f] \mapsto [g \circ f]$. Homotopic maps g, g' induce the same homomorphism $g_* = g'_*$. There is a boundary homomorphism $\partial : \Omega_m(X, A) \to \Omega_{m-1} A$ defined by associating the class of $f : (M, \partial M) \to (X, A)$ with the class $f|\partial M : \partial M \to A$. For any pair (X, A) of spaces, there is a long exact sequence of abelian groups

$$\to \Omega_{m+1}(X, A) \xrightarrow{\partial} \Omega_m A \to \Omega_m X \to \Omega_m(X, A) \to,$$

where $i : A \to X$ and $j : (X, \emptyset) \to (X, A)$ are the inclusions. The reduced bordism group $\tilde{\Omega}_m X$ is the kernel of the homomorphism ε_* induced by the projection ε of X onto a one point space. There is an isomorphism $\Omega_m(X, A) \approx \tilde{\Omega}_m(X/A)$ where $X/\emptyset$ is the disjoint union of

X and a one point set. Suppose that U is a subset in X such that its closure is in the interior of A. Then the inclusion $(X \setminus U, A \setminus U) \to (X, A)$ induces an isomorphism of bordism groups.

Cobordisms. We say that a proper map $f\colon M \to N$ of manifolds is oriented if the normal bundle of a map $f \times i\colon M \to N \times \mathbb{R}^k$ for an embedding $i\colon M \to \mathbb{R}^k$ with $k \gg 1$ is oriented. Two orientations defined by i and i' are equivalent, if they are compatible with an isotopy from i to i'. We say that two proper oriented maps $f_0\colon M_0 \to N$ and $f_1\colon M_0 \to N$ are *cobordant* if there is a proper oriented map $f\colon W \to N$ of a manifold with boundary $\partial W = M_0 \cup M_1$ extending f_0 and f_1 in such a way that the orientation of f agrees with the orientations of f_0 and f_1. The set of cobordism classes of maps of dimension $-q = \dim M - \dim N$ into N is a group denoted by $\Omega^q N$. [2]

[2] Note that oriented bordism classes are represented by maps of oriented closed manifolds, while oriented cobordism classes are represented by oriented proper maps of manifolds which are not necessarily oriented or closed.

Every smooth map $g\colon N' \to N$ of manifolds defines a homomorphism $g^*\colon \Omega^* N \to \Omega^* N'$ by $[f] \mapsto [g^* f]$. The homomorphism g^* depend only on the homotopy class of g. A proper oriented map $g\colon N' \to N$ of dimension d defines the so-called Gysin homomorphism $g_!\colon \Omega^m N' \to \Omega^{m-d} N$ by $[f] \mapsto [gf]$.

Products. The external products

$$\wedge\colon \Omega_m N \otimes \Omega_{m'} N' \to \Omega_{m+m'}(N \times N'),$$
$$\wedge\colon \Omega^d N \otimes \Omega_{d'} N' \to \Omega^{d+d'}(N \times N')$$

are defined by $[f_1] \wedge [f_2] = [f_1 \times f_2]$. There are also homomorphisms

$$/\colon \Omega^q(N \times N') \otimes \Omega_m N' \to \Omega^{q-m}(N),$$
$$\setminus\colon \Omega^q N \otimes \Omega_m(N \times N') \to \Omega_{m-q}(N').$$

The first homomorphism is defined by $[f]/[g] = [(\pi_1 \circ f \circ f^*(1_N \times g)]$. For example, suppose that both f and g are transverse embeddings. Then $f \circ f^*(1_N \times g)$ is the inclusion into $N \times N'$ of the intersection of the images of f and g. The map π_1 projects it to N. The second homomorphism is defined by $[f]\setminus[g] = [\pi_2 \circ g \circ g^*(f \times 1_{N'})]$.

The group $\Omega^* N$ is a ring with operation given by $[f_1] \cup [f_2] = \Delta^*(f_1 \wedge f_2)$ where $\Delta\colon N \to N \times N$ is the diagonal map. There is also the $\cap$-product

$$\cap\colon \Omega^q N \otimes \Omega_m(N, \partial N) \longrightarrow \Omega_{m-q}(N, \partial N),$$

defined by the composition

$$\Omega^q N \otimes \Omega_m(N, \partial N) \xrightarrow{1 \otimes \Delta_*} \Omega^q N \otimes \Omega_m(N \times (N, \partial N)) \xrightarrow{\setminus} \Omega_{m-q}(N, \partial N).$$

When N is a compact orientable manifold of dimension n, the class $[N]$ of the identity map of $(N, \partial N)$ is called the fundamental class of N in $\Omega_n(N, \partial N)$.

0.3 *Homology and Cohomology*

Let X be a CW-complex. We recall that it is constructed by induction beginning with a discrete set X^0. The n-th skeleton X^n is obtained from the $(n-1)$-skeleton by attaching to X^{n-1} closed discs D^n_α of dimension n by means of the attaching maps $\varphi_\alpha : \partial D^n_\alpha \to X^{n-1}$. The interior e^n_α of the attached disc D^n_α is called an open cell.

The algebraic counterpart of a CW-complex X is its chain complex C_*X. By definition, a chain complex C_* is a sequence of abelian groups $\{C_n\}$ together with homomorphisms $d_n : C_n \to C_{n-1}$ called the differentials such that $d_n \circ d_{n+1} = 0$. The condition $d_n \circ d_{n+1} = 0$ is equivalent to the requirement that the image of the homomorphism d_{n+1} belongs to the kernel of the homomorphism d_n. Then, the so-called homology groups $H_n(C_*) = \ker d_n / \operatorname{im} d_{n+1}$ can be defined.

The chain complex C_*X of a CW-complex X consists of the free abelian groups C_nX over the generators $[e^n_\alpha]$ corresponding to the n-cells of X. The boundary map $d_n : C_nX \to C_{n-1}X$ is defined by taking a generator $[e^n_\alpha]$ into the linear combination $\sum_\beta k_{\alpha,\beta}[e^{n-1}_\beta]$ of generators of $C_{n-1}X$ where $k_{\alpha,\beta}$ is the number of times that the boundary of the cell e^n_α wraps around the cell e^{n-1}_β. More precisely, the coefficient $k_{\alpha,\beta}$ is the degree of the map

$$S^{n-1} = \partial D^n_\alpha \xrightarrow{\varphi_\alpha} X^{n-1} \to X^{n-1}/(X^{n-1} \setminus e^{n-1}_\beta) \approx S^{n-1}.$$

The homology groups of the chain complex C_*X of a CW-complex X are simply denoted by H_nX.

The dual of a chain complex (C_*, d_*) is a cochain complex (C^*, δ_*). It consists of abelian groups $C^n = \operatorname{Hom}(C_n, \mathbb{Z})$, and coboundary homomorphisms $\delta_n : C^n \to C^{n+1}$. In other words, a cochain f in C^n is a linear function on chains in C_n. It is convenient to write $\langle f, x\rangle$ for the value $f(x)$ of the function f on x. In this notation, the coboundary homomorphism is defined by $\langle \delta_n f, x\rangle = \langle f, d_{n+1}x\rangle$. It follows that $\delta_{n+1} \circ \delta_n = 0$, and therefore the cohomology groups $H^n(C_*) = \ker \delta_n / \operatorname{im} \delta_{n-1}$ can be defined. The cohomology group of the chain complex C_*X of a CW-complex X are denoted by H^nX.

A map of chain complexes $f\colon C_* \to C'_*$ is a family of maps of abelian groups $f_n\colon C_n \to C'_n$ such that $d \circ f_n = f_{n-1} \circ d$. The requirement $d \circ f_n = f_{n-1} \circ d$ ensures that f gives rise to homomorphisms $H_n(C_*) \to H_n(C'_*)$ of homology groups and $H^n(C_*) \to H^n(C'_*)$ of cohomology groups.

A cellular map $f\colon X \to Y$ of CW-complexes is a map that takes n-cells of X to the n-th skeleton Y^n. By the so-called Cellular Approximation Theorem, every continuous map of CW-complexes can be continuously deformed to a cellular map.

A cellular map f defines a map $C_*X \to C_*Y$ of chain complexes by $f([e^n_\alpha]) = \sum k_{\alpha,\beta}[e^n_\beta]$ where $k_{\alpha,\beta}$ is the number of times the disc $f(e^n_\alpha)$ raps around the cell e^n_β. More precisely, the coefficient $k_{\alpha,\beta}$ is the degree of the map

$$S^n \approx X^n/(X^n \setminus e^n_\alpha) \xrightarrow{f} Y^n/(Y^n \setminus e^n_\beta) \approx S^n.$$

Dually, a cellular map $f\colon X \to Y$ of CW-complexes defines a map $C^*Y \to C^*X$ of cochain complexes.

Products.

When C and C' are chain complexes, a new chain complex $C \otimes C'$ can be defined; its n-th entry is $\sum_{i+j=n} C_i \otimes C'_j$, while the boundary homomorphism $d(c \otimes c') = dc \otimes c' + (-1)^k c \otimes dc'$ where k is the degree of c. For example, when X and Y are finite CW-complex, then $X \times Y$ is also a finite CW-complex that consists of products of cells in X and cells in Y. The chain complex $C_*(X \times Y)$ therefore is the tensor product $C_*X \otimes C_*Y$ of the chain complexes of X and Y. Similarly, the cochain complex $C^*(X \times Y)$ is the tensor product $C^*X \otimes C^*Y$ of cochain complexes.

Unfortunately, the inclusion of the diagonal $X \to X \times X$ is not a cellular map. However, it can be approximated by a cellular map $\Delta\colon X \to X \times X$. It has the property that its composition with any of the two projections $X \times X \to X$ is homotopic to the identity map. The cellular map Δ defines the *cup* product

$$\smile\colon C^nX \otimes C^mX \to C^{n+m}X$$

by identifying an element $x \otimes y$ with an element in $C^{m+n}(X \times X)$ and then setting $x \smile y = \Delta^*(x \otimes y)$. The cup product is a chain map and therefore defines a map on cohomology groups. The *slant* product

$$/\colon C_n(X \times Y) \otimes C^kY \to C_{n-k}X$$

is defined by first writing a chain $a \in C_n(X \times Y)$ as $\sum a_i \otimes a_i'$ where $a_i \in C_*X$ and $a_i' \in C_*Y$, and then setting $a/b = \sum b(a_i')a_i$. It is a chain map and therefore it defines a map on homology and cohomology groups. Together with the diagonal map Δ, the slant product defines the *cap* product:

$$\cap \colon C_n X \otimes C^k X \to C_{n-k} X$$

by $a \cap b = \Delta_* a/b$. Again, it is a chain map that induces a homomorphism $\cap \colon H_n X \otimes H^k X \to H_{n-k} X$.

Example 0.6. The homology groups of a closed oriented surface X of genus g can be identified with $\mathbb{Z}$ in degrees 0 and 2, and with the group $\mathbb{Z}\langle a_1, b_1, ..., a_g, b_g \rangle$ in degree 1. Determine all product maps for X. [3]

[3] Hint: Let $[1]$ denote a generator of $H_0 X$, and $[X]$ a generator of $H_2 X$. Since the composition of Δ with any of the two projections $X \times X \to X$ is homotopic to the identity map, we deduce that $\Delta_*[1] = [1] \otimes [1]$ and $\Delta_*[X] = [1] \otimes [X] + [X] \otimes [1] + \cdots$, where $\cdots$ stands for terms which do not involve $[1]$ and $[X]$. Note that every cell a_i maps under Δ into the diagonal in the torus whose meridian and parallel are copies of a_i. Therefore $\Delta_*(a_i) = [1] \otimes a_i + a_i \otimes [1]$. Similarly, $\Delta_*(b_i) = [1] \otimes b_i + b_i \otimes [1]$. Since

$$\langle \alpha_i \otimes \beta_j, \Delta_*[X] \rangle = \langle \Delta^*(\alpha_i \otimes \beta_j), [X] \rangle,$$

we deduce that

$$\Delta_*[X] = [1] \otimes [X] + [X] \otimes [1] + \sum [a_i] \otimes [b_i] - [b_i] \otimes [a_i].$$

For the cap product, we have, for example,

$$[X] \cap \beta_i = \Delta_*[X]/\beta_i = a_i.$$

1

Stable homotopy groups of spheres

A quick look at the table of homotopy groups of spheres suffices to observe that the groups $\pi_{m+k}S^k$ on the diagonals are the same for all sufficiently high values of k and any fixed m. For example, each of the groups $\pi_k S^k$ is isomorphic to $\mathbb{Z}$. These are the groups on the left-most non-zero diagonal in Table 1.1. The groups $\pi_{1+k}S^k$ on the next diagonal to the right are isomorphic to $\mathbb{Z}_2$ for $k > 2$, while the groups $\pi_{2+k}S^k$ are isomorphic to $\mathbb{Z}_2$ for $k > 1$. In fact, we will see that for any m the groups $\pi_{m+k}S^k$ are the same at least for $k > m+1$. These are so-called *stable homotopy groups* of spheres, denoted by π_m^S. In today's lecture we will recall the historically first approach to computing π_m^S due to Lev Pontryagin. Later we will use the Pontryagin construction to prove the observed stabilization phenomenon, and compute the stable groups π_0^S, π_1^S, π_2^S and π_3^S following the ideas of Pontryagin and Rokhlin.

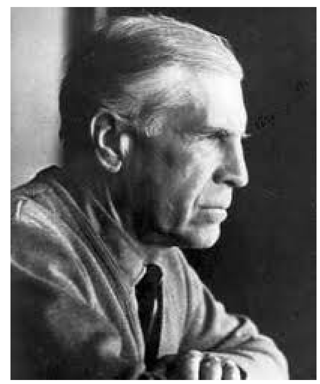


Figure 1.1: Lev Semenovich Pontryagin (1908–1988)

Table 1.1: For every m, the groups $\pi_{m+k}S^k$ on the m-th diagonal stabilize as k increases to infinity. The shaded areas corresponds to the stable range $k > m+1$.

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	$\mathbb{Z}_{15}$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

It is worth mentioning that the Pontryagin approach admits an extensive generalization—the Pontryagin-Thom construction—which plays an indispensable role in contemporary homotopy theory.

To explain the Pontryagin construction we will need to review the notions of a submanifold in a Euclidean space, smooth function on a submanifold, tangent and normal vectors, frame, exponential map, and tubular neighborhood of a manifold in a Euclidean space. The reader familiar with these notions may skip the following section.

1.1 *Framed submanifolds in Euclidean spaces*

In order to compute stable homotopy groups of spheres, Pontryagin came up with an idea of replacing stable homotopy groups of spheres with cobordism classes of framed smooth submanifolds in Euclidean spaces.

Informally, a smooth submanifold M of dimension m in $\mathbb{R}^{m+k}$ is a subset that locally looks like an open subset of $\mathbb{R}^m$, see Figure 1.2. More precisely, a subset $M \subset \mathbb{R}^{m+k}$ is a *smooth submanifold* of dimension m if for each point x in M there is a diffeomorphism Ψ of a neighborhood V of the origin in $\mathbb{R}^{m+k}$ into a neighborhood U of $x \in \mathbb{R}^{m+k}$ such that $\Psi^{-1}(M \cap U)$ is the intersection of V with the m-plane $\mathbb{R}^m \subset \mathbb{R}^{m+k}$ defined in the standard coordinates $(x_1, ..., x_{m+k})$ by the equations $x_{m+1} = \cdots = x_{m+k} = 0$.

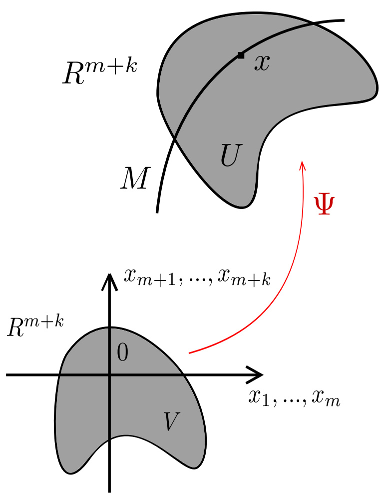


Figure 1.2: A submanifold M in $\mathbb{R}^{m+k}$.

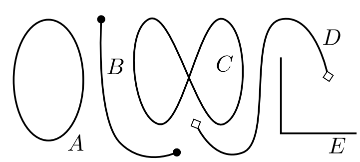


Figure 1.3: As subsets of $\mathbb{R}^2$, the circle A is a closed submanifold, while the open interval D is an open submanifold. The curves B, C and E are not submanifolds for their end points, double point, and the corner respectively.

It is also common to say that M is a *manifold* placed in $\mathbb{R}^{m+k}$, or, simply, a manifold in $\mathbb{R}^{m+k}$. A *closed* submanifold in $\mathbb{R}^{m+k}$ is one whose underlying set is compact, while an *open* manifold is one with no closed components. You may see examples of submanifolds and non-submanifolds in Figure 1.3.

Manifolds in $\mathbb{R}^{m+k}$ are ubiquitous. For example, almost all fibers of any smooth map $f\colon \mathbb{R}^{m+k} \to \mathbb{R}^k$ are smooth (possibly empty) manifolds in $\mathbb{R}^{m+k}$. Indeed, recall that a value $y \in \mathbb{R}^k$ is *regular* if the differential of f at x is an epimorphism for every point x in the inverse image of y. By the Sard theorem,[1] almost every point in the target is a regular value of f. On the other hand, by the Rank Theorem,[2] the inverse image of a regular value is a manifold in $\mathbb{R}^{m+k}$.

[1] **Sard Theorem.** *The set of non-regular values of a smooth map has measure zero.*

The restriction $\psi = \Psi|V \cap \mathbb{R}^m$ of the diffeomorphism Ψ in the definition of a submanifold is called a *coordinate chart* on M. A coordinate chart $\psi = \psi(\mathbf{x})$ locally parametrizes the submanifold M by means of m parameters $\mathbf{x} = (x_1, ..., x_m)$. In particular, for any function f on M, e.g., for $f\colon M \to \mathbb{R}^n$ with $n \geq 0$, the composition $f \circ \psi$ is a function in terms of $(x_1, ..., x_m)$; it is called the *coordinate representation* of f over the coordinate chart ψ. A function f on M is *smooth* if for any coordinate chart ψ, the partial derivatives of all orders of the function $f \circ \psi$ are continuous.

[2] **Rank Theorem.** *Let f be a map from an open subset U of $\mathbb{R}^{m+k}$ to $\mathbb{R}^k$ such that the differential of f is surjective at a point x in U. Then there are coordinates $(x_1, ..., x_{m+k})$ in a neighborhood of x and coordinates in a neighborhood of $f(x)$ such that $f(x_1, ..., x_{m+k}) = (x_1, ..., x_k)$.*

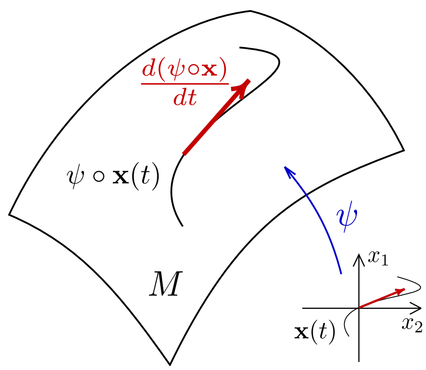


Figure 1.4: A curve $\mathbf{x}(t)$ and its image $\psi \circ \mathbf{x}(t)$.

Consider an arbitrary curve $\mathbf{x} = \mathbf{x}(t)$ on $V \cap \mathbb{R}^m$ passing at the time $t = 0$ through the origin with velocity vector $\dot{\mathbf{x}}(0) = (a_1, ..., a_m)$. Then the corresponding curve $\psi(\mathbf{x}(t))$ on M passes at the time $t = 0$ through $x = \psi(\mathbf{0})$. Its velocity vector at $t = 0$ is

$$\frac{d(\psi \circ \mathbf{x})}{dt}(0) = a_1 \frac{\partial \psi}{\partial x_1}(\mathbf{0}) + a_2 \frac{\partial \psi}{\partial x_2}(\mathbf{0}) + \cdots + a_m \frac{\partial \psi}{\partial x_m}(\mathbf{0}). \tag{1.1}$$

Such a vector in $\mathbb{R}^{m+k}$ is called a *tangent vector* of M at x. The expression (1.1) shows that the space $T_x M$ of tangent vectors at x is a vector space spanned by the m vectors $\frac{\partial \psi}{\partial x_i}(\mathbf{0})$ in $\mathbb{R}^{m+k}$.

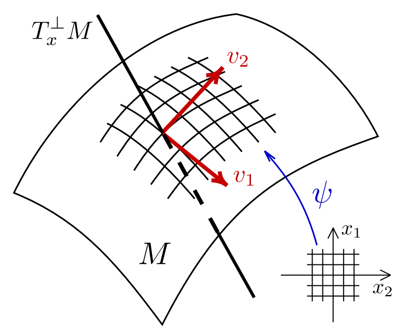


Figure 1.5: The tangent and the perpendicular spaces. Here, the tangent space is spanned by the vectors $v_1 = \frac{\partial \psi}{\partial x_1}(\mathbf{0})$ and $v_2 = \frac{\partial \psi}{\partial x_2}(\mathbf{0})$.

Exercise 1.1. Show that the vectors $\frac{\partial \psi}{\partial x_i}(\mathbf{0})$ are linearly independent and form a basis for the tangent space $T_x M$. Conclude that the dimension of the tangent space $T_x M$ is the same as the dimension of M, i.e., m.[3]

[3] *Hint to Exercise 1.1.* The vectors $\frac{\partial \psi}{\partial x_i}(\mathbf{0})$ are the first m column vectors in the invertible Jacobi matrix of Ψ at $\mathbf{x} = \mathbf{0}$. By the equation 1.1, the vectors $\frac{\partial \psi}{\partial x_i}(\mathbf{0})$ span $T_x M$.

A vector v in $\mathbb{R}^{m+k}$ at the point x of M is a *perpendicular vector* if it is orthogonal to the tangent space $T_x M$. The set of perpendicular vectors at x also forms a vector space, denoted by $T_x^{\perp} M$. We note that for any point $x \in M$, there is a canonical isomorphism $T_xM \oplus T_x^\perp M \approx \mathbb{R}^{m+k}$ of vector spaces.

Choosing a basis for $T_x^\perp M$ is equivalent to choosing an isomorphism $\tau_x : \mathbb{R}^k \to T_x^\perp M$ of vector spaces.[4] Suppose that such an isomorphism τ_x is chosen for each point x. Then for each point x, the image $\tau_x(v)$ of any vector $v \in \mathbb{R}^k$ is a linear combination

$$\tau_x(v) = \alpha_1(x)e_1 + \alpha_2(x)e_2 + \cdots + \alpha_{m+k}(x)e_{m+k}$$

of basis vectors $\{e_1, ..., e_{m+k}\}$ of $\mathbb{R}^{m+k}$. If the coefficient functions α_i are continuous (respectively, smooth) for every vector v, then we say that the manifold M in $\mathbb{R}^{m+k}$ is *continuously* (respectively, smoothly) framed. To summarize, a continuous (respectively, smooth) *frame* on a manifold M is a choice of a basis in each perpendicular vector space $T_x^\perp M$ that changes continuously (respectively, smoothly) with x.

[4] More precisely, a basis $\{\tau_1, ..., \tau_k\}$ for $T_x^\perp M$ defines an isomorphism τ_x by $\tau_x(e_i) = \tau_i$ for $i = 1, ..., k$, where $\{e_i\}$ is the standard basis for $\mathbb{R}^k$.

Up to continuous deformations, the set of continuous frames over M is isomorphic to the set of smooth frames over M. For this reason we will turn to smooth frames whenever it is necessary, and use continuous frames otherwise.

Given a (smoothly) framed manifold M with a frame τ, there is an *exponential map* $\exp : M \times D_\varepsilon \to \mathbb{R}^{m+k}$ where D_ε is the open disc of radius ε in $\mathbb{R}^k$ centered at the origin. The exponential map is defined by $\exp(x, v) = x + \tau_x(v)$. For example, when $\varepsilon = \infty$, the open disc D_ε coincides with $\mathbb{R}^k$ and the image of $x \times D_\varepsilon$ is actually the perpendicular space $T_x^\perp M \subset \mathbb{R}^{m+k}$. In general, perpendicular spaces $T_x^\perp M$ and $T_y^\perp M$ at different points x and y of M may have common points in $\mathbb{R}^{m+k}$. However, if M is compact, then, by the Tubular Neighborhood Theorem,[5] for small ε the exponential map is a diffeomorphism onto a neighborhood of M, called a *tubular neighborhood* of M. In other words, for small values of ε, the exponential map identifies a neighborhood of M in $\mathbb{R}^{m+k}$ with a tube $M \times D_\varepsilon$.

[5] **Tubular Neighborhood Theorem:** *If M is compact, then for small ε, the map* exp *is a diffeomorphism onto a neighborhood of M.*

1.2 *Cobordisms of framed manifolds*

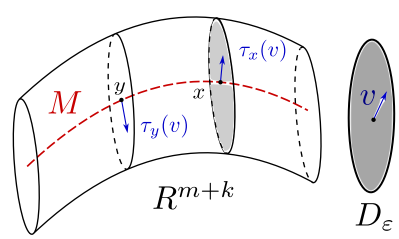


Figure 1.6: A tubular neighborhood of a framed manifold M consists of ε-discs centered at points x of M and orthogonal to T_xM.

The Pontryagin construction shows that a class in $\pi_{m+k}S^k$ defines a (closed) framed manifold M uniquely up to a cobordism. In fact, there is a bijective correpsondence between framed manifolds of dimension m in $\mathbb{R}^{m+k}$ up to cobordisms and pointed maps $S^{m+k} \to S^k$ up to homotopy, where cobordisms of m-manifolds in $\mathbb{R}^{m+k}$ is an equivalence relation which we define next.

Definition 1.2. For a compact subset W in the slice $\mathbb{R}^{m+k} \times [0,1]$ of the Euclidean space $\mathbb{R}^{m+k+1}$ and for $i = 0, 1$, let us denote the set of points of W on the hyperplane $\mathbb{R}^{m+k} \times \{i\}$ by $\partial_i W$. We say that the space W is a *cobordism* between $\partial_0 W$ and $\partial_1 W$ if the complement to $\partial_0 W \cup \partial_1 W$ in W is a manifold in $\mathbb{R}^{m+k+1}$, and there is some $\varepsilon > 0$ such that

$$W \cap \left(\mathbb{R}^{m+k} \times [0,\varepsilon)\right) = \partial_0 W \times [0,\varepsilon),$$
$$W \cap \left(\mathbb{R}^{m+k} \times (1-\varepsilon,1]\right) = \partial_1 W \times (1-\varepsilon,1].$$

The union ∂W of the two sets $\partial_i W$ is called the *boundary* of W, while W is called a *manifold with boundary*. Note that the boundary ∂W has a nice neighborhood in W, namely the union of $\partial_0 W \times [0,\varepsilon)$ and $\partial_1 W \times (1-\varepsilon, 1]$, called a *collar neighborhood*. In particular, the boundary components $\partial_0 W$ and $\partial_1 W$ are smooth manifolds.

For example, for a manifold M in $\mathbb{R}^{m+k}$ the manifold with boundary $W = M \times [0,1]$ in $\mathbb{R}^{m+k} \times [0,1]$ is a cobordism between two copies of M; it is called the *trivial cobordism* of M. On the other hand, $W = M \times [0, 1/2)$ is not a cobordism between M and an empty set since W is not compact.

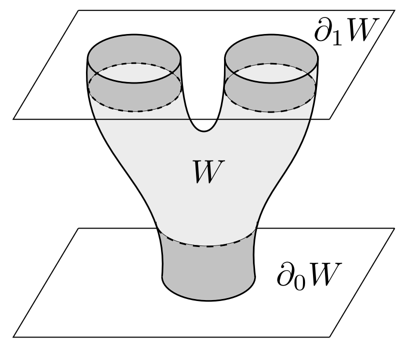


Figure 1.7: A manifold with boundary, and the collar neighborhood of its boundary.

The technical requirement that the boundary of W possesses a collar neighborhood is often omitted. However, the existence of collar neighborhoods simplifies proofs. For example, in the presence of collar neighborhoods, the cobordism relation is clearly transitive.

Suppose now that the manifold W is framed, i.e., at each point $(x,t) \in \mathbb{R}^{m+k} \times [0,1]$ on the manifold W there is a chosen frame $\tau_{(x,t)}$. Since the vectors in $\tau_{(x,t)}$ are perpendicular to $T_{(x,t)}W$, the frame $\tau_{(x,t)}$ restricted to $\partial_0 W$ and $\partial_1 W$ turns the two boundary components into framed manifolds in $\mathbb{R}^{m+k}$. Furthermore, suppose that $\tau_{(x,t)} = \tau_{(x,1)}$ if t is ε-close to 1 and $\tau_{(x,t)} = \tau_{(x,0)}$ if t is ε-close to 0 for some $\varepsilon > 0$. Then we say that W is a *(framed) cobordism* between the framed manifolds $\partial_0 W$ and $\partial_1 W$.

The framed cobordism relation is an equivalence relation. The equivalence class of a framed manifold M will be denoted by $[M]$.

__Exercise__ **1.3.** Show that a framed manifold M_0 is cobordant to the framed manifold M_1 obtained from M_0 by a parallel translation.[6]

Finally, we note that the set of all cobordism classes of closed framed manifolds of dimension m in $\mathbb{R}^{m+k}$ forms an abelian group. Indeed, if M_0 and M_1 are two framed manifolds representing cobordism classes $[M_0]$ and $[M_1]$, we can use a parallel translation to shift M_0 into the left

[6] *Hint to Exercise 1.3.* Let $f\colon \mathbb{R}^{m+k} \to \mathbb{R}^{m+k}$ be the parallel translation that brings M_0 to M_1. Let $j\colon [0,1] \to [0,1]$ be a smooth function that equals 0 for $t < \varepsilon$, increases on the interval $(\varepsilon, 1-\varepsilon)$ and equals 1 for $t > 1-\varepsilon$. Then

$$F_t(x) = (1 - j(t))x + j(t)f(x)$$

is a smooth deformation of M_0 to M_1 where $t \in [0,1]$ and $x \in M_0$, while the trace $W = \{(F_t(x), t)\}$ of the deformation is a cobordism from M_0 to M_1. The frame over W at x is constructed in two steps. First, one applies $d_x(F_t)$ to the frame of M_0 at x to get k vectors at $F_t(x)$, and, second, one projects the k vectors to $T_x^{\perp} W$.

half space $L = (-\infty, 0) \times \mathbb{R}^{m+k-1}$ of $\mathbb{R}^{m+k}$ and M_1 into the right half space $R = (0, \infty) \times \mathbb{R}^{m+k-1}$, and get a new framed manifold $M_0 \sqcup M_1$ in $\mathbb{R}^{m+k}$ representing the sum $[M_0] + [M_1]$.[7] The so-defined operation (addition) is well-defined: changing representatives in the equivalence classes $[M_0]$ and $[M_1]$ does not change the resulting class $[M_0] + [M_1]$.

[7] If M_0 and M_1 do not share common points in $\mathbb{R}^{m+k}$, then we may define the sum $[M_0] + [M_1]$ to be $[M_0 \cup M_1]$ without taking parallel translations. However, if the intersection of M_0 and M_1 is non-empty, the subset $M_0 \cup M_1$ of $\mathbb{R}^{m+k}$ may not be a submanifold.

The addition is clearly associative and commutative. The zero is represented by an empty manifold. To construct the inverse of the cobordism class of a framed manifold M, place M into L and rotate L in $\mathbb{R}^{m+k+1}_+ = \mathbb{R}^{m+k} \times [0, \infty)$ along $\mathbb{R}^{m+k-1}$ till L turns into R. Then the framed manifold M traces a framed cobordism W from $M \subset L$ to a framed manifold $-M \subset R$.[8] In particular, $[M] + [-M] = 0$, and therefore $-M$ represents the negative of $[M]$, see Figure 1.8.

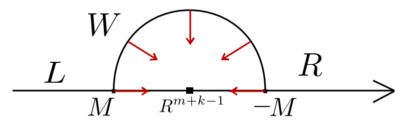


Figure 1.8: The inverse to the class of the framed manifold M is the class of the reflection of M

[8] To be more precise, we add collar neighborhoods to the boundary of W, and shrink $\mathbb{R}^{m+k+1}$ along the $(m+k+1)$-st direction so that W is a subset of $\mathbb{R}^{m+k} \times [0, 1)$.

To summarize, we have shown that the cobordism classes of (closed) framed manifolds of dimension m in $\mathbb{R}^{m+k}$ form an abelian group.

1.3 *The Pontryagin construction*

Pontryagin observed a beautiful relation between framed manifolds M and homotopy groups of spheres. To see the relation, let us identify the complement in S^n to its south pole with $\mathbb{R}^n$ so that $0 \in \mathbb{R}^n$ is the north pole of S^n; the south pole ∞ will be the base point of S^n. Let U denote an ε-tubular neighborhood of a closed framed manifold M of dimension m in $\mathbb{R}^{m+k}$. Define $f|U$ to be the map from U to $\mathbb{R}^k \subset S^k$ by $\exp(x, v) \mapsto v/(\varepsilon - |v|)$. It takes the manifold M to the north pole of S^k and identifies each fiber D_ε of the tubular neighborhood with the Euclidean space $\mathbb{R}^k \subset S^k$. Then, define the restriction $f|S^{m+k} \setminus U$ to be the constant map to $\infty \in S^k$. The obtained continuous map f sends the south pole of S^{m+k} to the south pole of S^k, and, therefore, represents an element in $\pi_{m+k} S^k$. Choosing a different value for ε leads to the same element in $\pi_{m+k} S^k$. Thus, every closed framed manifold M in $\mathbb{R}^{m+k}$ determines a homotopy class in $\pi_{m+k} S^k$.

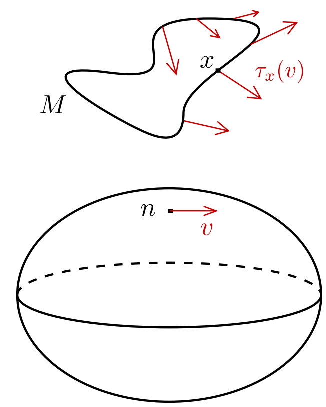


Figure 1.9: The Pontryagin construction can alternatively be described as follows: for each point $x \in M$ consider its open ε-disc neighborhood D_x in the perpendicular space $T_x^\perp M$. These discs form a neighborhood U of M. To define the map $f|U$ represent the sphere S^k as the the disc D^k in which all points of the boundary ∂D^k are identified. The map f sends D_x isomorphically onto D^k in such a way that the frame vectors $\tau_1, ..., \tau_k$ in D_x at x are send to the standard vectors $e_1, ..., e_k$ on D^k at 0. Finally the map f is extended over S^{m+k} by sending the complement to U onto the south pole of S^k.

The corresondence $[M] \to [f]$ is well-defined as cobordant closed framed manifolds M_0 and M_1 define homotopic maps $f_0, f_1 : S^{m+k} \to S^k$. Indeed, the Pontryagin construction applied to a cobordism $W \subset \mathbb{R}^{m+k} \times [0, 1]$ between M_0 and M_1 results in a homotopy $S^{m+k} \times [0, 1] \to S^k$ between f_0 and f_1.

Conversely, let $f : S^{m+k} \to S^k$ be a smooth representative of an element in $\pi_{m+k} S^k$. In view of the Sard theorem, by slightly perturbing the map f, we may assume that the north pole of S^k is a regular value of f. Then the inverse image of the north pole is a manifold M in $\mathbb{R}^{m+k} \subset S^{m+k}$, see Figure 1.9. Furthermore, at each point x in M the differential $d_x f$ takes the perpendicular space $T_x^\perp M$ isomorphically onto the tangent space $T_0\mathbb{R}^k = T_0 S^k$ at the origin. [9] Thus df composed with the canonical isomorphism $T_0\mathbb{R}^k = \mathbb{R}^k$ defines a frame on M. In other words M is a framed manifold. However, the framed manifold M is not uniquely determined by the class in $\pi_{m+k}S^k$; choosing a different representative f of the class leads to a different framed manifold M.

[9] Let us show that $d_x f|T_x^\perp M$ is an isomorphism of the perpendicular space $T_x^\perp M$ and the tangent space $T_0\mathbb{R}^k$. To begin with, the differential $d_x f$ is surjective since the north pole of S^k is a regular value of f. On the other hand, the map f takes the entire manifold M to 0, and therefore the differential $d_x f$ is trivial on $T_x M$. Consequently, the restriction $d_x f|T_x^\perp M$ is surjective. Finally, the dimension of the perpendicular space $T_x^\perp M$ is the same as the dimension of $T_0\mathbb{R}^k$, which implies that the surjective homomorphism $d_x f|T_x^\perp M$ is actually an isomorphism.

To analyze the indeterminacy, let f_0 and f_1 be two different smooth representatives of the same element in $\pi_{m+k}S^k$ determining two framed manifolds M_0 and M_1. Then f_0 is homotopic[10] to f_1 through a smooth homotopy $f_t \colon S^{m+k} \to S^k$ parametrized by $t \in [0,1]$. Define a map $f \colon S^{m+k} \times [0,1] \to S^k$ by $f(x,t) = f_t(x)$. We may assume that the north pole of S^k is a regular value of f_0, f_1 and f, and that the deformation f is trivial when the time parameter is close to 0 and 1. Then, as above, the map f determines a framed manifold $W = f^{-1}(0)$ in $\mathbb{R}^{m+k} \times [0,1]$ with boundary. Furthermore, the boundary of W coincides with the union of M_0 and M_1, and the frame of W restricts to the frames of M_0 and M_1. Thus W is a cobordism between the framed manifolds M_0 and M_1.

[10] We say that a map g is *homotopic* to h if there is a continuous family of maps f_t parametrized by $t \in [0,1]$ such that $f_0 = g$ and $f_1 = h$.

To summarize, we have shown that the homotopy classes in $\pi_{m+k}S^k$ are in bijective correspondence with the cobordism classes of framed manifolds.

Futhermore, the sum of cobordism classes of framed manifolds under the Pontryagin construction corresponds to addition in $\pi_{m+k}S^k$. Thus, we proved the Pontryagin Theorem.

Theorem 1.4 (Pontryagin)**.** *The group of cobordism classes of framed manifolds $M^m \subset \mathbb{R}^{m+k}$ is isomorphic to the homotopy group $\pi_{m+k}S^k$.*

In the remainder of this section we will describe the stability phenomenon, and state the Pontryagin theorem in the form that we discussed in the beginning of the lecture.

A framed manifold in $\mathbb{R}^{m+k}$ is naturally a framed manifold in a bigger space $\mathbb{R}^{m+k+1}$; in the bigger space the frame at a point $x \in M$ contains the old k frame vectors in $\mathbb{R}^{m+k}$ and the additional new basis vector e_{m+k+1}. Furthermore, any two cobordant framed manifolds in $\mathbb{R}^{m+k}$ are also cobordant as framed manifolds in $\mathbb{R}^{m+k+1}$. Consequently, in view of the Pontryagin theorem, there is a so-called *Freudenthal homo-*

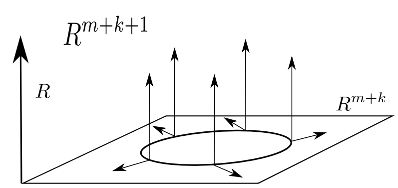


Figure 1.10: A framed manifold $M \subset \mathbb{R}^{m+k}$ also belongs to $\mathbb{R}^{m+k+1}$. Its frame in $\mathbb{R}^{m+k}$ can be completed by e_{m+k+1} to a frame in $\mathbb{R}^{m+k+1}$.

morphism $\pi_{m+k}S^k \to \pi_{m+k+1}S^{k+1}$. We will see later that it is an isomorphism provided that $k > m+1$ and it is an epimorphism for $k > m$, see the Freudenthal Theorem 2.4. Thus there is a characterization of stable homotopy groups of spheres in terms of framed manifolds.

Corollary 1.5 (Pontryagin)**.** *The group of cobordism classes of framed manifolds $M^m \subset \mathbb{R}^{m+k}$ is isomorphic to π_m^S provided that $k > m+1$.*[11]

[11] Recall that π_m^S denotes the m-th stable homotopy group of spheres. It is isomorphic to the group $\pi_{m+k}S^k$ for any $k > m+1$.

1.4 *Example: The stable group π_0^S*

The Pontryagin construction identifies the stable homotopy group π_0^S with the cobordism group of (closed) framed manifolds of dimension 0 in $\mathbb{R}^k$ for $k > 1$. Being compact, such a manifold consists of finitely many points p, each of which equipped with a frame, i.e., a basis for $T_p\mathbb{R}^k \approx \mathbb{R}^k$. As parallel translations do not change the cobordism class of a framed manifold (see Exercise 1.3), the exact location of points p is not essential.

A frame at any point p can be smoothly deformed[12] into the standard positive basis $\{e_1, ..., e_k\}$ or the standard negative basis $\{-e_1, e_2, ..., e_k\}$. In particular, every class in π_0^S can be represented by a union of m positively framed points and n negatively framed points for some $m, n \geq 0$. We claim that the homomorphism $H\colon \pi_0^S \to \mathbb{Z}$ given by $(m,n) \mapsto m-n$ is well-defined and it is an isomoprhism.

[12] A frame $\tau_1, ..., \tau_k$ at a point p determines a homomorphism of vector spaces

$$\mathbb{R}^k \longrightarrow \mathbb{R}^k$$
$$e_i \mapsto \tau_i.$$

The group $\mathrm{GL}_k(\mathbb{R})$ of homomorphisms has two path compoments; the component of homomorphisms f with $\det f > 0$ and the component of homomorphisms f with $\det f < 0$. Therefore, each frame $\{\tau_1, ..., \tau_k\}$ can be smoothly deformed either to the basis $\{e_1, ..., e_k\}$ or to the basis $\{-e_1, ..., e_k\}$.

Indeed, a cobordism W between two framed manifolds $\partial_0 W$ and $\partial_1 W$ of dimension 0 is a manifold with boundary of dimension 1, i.e., a union of finitely many circles and segments. We may discard all the circles from W, and still have a cobordism from $\partial_0 W$ to $\partial_1 W$. Any remaining segment in W has two boundary components p and q. If both p and q belong to the same boundary component of W, i.e., either both belong to $\partial_0 W$ or $\partial_1 W$, then the signs of their frames are opposite. On the other hand if p and q belong to the different components of W, then the signs of their frames are the same. This implies that H is well-defined as a map of sets. The disjoint union of two framed manifolds (m,n) and (m', n') is a framed manifold $(m+m', n+n')$. The equalities

$$H(m+m', n+n') = (m+m') - (n+n') = H(m,n) + H(m', n')$$

show, then, that the map H is actually a homomorphism.

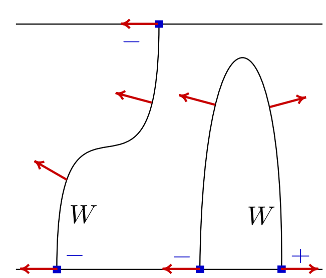


Figure 1.11: A cobordism of framed points.

The homomorphism H is an epimorphism since every positive integer m is the image of the cobordism class of m positively framed points, while every negative integer n is the image of the cobordism class of n negatively framed points. Finally, the cobordism class of m positively framed points and m negatively framed points is trivial, which implies that the homomorphism H is injective. This completes the proof that H is an isomorphism of the stable homotopy group π_0^S onto $\mathbb{Z}$. [13]

[13] The homomorphism H assigns, of course, to the homotopy class of a map $S^k \to S^k$ its degree.

1.5 Further reading

We only discussed manifolds of dimension m in $\mathbb{R}^{m+k}$. These are usually defined in advanced calculus courses, e.g., see **Advanced calculus of several variables** by *C. H. Edwards, JR.* In more advanced courses manifolds are defined without referring to the ambient space $\mathbb{R}^{m+k}$, e.g., see the textbook **Differential topology** by *M. W. Hirsch* and **An introduction to differentiable manifolds and Riemannian geometry** by *W. M. Boothby*. Early works of *L. S. Pontryagin* on stable homotopy groups of spheres include **Homotopy classification of the mappings of an (n+2)-dimensional sphere on an n-dimensional one** (1950) and **Smooth manifolds and their applications in homotopy theory** (1955, 1976). Pontryagin applied his construction to compute π_1^S and π_2^S. The Thom's generalization of the Pontryagin construction can be found in his excellent paper **Quelques propriétés globales des variétés différentiables** (1954), which lays the foundation of the cobordism theory. The J-homomorphism was defined by *G. W. Whitehead* in **On the homotopy groups of spheres and rotation groups** (1942).

The homotopy theoretic calculation of $\pi_3 S^2$ as well as π_1^S can be found in the textbook **Topology and geometry** by *G. E. Bredon* [Br93, page 465]. It is remarkable that even before the invention of higher homotopy groups by E. Čech in **Höherdimensionale Homotopiegruppen** [Ce32] and W. Hurewicz in **Beiträge zur Topologie der Deformationen** [Hu35], Heinz Hopf proved in **Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche** [Ho30] that there are infinitely many mutually non-homotopic pointed maps $f\colon S^3 \to S^2$. Hopf used the linking number of links $f^{-1}x$ and $f^{-1}y$ for any two regular values x, y of f to distinguish the homotopy classes of maps f. Note that if x is the north pole of S^2 and y a point near x, then $f^{-1}x$ and $f^{-1}y$ play the roles of the manifold M and its frame in the Pontryagin construction.

2

Immersion Theory

A manifold placed into $\mathbb{R}^n$ with possible self-intersection points is called an immersed manifold. The 50s and 60s saw a rapid development of the study of immersed manifolds. Historically, the birth of the Immersion Theory is commonly associated with the works of Stephen Smale who classified immersions of spheres, and, in particular, proved a peculiar, counterintuitive statement that the standard sphere S^2 in $\mathbb{R}^3$ can be turned inside out by a deformation through immersions. The work of Smale was later extended by Hirsch to a classification of immersions of arbitrary manifolds.

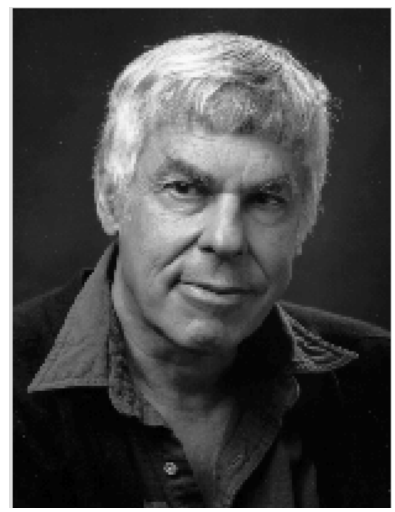


Figure 2.1: Steve Smale (b. 1930) https://math.berkeley.edu/ smale/

In this lecture we will give an exposition of a modern approach to classical theorems of immersion theory, due to Rourke and Sanderson. In the prerequisite section §2.1 we introduce an ambient isotopy, which is a deformation of a manifold in $\mathbb{R}^{m+k}$. In §2.2 we review the compression technique of Rourke and Sanderson. Smale-Hirsch theorem is one of its applications (§99.2.2). Another application of the Rourke-Sanderson technique is the Freudenthal suspension theorem (§2.3) which we have discussed in chapter 1. We will conclude the lecture with two examples: the Smale paradox (§99.2.3), and the interpretation of stable homotopy groups in terms of cobordism groups of orientable immersed manifolds of dimension m in $\mathbb{R}^{m+1}$ (§99.2.4).

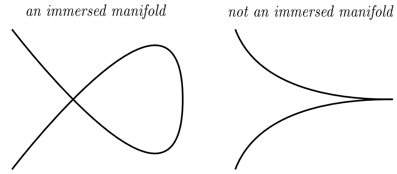


Figure 2.2: Immersion vs. non-immersion.

To put the results of the present chapter into perspective, let's recall that according to the Pontryagin theorem, every element of the stable homotopy group $\pi_{m+k}S^k$ is represented by a framed manifold M in $\mathbb{R}^{m+k}$. We would like to simplify the representing manifold M as much as possible. In the present chapter we will use a deformation (ambient isotopy) to bring the manifold M into $\mathbb{R}^{m+k-1} \subset \mathbb{R}^{m+k}$ when possible. In later chapters we will attempt to replace M with a sphere by performing spherical surgeries on M.

2.1 *Ambient isotopy*

In this section we will make precise the somewhat vague notion of a deformation of a manifold in $\mathbb{R}^{m+k}$. The correct term is actually an ambient isotopy. When possible, we will use an ambient isotopy to simplify a framed manifold M in $\mathbb{R}^{m+k}$ by deforming it into the Euclidean hyperspace $\mathbb{R}^{m+k-1}$.

A smooth vector field w on $\mathbb{R}^{m+k}$ defines a first order differential equation $\dot\gamma(t) = w(\gamma(t))$ with smooth coefficients;[1] its solution is a curve $\gamma\colon \mathbb{R} \to \mathbb{R}^{m+k}$ whose velocity vector at the moment t is precisely the vector w at the point $\gamma(t) \in \mathbb{R}^{m+k}$. Informally, if we imagine that the space $\mathbb{R}^{m+k}$ consists of particles, then the vector field w shows the direction for the flow of the particles, see Fig. 2.3.

[1] Let's take a look at a simple example of how a smooth vector field turns into a differential equation. Suppose that $w(x,y)$ is a vector field on $\mathbb{R}^2$ with components (x^2+y, y^3), while $\gamma(t)$ is a curve on $\mathbb{R}^2$ with components $(x(t), y(t))$. Then the velocity vector of $\gamma(t)$ has components $(\dot x(t), \dot y(t))$, and the differential equation $\dot\gamma(t) = w(\gamma(t))$ corresponding to the vector field $w(x,y)$ takes the form

$$\begin{cases} \dot x = x^2 + y \\ \dot y = y^3 \end{cases}$$

where $x = x(t)$ and $y = y(t)$ are functions in t.

We will often assume that the vectors in the vector field w are bounded in length, i.e., there is a positive number l such that $|w(x)| < l$ for all $x \in M$. By the Existence and Uniqueness Theorem[2], if the vectors $w(x)$ of the vector field w are bounded in length, then there is a unique solution $\gamma(t) = F(x,t)$ for any initial condition $x = \gamma(0)$. In other words, a vector field defines a one parametric deformation F of the Euclidean space $\mathbb{R}^{m+k}$:

$$F_t\colon \mathbb{R}^{m+k} \to \mathbb{R}^{m+k},$$
$$F_t\colon x \mapsto F(x,t),$$

[2] **Existence and Uniqueness Theorem for first order differential equations.** *Suppose that the function F is continuous. Then the differential equation $\dot y(t) = F(t, y(t))$ with an initial value $y(t_0) = y_0$ has a unique solution on the interval $[t_0 - \varepsilon, t_0 + \varepsilon]$ for some real number $\varepsilon < 0$.*

with F_0 being the identity map of $\mathbb{R}^{m+k}$. It is known that the deformation (or, flow) $F(x,t)$ associated with a smooth vector field is smooth in x and t. In fact, for each moment of time t, the map F_t is a a *diffeomorphism* of $\mathbb{R}^{m+k}$, i.e., a smooth homeomorphism whose inverse is also smooth.

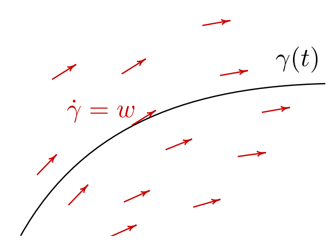


Figure 2.3: A vector field w and the trace $\gamma(t)$ of isotopy.

Note that if we do not require that the vectors $w(x)$ in the vector field w are bounded in length, then it may happen that the flow carries a point $x = \gamma(0)$ to infinity in finite time, and therefore, the position $\gamma(t)$ for large t may not be well-defined.

A smooth *time dependent* vector field w_t smoothly[3] associates at each moment of time t a vector $w_t(x)$ to each $x \in \mathbb{R}^{m+k}$. Again, under the assumption that all vectors $w_t(x)$ are bounded in length, there exists a

[3] In other words, a smooth vector field is a smooth function

$$w\colon \mathbb{R} \times \mathbb{R}^{m+k} \to \mathbb{R}^{m+k},$$
$$w\colon (t,x) \mapsto w_t(x).$$

We may think of w_t as of a smooth family of vector fields parametrized by t.

flow $F_t : \mathbb{R}^{m+k} \to \mathbb{R}^{m+k}$ along w_t in the sense that for each x the curve $t \mapsto F_t(x)$ has velocity $w_t(F_t(x))$.

The flow F of a (possibly, time dependent) vector field is an (ambient) isotopy, i.e., it is a smooth parametric family F_t of diffeomorphisms of $\mathbb{R}^{m+k}$ such that F_0 is the identity diffeomorphism. In practice, an ambient isotopy is viewed as a parametric deformation of M in $\mathbb{R}^{m+k}$, as for each moment of time t, the set $M_t = F_t(M)$ is also a manifold placed in $\mathbb{R}^{m+k}$. We even say that F_t is an ambient isotopy of M. However, as the word "ambient" suggests, the flow F actually deforms not only M, but the entire background space $\mathbb{R}^{m+k}$.

A smooth (respectively, continuous) *normal vector field* v on a manifold M in $\mathbb{R}^{m+k}$ is a function that associates to each point x in M a vector at x not in T_xM such that the components of the vector $v(x)$ change smoothly (respectively, continuously) with the parameter x. Projecting each vector $v(x)$ to the corresponding perpendicular plane $T_x^{\perp}M$ produces a vector field that is actually *perpendicular* to T_xM. Every perpendicular vector field is normal. Not ever normal vector field is perpendicular, but every normal vector field can be linearly deformed to a perpendicular one.[4] Exercise 2.1 shows why we prefer to work not only with perpendicular vector fields.

[4] Let v_0 be a normal vector field over M, and v_1 be the vector field obtained from v by projecting each vector $v_0(x)$ to the perpendicular space $T_x^{\perp}M$. The linear deformation of v_0 to v_1 is the family of vector fields $v_t = (1-t)v_0 + tv_1$ parametrized by $t \in [0,1]$.

Exercise 2.1. Under isotopy a manifold with a normal vector field flows into a manifold with a normal vector field. On the other hand, a manifold with a perpendicular vector field may not flow into a manifold with a perpendicular vector field. [5]

[5] **Hint for Exercise 2.1.** Note that the differential dF_t of the diffeomorphism F_t is invertible. In particular, if a vector $v(x)$ is not in T_xM, then the vector $dF_t(v(x))$ is not in T_yM_t, where $M_t = F_t(M)$ and $y = F_t(x)$. This implies that normal vector fields over M flow into normal vector fields over M_t. On the other hand, the differential dF_t may not preserve the angles, i.e., perpendicular vector fields over M may not flow into perpendicular vector fields over M_t.

Exercise 2.2. Let M be a manifold of dimension m in $\mathbb{R}^{m+k}$. Suppose that normal vector fields v_0 and v_1 over M are homotopic through normal vector fields. Show that v_0 is isotopic to v_1, i.e., there is an ambient isotopy F_t with $t \in [0,1]$ such that $F_0 = \mathrm{id}$, $F_t(x) = x$ for all x in M, and $dF_1(v_0) = v_1$.

2.2 *The Global Compression Theorem*

We say that a vector field v along a manifold M in $\mathbb{R}^{m+k}$ is *vertical up* if for every point $x \in M$ the vector $v(x)$ is a positive multiple of the last basis vector e_{m+k}; we regard $\mathbb{R}^{m+k-1} \times \{0\} \subset \mathbb{R}^{m+k}$ as *horizontal*.

Compression Theorem 2.1 (Rourke-Sanderson). *Every normal vector field v on a closed manifold M of dimension m in $\mathbb{R}^{m+k}$ with $k > 1$ can* *be made vertical up by an ambient isotopy of the manifold M and a smooth deformation of the vector field v through normal vector fields.*

The condition $k > 1$ in the Compression theorem is important. For example, the (radial) unit perpendicular vector field v over the standard circle $M = S^1$ in $\mathbb{R}^2$ can not be straightened up by an ambient isotopy of M and a deformation of v through normal vector fields. We recommend that the reader proves this fact and finds a straightening up deformation of v in $\mathbb{R}^3 \supset \mathbb{R}^2$ before reading the proof of the Compression theorem, see Figure 2.4.

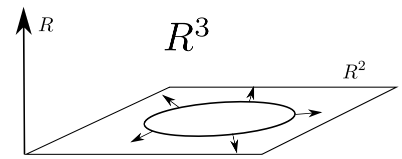


Figure 2.4: A normal frame over $M = S^1$ in $\mathbb{R}^2$ can not be vertical up since at the right most point, i.e., at the point (x, y) in M with maximal value of y, the vertical up direction is tangent to M. In $\mathbb{R}^3$ every vector $v(x)$ can be rotated to the vertical up position.

On the other hand, in view of Exercise 2.2 we may prohibit deformations of vector fields in the Compression Theorem: Under the hypothesis of the Compression Theorem, there is an ambient isotopy F_t with $t \in [0, 1]$ such that F_0 is the identity map of $\mathbb{R}^{m+k}$ and $dF_1(v) = e_{m+k}$ over $F_1(M)$.

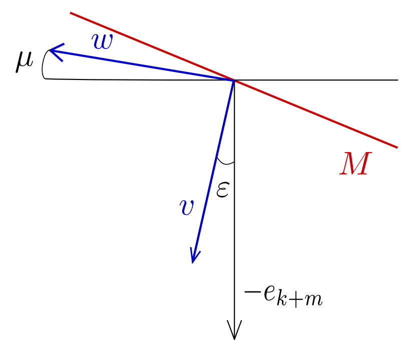


Figure 2.5: A rotation of v to w.

Proof. To begin with we deform v into the vector field of unit length perpendicular to M. Then each vector $v(x)$ determines a point on the unit sphere S^{m+k-1}; namely, if we translate $v(x)$ so that its initial point is the origin in $\mathbb{R}^{m+k-1}$, then its terminal point specifies a point on the unit sphere S^{m+k-1}. In other words, the vector field v defines a so-called *Gauss map* $M \to S^{m+k-1}$. By the Sard Theorem, almost every value in the target of the Gauss map is regular. On the other hand, if $k > 1$, then regular values are precisely those with empty inverse image. Thus, if $k > 1$, then by slightly rotating the manifold M together with the vector field, we may assume that the image of the Gauss map avoids the south pole. That is to say, the vector field v is never vertical down.

Next we observe that since M is compact, the angle between $v(x)$ and the vertical direction $-e_{m+k}$ is at least $\epsilon > 0$. Choose a positive real number $\mu < \epsilon$, and rotate each vector $v(x)$ in the plane P spanned by $v(x)$ and e_{k+m} in the direction vertically up till its last component is positive and $v(x)$ has angle at least μ with the horizontal plane. Such a rotation of the vector field v can be chosen to be through normal vector fields over M. Indeed, the intersection of T_xM with the plane of rotation P is either a point or a line whose angle with the horizontal plane is at least ϵ; therefore we stop rotating $v(x)$ before reaching T_xM. Let w denote the smoothing of the rotated unit vector field over M.

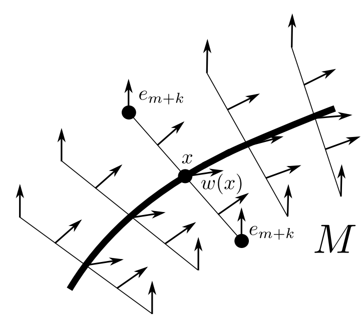


Figure 2.6: The neighborhood U of M consists of δ-discs perpendicular to M and centered at points x in M. Over each of the discs we extend w so that it changes linearly along radial lines from $w(x)$ at the center x of the disc to e_{m+k} at the boundary of the disc.

The vector field w can be extended over $\mathbb{R}^{k+m}$ so that the angle of w with the horizontal plane is at least μ and outside a δ-neighborhood

U of M the vector field w is vertical up, for some small δ. Indeed, the neighborhood U consists of δ-discs perpendicular to M and centered at points x in M. Over each of the discs we extend w so that it changes linearly along radial lines from $w(x)$ at the center x of the disc to e_{m+k} at the boundary of the disc. Finally, we extend w over the rest of $\mathbb{R}^{m+k}$ by vertical up vector filed, smooth the resulting vector field w, and multiply it by a bump function with (a very big) compact support. The flow of w isotopes M to a manifold M' outside U in finite time.[6] The modified normal vector field on M' is vertical up. □

[6] Note that since the manifold M is compact, it lies in a ball of finite radius. On the other hand, since the angle of w with the horizontal hyperspace is at least μ, the flow along w lifts each point by at least $T \sin \mu$ units at time T.

The Compression Theorem allows us in certain cases to deform M to the hyperplane $\mathbb{R}^{m+k-1}$. Indeed, suppose that a compact manifold M is equipped with a vertical up normal vector field v. Furthermore, suppose that none of the lines in the direction v intersects M at more than one point. Then the projection $\pi\colon \mathbb{R}^{m+k} \to \mathbb{R}^{m+k-1}$ along e_{m+k} places M into the hyperplane $\mathbb{R}^{m+k-1}$.

Exercise 2.3. Show that there is an ambient isotopy that moves all points of $\mathbb{R}^{m+k}$ in the vertical direction and brings M to $\pi(M)$. [7]

[7] **Hint for Exercise 2.3.** To define such an isotopy, let f be a function on $\pi(M)$ that assigns to a point $\pi(x)$ the last coordinate x_{m+k} of the point $x \in M$. As a smooth function on M, the function f admits an extension to a smooth function on all $\mathbb{R}^{m+k-1}$. Now, define a vector field w on $\mathbb{R}^{m+k}$ by $w(x,y) = -f(x)e_{m+k}$ where $(x,y) \in \mathbb{R}^{m+k-1} \times \mathbb{R}$. Show that the flow of the vector field w is along the direction $\pm e_{m+k}$ and brings M to $\pi(M)$ in time 1.

We say that the ambient isotopy of Exercise 2.3 *compresses* the manifold M to $\mathbb{R}^{m+k-1}$.

The Compression Theorem admits various generalizations, which we discuss in the chapter of additional topics. For example, the Multi-compression theorem asserts that if a manifold M of dimension m in $\mathbb{R}^{m+k}$ is equipped with $n < k$ linearly independent perpendicular vector fields $v_1, ..., v_n$, then there is an ambient isotopy F_t that straightens the vectors $v_1, ..., v_n$ up in the sense that $F_0 = \text{id}$ and $dF_1(v_i) = e_i$ for $i = 1, ..., n$.

2.3 *The Freudenthal suspension theorem*

Recall that the Pontryagin construction identifies the homotopy groups $\pi_{m+k}S^k$ of spheres with the cobordism classes of framed manifolds M of dimension m in the Euclidean space $\mathbb{R}^{m+k}$. Such a manifold M also lies in a bigger space $\mathbb{R}^{m+k+1}$. Furthermore, its frame in $\mathbb{R}^{m+k}$ can be augmented with an additional vector e_{m+k+1} to produce a frame in $\mathbb{R}^{m+k+1}$, see Figure 2.7. In other words, each framed manifold in $\mathbb{R}^{m+k}$ may also be considered to be a framed manifold in a bigger space $\mathbb{R}^{m+k+1}$. Furthermore, cobordant framed manifolds in $\mathbb{R}^{m+k}$ are

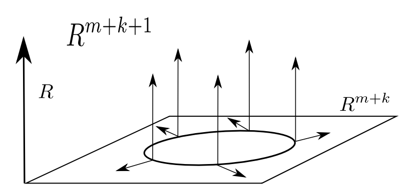


Figure 2.7: A framed manifold $M \subset \mathbb{R}^{m+k}$ also belongs to $\mathbb{R}^{m+k+1}$. Its frame in $\mathbb{R}^{m+k}$ can be augmented by e_{m+k+1} to form a frame in $\mathbb{R}^{m+k+1}$.

still cobordant as framed manifolds in $\mathbb{R}^{m+k+1}$. In view of the Pontryagin construction, the correspondence between framed manifolds in $\mathbb{R}^{m+k}$ and $\mathbb{R}^{m+k+1}$ defines a so-called Freudenthal suspension homomorphism[8] of homotopy groups of spheres $\pi_{m+k}S^k \to \pi_{m+k+1}S^{k+1}$.

[8] **Freudenthal homomorphism (classical definition):** Every map $f\colon S^{m+k} \to S^k$ defines the suspension map $S^{m+k+1} \to S^{k+1}$ by taking the poles of S^{m+k+1} into the respective poles of S^{k+1} and taking the longitude through any point x in S^{m+k} homeomorphically into the longitude through the point $f(x)$. Taking suspensions of representatives, gives another definition of the Freudenthal homomorphism $\pi_{m+k}S^k \to \pi_{m+1}S^{k+1}$.

Theorem 2.4. *The Freudenthal homomorphism is an isomorphism for $k > m+1$ and an epimorphism for $k > m$.*

It follows that the homotopy groups of spheres $\pi_{m+k}S^k$ are the same for a fixed m and $k > m+1$. These are stable homotopy groups π_m^S.

Proof of the Freudenthal theorem. Suppose that $k > m$. Recall that a normal frame $v_1, .., v_{k+1}$ on a manifold M in $\mathbb{R}^{m+k+1}$ is a set of $k+1$ normal vector fields over M that project to a basis of $T_x^{\perp}M$ for each point $x \in M$. In view of the Compression theorem, we may assume that the $(k+1)$-st normal vector field $v = v_{k+1}$ is vertical up.[9]

[9] To show surjectivity of the Freudenthal homomorphism, we need to show that the framed manifold $M \subset \mathbb{R}^{m+k+1}$ is cobordant to a manifold in $\mathbb{R}^{m+k} \subset \mathbb{R}^{m+k+1}$ with v being vertically up. Recall also that isotopies of framed manifolds as well as deformations of normal vector fields imply cobordisms.

We claim that the manifold M can be rotated slightly in $\mathbb{R}^{m+k+1}$ so that any line in the direction e_{m+k+1} intersects M at most at one point. Indeed, for any two distinct points x and y in M, the unit vector $w(x,y)$ in the direction $x-y$ points to a point in S^{m+k}. Since the dimension of pairs of distinct points is $2m$, and the dimension of the sphere S^{m+k} is at least $2m+1$, the set of vectors $w(x,y)$ is of measure zero by the Sard theorem, see Figure 2.8. Therefore by slightly rotating M we may make sure that e_{m+k-1} is not among the vectors $w(x,y)$, which precisely means that any line in the direction e_{m+k+1} intersects M at most at one point. Of course, after the rotation v may not be vertically up any more. However, since we may choose the rotation to be arbitrarily small, the rotated v can be deformed back to the vertical up position.

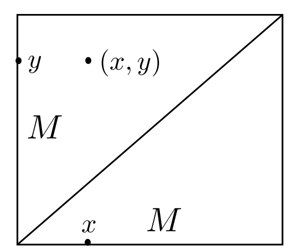


Figure 2.8: The manifold of distinct pairs (x,y) is obtained from the manifold $M \times M$ by removing the diagonal $\{x = y\}$.

Now we may use the vector field v to compress the framed manifold $M \subset \mathbb{R}^{m+k+1}$ to a framed manifold in $\mathbb{R}^{m+k}$, see Exercise 2.3. The compression can be iterated consecutively using the vectors $v_{k+1}, v_k, ...$ as long as the index of the vector v_i under consideration satisfies $i > m$. This proves surjectivity of the Freudenthal homomorphism.

To show that the Freudenthal homomorphism is injective,[10] suppose that a framed manifold $M \subset \mathbb{R}^{m+k}$ is zero cobordant after applying the Freudenthal suspension. In other words, after including M into $\mathbb{R}^{m+k+1}$ and adding the vector field e_{m+k+1} to its frame, the manifold M becomes zero cobordant, i.e., there is a framed manifold W in $\mathbb{R}^{m+k+1} \times [0,1]$ with boundary $M \subset \mathbb{R}^{m+k+1} \times \{0\}$. We know that the last vector field v in the frame of W restricts to e_{m+k+1} over M. In fact,

[10] Note that the proof of injectivity of the Freudenthal homomorphism does not follow immediately from the proof of surjectivity. We begin with a framed manifold M in $\mathbb{R}^{m+k}$ that is zero cobordant in $\mathbb{R}^{m+k+1}$ by means of a framed cobordism W. Of course, we may find a flow F that straightens up the last vector v of the framed cobordism W, and then we can compress W to $\mathbb{R}^{m+k}$. However if F displaces M, the resulting compressed cobordism W is a cobordism to zero of a displaced manifold M, not the original manifold M.

we may assume that it restricts to e_{m+k+1} over an ε-collar neighborhood U of M in W.

Now let us carefully examine the argument in the proof of the Compression Theorem. We first deform v over W to a vector field so that the last component of each vector $v(x)$ is positive. We may assume that during the deformation v stays vertically up over the collar neighborhood U, see Figure 2.9. Next we extend v to a vector field on $\mathbb{R}^{m+k+1} \times [0,1]$. We can extend v as in the Compression Theorem so that v is vertically up outside a neighborhood of W. In addition we may assume that v is vertically up over the slices $\mathbb{R}^{m+k+1} \times [0,\varepsilon)$ and $\mathbb{R}^{m+k+1} \times (1-\varepsilon, 1]$ whose union we will denote by V.[11] Then the flow F along the extended vector field is well-defined as the condition that $v \equiv e_{m+k+1}$ over V prevents the points of $\mathbb{R}^{m+k+1} \times [0,1]$ from flowing out of the region of definition of F. The flow F takes W outside its neighborhood in a finite time T and therefore straightens up the vector field v. Note that all points in U travel the distance T in the direction $v = e_{m+k+1}$. Therefore, if we postcompose the ambient isotopy along v with translation along $-Te_{m+k+1}$, then we get a framed cobordism W' bounding the original manifold M and such that v is vertical up.

[11] We can extend v vertically up over the slice $\mathbb{R}^{m+k+1} \times [0,\varepsilon)$ since, by the definition of a framed cobordism, the only points of W that are in $\mathbb{R}^{m+k+1} \times [0,\varepsilon)$ are the points of the collar neighborhood U.

Finally, when $k > m+1$, we can use the same argument as in the proof of surjectivity to show that W' compresses to a cobordism of M in $\mathbb{R}^{m+k} \times [0,1]$. Thus, a framed manifold $M \subset \mathbb{R}^{m+k}$ is cobordant to zero after the Freudenthal suspension only if it is cobordant to zero itself. □

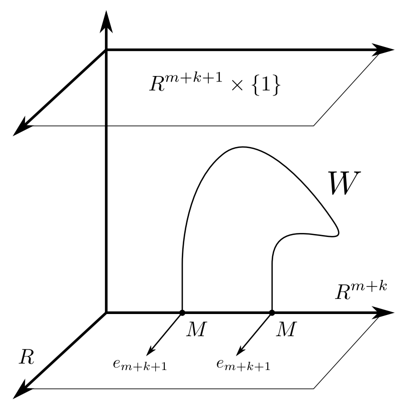


Figure 2.9: The framed manifold M is placed in $\mathbb{R}^{m+k}$ which is depicted as the direction to the right. It is zero cobordant in the horizontal space $\mathbb{R}^{m+k+1}$. The cobordism W itself is a manifold with boundary in $\mathbb{R}^{m+k+1} \times [0,1]$. If we can compress W to $\mathbb{R}^{m+k} \times [0,1]$ without displacing M, then we get a cobordism to zero of the original framed manifold M in $\mathbb{R}^{m+k}$.

2.4 *Further reading*

We borrowed the proof of the Compression Theorem from the original paper **The compression theorem I** [RS01] of *C. Rourke* and *B. Sanderson*. The theorem has many interesting applications some of which are explained in **The compression theorem II: directed embeddings** [RS01a] and **The compression theorem III: applications** [RS03]. We gave one more application here: the Freudenthal theorem.

The original proof of Freudenthal suspension theorem in **Über die Klassen der Sphärenabbildungen. I. Große Dimensionen** [Fr38] was deemed hard. The modern homotopy theoretic proof is based on the Blackers-Massey excision theorem proved in **The homotopy groups of a triad** [BM51]. Recall that in general the Freudenthal homomorphism E is not an isomorphism below the stable range. In the paper **On the**

Freudenthal theorem [Wh53], Whitehead fitted E into the so-called EHP sequence[12]

$$\cdots \to \pi_q S^n \xrightarrow{E} \pi_{q+1} S^{n+1} \xrightarrow{H} \pi_{q-1} S^{2n-1} \xrightarrow{P} \pi_{q-1} S^n \to \cdots .$$

[12] To find the EHP exact sequence, Whitehead observed that E is the homomorphism induced by an embedding $S^n \to \Omega S^{n+1}$, and therefore E fits the homotopy long exact sequence of the pair $(\Omega S^{n+1}, S^n)$.

Promptly James proved in **On the iterated suspension** [J54] the existence of a similar long exact sequence for an iterated Freudenthal homomorphism $E^m \colon \pi_q S^n \to \pi_{q+m} S^{n+m}$. Its geometric interpretation in terms of Pontryagin construction can be found in the paper **Geometric interpretations of the generalized Hopf invariant** [KS77] by Koschorke and Sanderson.

The original proof of the Smale theorem appeared in his papers **A classification of immersions of the two-sphere** and **The classification of immersions of spheres in Euclidean spaces**, [Sm58, Sm59]. Smale's theorems were generalized by Hirsch in the paper **Immersions of manifolds**[Hi59]. It was observed later that not only immersions can be replaced with their formal counterparts. A number of remarkable discoveries culminated in the so-called homotopy principle (h-principle, for short). Besides the classical reference of **Partial differential relations** [Gr86] by Gromov, we recommend the **Introduction to the h-principle** [EM02] by Eliashberg and Mishachev.

3

Spherical surgeries

Under the Pontryagin construction, every element of the homotopy group $\pi_{m+k}S^k$ is identified with the cobordism class of framed manifolds of dimension m in $\mathbb{R}^{m+k}$. The choice of a representing framed manifold in the cobordism class is far from being unique, and it is our goal to find a simple representative. Our strategy is to begin with an arbitrary (a priory complicated) representative and then use a framed cobordism to simplify it as much as possible.


Figure 3.1: Marston Morse (1892–1977)

A general cobordism on a manifold W_0 could be overly complex: it is hard to describe such a cobordism and its action on the homotopy groups $\pi_i W_0$. However, we will see that every non-trivial cobordism breaks into a composition of elementary, so-called *spherical cobordisms*. A spherical cobordism on a manifold W_0 performs a *spherical surgery* on W_0. A spherical surgery is relatively easy to describe in terms of (locally flat) topological submanifolds of $\mathbb{R}^{m+k}$ which are almost everywhere smooth. We will discuss a general Cairns-Whitehead technique of smoothing topological manifolds, and apply it to smooth corners and more complicated singularities produced by an embedded spherical surgery.

Next, we will learn how to efficiently encode a spherical surgery in terms of a *base of surgery*. We will determine how a spherical surgery on a manifold W_0 changes its homotopy groups. As a demonstration of the introduced surgery technique, we will calculate the homotopy group $\pi_3 S^2$ as well as the stable homotopy groups π_1^S and π_2^S.

3.1 *Surgery and cobordisms*

Trivial cobordisms

Recall that a cobordism of framed manifolds of dimension m in $\mathbb{R}^{m+k}$ is a framed closed manifold W of dimension $m+1$ in $\mathbb{R}^{m+k} \times [0,1]$. The projection f of W onto the last coordinate is a so-called *hight* function. We say that a point $x \in W$ is *regular* if $d_x f$ is surjective, and *critical* otherwise.[1] Geometrically, the critical points of a height function are those at which the tangent plane is horizontal, see Figure 3.2.

[1] Recall, the differential $d_x f$ of the height function linearly projects the space $T_x W \subset \mathbb{R}^{m+k} \times \mathbb{R}$ to the last component.

There is a simple coordinate description of regular and critical points. Namely, by the Rank theorem, in a neighborhood of a regular point, there are coordinates $(x_1, ..., x_m, x_{m+1})$ on W such that $f(x) = x_{m+1}$. On the other hand, after slightly perturbing W, near each critical point there are coordinates such that

$$f(x_1, ..., x_{m+1}) = -x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots x_{m+1}^2,$$

see Figure 3.3. The latter statement—which we will not prove here—is known as the Morse lemma, while the perturbed function is called a *Morse function*.

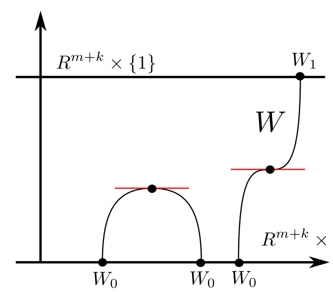


Figure 3.2: A cobordism W between W_0 and W_1 with two critical points. The tangent planes at critical points are horizontal.

The integer i in the coordinate representation of f near a critical point p may be different for different critical points. It is called the *index* of the critical point. The index of a critical point p does not depend on the choice of a coordinate chart about p. Note that if p is a critical point of f of index i, then p is also a critical point of $-f$ of index $j = m+1-i$.

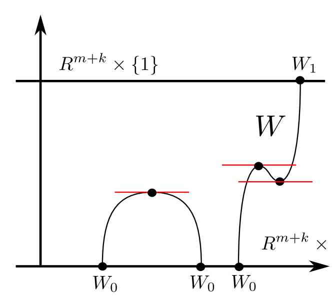
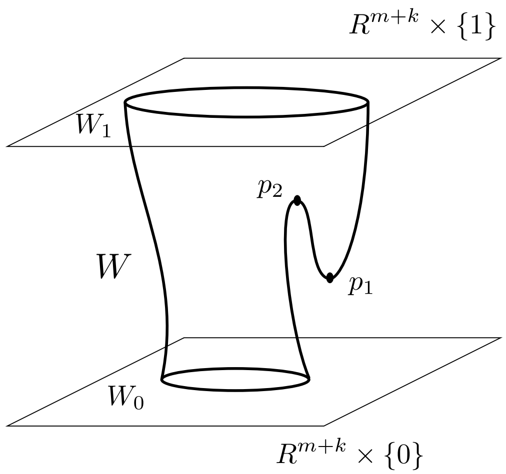


Figure 3.3: After slightly perturbing W from Figure 3.2, the height function has only Morse critical points.

For example, in the Figure below the cobordism has two Morse critical points, the points p_1 and p_2. At the point p_1 the height function f has a local minimum; in a neighborhood of p_1 in appropriate coordinates f can be written as $f(x_1, x_2) = x_1^2 + x_2^2$. The index of the critical point p_1 is 0. The point p_2 is a saddle point. In its neighborhood in appropriate coordinates the height function has the form $f(x_1, x_2) = -x_1^2 + x_2^2$. In particular, the index of p_2 is 1.

According to Lemma 3.1, as t changes from 0 to 1, the *regular levels* $W_t = f^{-1}(t)$, i.e., levels with no critical points of f, change by isotopy.



Lemma 3.1. *Suppose that f has no critical points. Then the cobordism W is trivial, that is there is a diffeomorphism $\varphi\colon W_0 \times [0,1] \to W$ that identifies $W_0 \times \{t\}$ with the t-th level W_t of f for all t.*

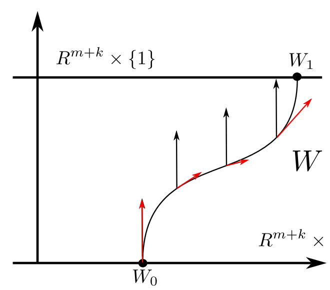


Figure 3.4: The vertically up vector field e_{m+k+1} and its projection v to W.

Sketch of the proof. Consider the vector field e_{m+k+1} in the slice $\mathbb{R}^{m+k} \times [0,1]$ of a Euclidean space. Since the hight function f has no critical points, over W the projection v of e_{m+k+1} to W is never horizontal, i.e., the last component v_{m+k+1} of the vector field v is never zero. If we scalar multiply v by the smooth function $1/v_{m+k+1}$, then we obtain a vector field w over W with last component 1. The flow F of the manifold W_0 along the scaled vector field w carries W_0 along W and brings it to W_t at the time t. In fact, F defines a desired diffeomorphism φ by taking a point (x, t) to $F_t(x)$. $\square$

Note that in the proof of Lemma 3.1 we need to scale the vector field v in order to make sure that for any point $x \in \partial_0 W$ the last coordinate of the curve $t \mapsto F_t(x)$ increases with speed 1 so that the flow F_t indeed brings W_0 to the t-th level at the time t.

Exercise 3.2. Show that the map φ is a diffeomorphism that identifies $W_0 \times \{t\}$ with W_t. In particular, show that φ^{-1} is smooth.[2]

Spherical cobordisms

In view of Lemma 3.1 as t increases from 0 to 1 the level W_t essentially changes only when t passes a *critical value*, i.e, the value $f(x)$ of a critical point x. In appropriate coordinates, the gradient of f in a neighborhood of a Morse critical point is

$$df(x_1, ..., x_{m+1}) = 2(-x_1, ..., -x_i, x_{i+1}, ..., x_{m+1}),$$

[2] *Hint to Exercise 3.2* We have defined the map

$$\varphi\colon W_0 \times [0,1] \to W$$

in the proof of Lemma 3.1 by $\varphi(x,t) = F_t(x)$. The flow F carries points along the scaled vector field w. Since the last component of the scaled vector field w is 1, the last component of each point increases with velocity 1. Thus, in time t, the map F_t lifts each point of W_0 to the hight t. In other words, $\varphi(W_0 \times \{t\}) \subset W_t$. Since the flow F^{-1} along the negative of the scaled vector field w is smooth and brings W_t to W_0 we deduce that $\varphi|W_0 \times \{t\}$ is a diffeomorphism. In particular, φ identifies $W_0 \times \{t\}$ with W_t. Since φ is a homeomorphism and $d\varphi$ is of full rank, it follows that φ is a diffeomorphism, e.g., see [Lee13].

which means that the set of critical points of a Morse function is discrete.[3] It is even finite, since W is compact. Furthermore, by slightly perturbing W, we may always assume that each level W_t contains at most one critical point of f, see Figure 3.5.

[3] Note that if

$$f(x_1, ..., x_{m+1}) = -\sum x_i^2 + \sum x_j^2,$$

then $df(x_1, ..., x_{m+1})$ is

$$(-2x_1, ..., -2x_i, 2x_{i+1} + 2x_{m+1}),$$

or, in other words, $df(x) = 0$ only if $x = 0$.

Thus, every non-trivial cobordism is a composition of *spherical cobordisms* W whose height function has at most one critical point. Under a spherical cobordism, the manifold $\partial_0 W = W_0$ is modified into a manifold $\partial_1 W = W_1$ by the so-called *spherical surgery*.

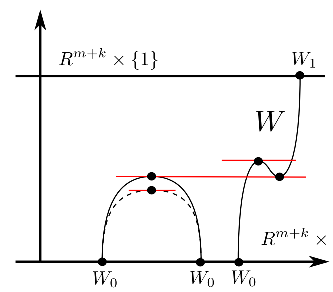


Figure 3.5: By perturbing W, we may assume that each level W_t contains at most one critical point of the hight function.

Exercise 3.3. Let W be a non-trivial cobordism such that each level W_t contains at most one critical point of its hight function. Show that W is diffeomorphic to a composition of spherical cobordisms.[4]

[4] *Hint for Exercise 3.3.* Suppose that the hight function on the cobordism W has n critical points $p_1, ..., p_n$. We may stretch $\mathbb{R}^{m+k} \times [0, 1]$ along the $(m+k+1)$-st coordinate in such a way that for the new hight function f on W we have $f(p_i) \in (i-1, i)$. Next we may modify W near W_i by ambient isotopy so that each level W_t near W_i for $i = 1, ..., n-1$ is obtained from W_i by a vertical translation. Then W is a composition of spherical cobordisms, and it is diffeomorphic to the original cobordism.

Let's describe the result of a spherical surgery. The projection v of the vertical up vector field over W to W is a smooth vector field. It is zero only at the critical point p of the height function since only at the critical point p the tangent space T_pW is horizontal. Let F^v denote the flow along the projected vector field v. The set of points x in W with $F^v_t(x) \to p$ as $t \to \infty$ is called the *core disc* of the spherical surgery, see Figure 3.6. Similarly, the set of points x with $F^v_t(x) \to p$ as $t \to -\infty$ is called the *belt disc* of the spherical surgery.[5] We will see shortly that the core and belt discs are indeed discs D^i of dimension i and D^j of dimension $j = m = 1 - i$ respectively. The boundary ∂D^i is called the *attaching sphere* of the surgery, while ∂D^j is called the *belt sphere*. The attaching sphere S^{i-1} is a sphere on the level W_0, while the belt sphere S^{j-1} is one on the level W_1.

[5] Near the critical point of index i, in Morse coordinate neighborhood, the core disc of a spherical surgery consists of the points $(x_1, ..., x_i, 0, ..., 0)$, while the belt disc consists of the points $(0, ..., 0, x_{i+1}, ..., x_{m+1})$.

All points on $W_0 \setminus S^{i-1}$ are carried by the flow $F^v_t(x)$ to points on $W_1 \setminus S^{j-1}$. The flow of the vector field $-v$ defines the inverse map showing that the manifold $W_0 \setminus S^{i-1}$ is diffeomorphic to $W_1 \setminus S^{j-1}$. In fact, we may choose neighborhoods h_i and h_j of the core and belt spheres so that the flow defines a diffeomorphism between $W_0 \setminus h_i$ and $W_1 \setminus h_j$. Thus, we determined a geometric description of a spherical surgery: up to isotopy the manifold W_1 is obtained from W_0 by replacing h_i with h_j, see Figure 3.7.

To identify h_i and h_j, we may assume that the unique critical point p of the hight function is on the level $W_{1/2}$. Since f has no other critical points, we already know that all levels W_t for $t \in [0, 1/2 - \varepsilon]$ are mutually diffeomorphic. The same is true for levels W_t with $t \in [1/2 + \varepsilon, 1]$. In fact, we may remove $f^{-1}[0, 1/2 - \varepsilon)$ and $f^{-1}(1/2 + \varepsilon, 1]$ from the cobordism W without changing its diffeomorphism type. To simplify notation, we will denote the parts of the core and the cocore discs D^i and D^j remaining in W by the same symbols, and call their boundaries the attaching and belt spheres.

The manifolds $W_{1/2+\varepsilon}$ and $W_{1/2-\varepsilon}$ in a neighborhood of the critical point p are depicted in Figure 3.8. The integral curves of the flow F_t are perpendicular to the level sets W_t, see Figure 3.9. In particular, the core disc D^i is a disc in the plane $(x_1, ..., x_i, 0, ..., 0)$, while the cocore disc D^j is a disc in the plane $(0, ..., 0, x_{i+1}, ..., x_{m+1})$, see Figure 3.8. The neighborhood h_i of the attaching sphere S^{i-1} in $W_{1/2-\varepsilon}$ is diffeomorphic to $S^{i-1} \times D^j$. The rest of $W_{1/2-\varepsilon}$ flows under F_t to the complement in $W_{1/2+\varepsilon}$ to a neighborhood h_j of the belt sphere S^{j-1}. The neighborhood h_j is diffeomorphic to $D^i \times S^{j-1}$, and $W_{1/2+\varepsilon}$ is obtained from $W_{1/2-\varepsilon}$ by removing h_i, transferring the remaining part by F_t to the level $1/2 + \varepsilon$, and, finally, attaching h_j.

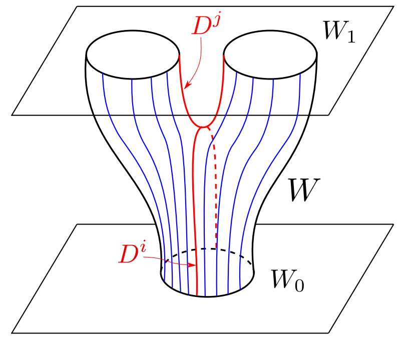


Figure 3.6: The belt disc D^j is above the critical point (red), the core disc D^i is below the critical point (red). The traces of the flow F outside the belt and core disc (blue) run from W_0 to W_1.

Example 3.4. There is a spherical surgery of index 1 on a sphere S^2 that results in a torus, see Figure 3.10. A spherical surgery of index 1 removes a neighborhood $h_1 = S^0 \times D^2$ of an attaching sphere S^0, and then attaches a handle $h_2 = D^1 \times S^1$, which is a cylinder. If we turn the corresponding cobordism W upside down, then we obtain a cobordism corresponding to a spherical surgery turning a torus into a sphere S^2, see Figure 3.11. Now the spherical surgery is of index 2. It removes a neighborhood $h_2 = S^1 \times D^2$ of an attaching sphere S^1, and then attaches a handle $h_1 = D^2 \times S^0$.

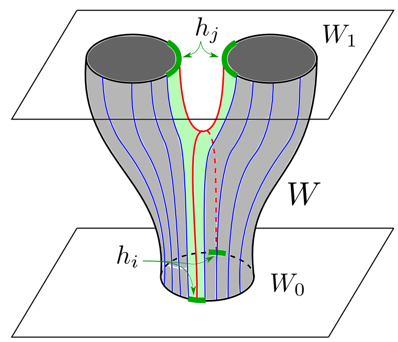


Figure 3.7: The handles h_i and h_j are in green. The flow F carries $W_0 \setminus h_i$ to $W_1 \setminus h_j$.

Conversely, let W_0 be a manifold of dimension m in $\mathbb{R}^{m+k}$, and let D^i be a disc of dimension i in $\mathbb{R}^{m+k}$ together with j linearly independent perpendicular vector fields $v_1, ..., v_j$ over D^i such that ∂D^i is a sphere in W_0 while all vector fields $v_\ell | \partial D^i$ are tangent to W_0. Suppose that D^i intersects the manifold W_0 only along ∂D^i. Then, we say that the disc D^i together with the j perpendicular vector fields is a *base of spherical surgery*. We call D^i the *core disc*.

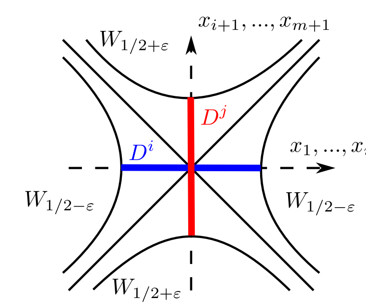


Figure 3.8: The manifolds $W_{1/2-\varepsilon}$ and $W_{1/2+\varepsilon}$.

In view of the exponential map, the disc D^i together with the perpendicular vector fields v_ℓ are often replaced with a so-called embedded i-handle $H^i \subset \mathbb{R}^{m+k}$ which is a thickening $D^i \times D^j$ of the core disc D^i in $\mathbb{R}^{m+k}$, see Figure 3.12. In fact, given a base of surgery $(D_i, v_1, ..., v_j)$, we may choose an embedding $\psi \colon D^i \times D^j \to \mathbb{R}^{m+k}$ of the i-handle H^i so that $\psi(D^i \times \{0\})$ is the core disc, the i-handle H^i intersects W_0 along $\partial D^i \times D^j$, and $\psi(\{0\} \times e_i) = \varepsilon v_i$ for some small real number ε and every basis vector e_i in the standard unit disc $D^j \subset \mathbb{R}^j$.

It can be shown that in the presence of a base of spherical surgery, there exists a spherical surgery of W_0 to a manifold W_1 such that h_i

is given by $\partial H^i \cap W_0$, while W_1 is obtained from $W_0 \setminus h_i$ by taking its union with $h_j = \partial H^i \setminus h_i$ and smoothing the corners. To construct the corresponding spherical cobordism W, we begin with a collar neighborhood $W_0 \times [0, 1/2]$ in $\mathbb{R}^{m+k} \times [0, 1/2]$, and attach to it a copy of H^i in $\mathbb{R}^{m+k} \times \{1/2\}$, see Figure 3.13. The interior of the obtained space is a manifold with corners. Its boundary is a copy of a manifold with corners W_1. Finally, we attach to the constructed space the final part $W_1 \times [1/2, 1]$ in $\mathbb{R}^{m+k} \times [1/2, 1]$. The resulting cobordism

$$W = (W_0 \times [0, 1/2]) \cup H_i \cup (W_1 \times [1/2, 1])$$

is a so-called topological manifold. We can smooth it, and turn W into a smooth cobordism between smooth manifolds W_0 and W_1.

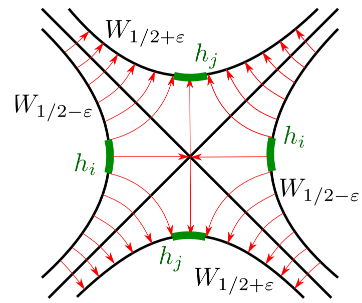


Figure 3.9: The curves of the flow F_t.

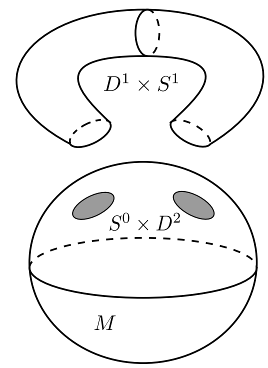


Figure 3.10: A surgery turning a 2-sphere into a torus.

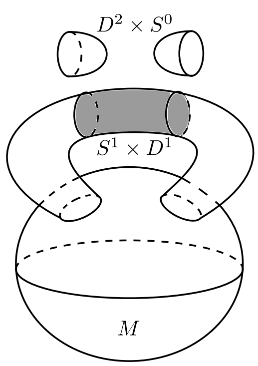


Figure 3.11: A surgery turning a torus into a 2-sphere.

Example 3.5. The spherical surgery in Figure 3.10 corresponds to the base of surgery in Figure 3.14. Note that the base of surgery of index n consists of a disc D^n of dimension n together with $\dim W_0 - \dim \partial D^n$ perpendicular vector fields along D^n. The base of surgery of index 1 in Figure 3.14 consists of the disc of dimension 1 as well as 2 perpendicular vector fields. In Figure 3.15, the surgery is of index 2. Therefore the base of surgery consists of a disc of dimension 2 as well as 1 normal vector field.

3.2 *Framed base of surgery*

In the previous section we have seen how a base of surgery defines a spherical surgery. However, we are still to answer the question, "Given a base of surgery on a framed manifold, does it correspond to a framed surgery?" Recall that we are interested in framed surgeries rather than surgeries as the Pontryagin construction identifies homotopies of maps of spheres with framed cobordisms rather than cobordisms.

We will see that the answer to the question is negative in general: in order to define a framed surgery, we need a base of framed surgery rather than a base of surgery.

We will say that $\mathbb{R}^{m+k} = \mathbb{R}^{m+k} \times \{0\}$ is the horizontal space in $\mathbb{R}^{m+k+1}$ while the last factor in $\mathbb{R}^{m+k+1}$ will be referred to as the vertical space. Let W_0 be a manifold of dimension m in the horizontal space $\mathbb{R}^{m+k} \times \{0\}$ framed by vector fields $\tau_1, ..., \tau_k$. Let D^i be a disc in $\mathbb{R}^{m+k} \times [0, 1)$ which bounds a sphere ∂D^i in W_0 in such a way that near the boundary ∂D^i the disc D^i is a vertical cylinder $\partial D^i \times [0, \varepsilon)$ in $\mathbb{R}^{m+k} \times [0, \varepsilon)$. Let $\tau_{k+1}, ..., \tau_{k+i}$ be perpendicular vector fields over D^i. Suppose that the vector fields $\tau_1, ..., \tau_k$ over $\partial D^i \subset W_0$ extend to perpendicular vector fields $\tau_1, ..., \tau_k$ over D^i. Furthermore, suppose that the projection of the disc D^i together with vector fields $\tau_{k+1}, ..., \tau_{k+j}$ to the horizontal space $\mathbb{R}^{m+k}$ is a base of surgery. Then we say that the disc D^i together with the vector fields $\tau_1, ..., \tau_{k+j}$ is a *base of framed surgery*.[6]

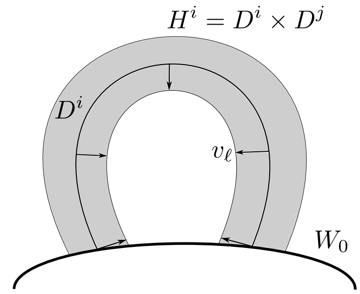


Figure 3.12: A base of surgery $(D^i; \{v_l\})$, as well as its replacement H^i.

Let $\pi\colon \mathbb{R}^{m+k+1} \to \mathbb{R}^{m+k}$ be the projection. Then a base of framed surgery $\mathbf{D} = (D^i, \tau_1, ..., \tau_{k+j})$ is a lift of the base of surgery $\pi\mathbf{D} = (\pi D^i, \pi\tau_{k+1}, ..., \pi\tau_{k+j})$. The surgery and cobordism associated with the base of framed surgery $\mathbf{D}$ is constructed as the surgery and cobordism W associated with the base of surgery $\pi\mathbf{D}$, see Figure 3.13. Furthermore, the cobordism W is framed as the vector fields $\tau_1, ..., \tau_k$ over W_0 extend to perpendicular vector over W in, essentially, a canonical way. Namely, we first extend $\tau_1, ..., \tau_k$ over a collar of W_0 in W by translation. Then we extend $\tau_1, ..., \tau_k$ over the handle H_i as prescribed by the base of framed surgery, and finally we complete the construction by observing that there is essentially a unique extension of the vector fields $\tau_1, ..., \tau_k$ over $W_1 \times [1/2, 1]$.

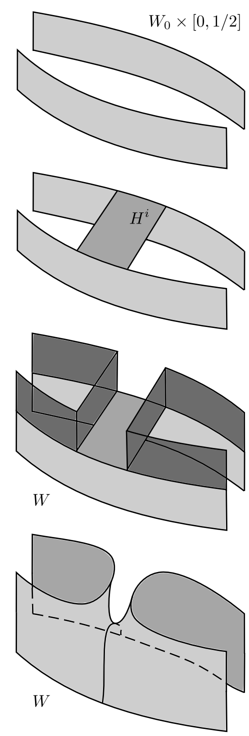


Figure 3.13: Construction of the spherical cobordism W: Take a collar neighborhood $W_0 \times [0, 1/2]$. Attach H^i. Attach $W_1 \times [1/2, 1]$. Smooth the corners.

3.3 *Reduction of homotopy groups*

Recall that an element in the homotopy group $\pi_{m+k}S^k$ is represented by a framed manifold. Furthermore, we have seen that two framed manifolds represent the same element in $\pi_{m+k}S^k$ if and only if one framed manifold can be obtained from the other one by performing finitely many spherical surgeries. In this section we will use spherical surgeries to simplify a given framed manifold as much as possible.

To begin with, if a given framed manifold M representing an element $x \in \pi_{m+k}S^k$ is not path connected, we may pick points p and q on different components in M and then perform a framed spherical surgery of index 1 along the attaching sphere $S^0 = \{p, q\}$. The resulting framed manifold still represents the same element x, but has less path components than the initial manifold M. After finitely many surgeries of index 1, the framed manifold M is path connected. If it is not simply connected, then there is a circle γ representing a non-trivial element in $\pi_1 M$. We will see that if a surgery of index 2 along γ exists, then it reduces the fundamental group of M, see Figure 3.15. After the fundamental group of M is reduced to zero, we move to perform surgeries of index 3 along spheres in M representing non-trivial elements in $\pi_2 M$; and proceed by induction.

We need to make sure that a surgery over a given representative of $\pi_n M$ exists, and that an appropriate surgery indeed reduces the group $\pi_n M$. Lemma 11.4 shows that a spherical surgery with attaching sphere ∂D^i representing an element $x \in \pi_n W_0$ kills the element x and does not create new elements in lower dimensional homotopy groups $\pi_i W_0$.

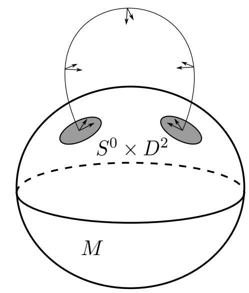


Figure 3.14: The base of surgery in Figure 3.10.

Lemma 3.6. *Let W be a spherical cobordism between W_0 and W_1 of index $n+1 \leq m/2$, where m is the dimension of the manifolds W_0 and W_1. Suppose that the attaching sphere of the corresponding surgery represents an element $x \in \pi_n W_0$. Then $\pi_i W_0 = \pi_i W_1$ for $i < n$, while $\pi_n W_1$ is a factor group of $\pi_n W_0$ by a subgroup containing x.*

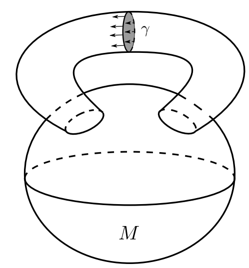


Figure 3.15: The base of surgery in Figure 3.11. The curve γ represents a nontrivial element in $\pi_1 M$. However, after the surgery, the manifold is simply connected.

Proof. Let S_b denote the belt sphere of the surgery; it is of dimension $m-n-1$.[7] Note that the complement in W_1 to the belt sphere can be deformed into $W_0 \setminus h_i \simeq W_1 \setminus h_j$, see Figure 3.16. An element of $\pi_i W_1$ with $i \leq n$ can be represented by a sphere Σ in W_1 of dimension i. Since $\dim \Sigma + \dim S_b < \dim W_1$, we may choose Σ so that it does not intersect S_b and, therefore, can be deformed to $W_0 \setminus h_i \subset W_0$, see Figure 3.17. When $i < n$, we may deform to W_0 not only representatives of $\pi_i W_1$, but also their homotopies. Therefore, in this range there is a well-defined homomorphism $\psi \colon \pi_i W_1 \to \pi_i W_0$.

Similarly, the attaching sphere S_a in W_0 is of dimension n. Therefore, every sphere representing an element in $\pi_i W_0$ with $i \leq n+1$ can be deformed into one in the complement to S_a, and, then, to W_1. Such a deformation is unique up to homotopy if $i \leq n$. Thus, in the range $i \leq n$, there is a well-defined homomorphism $\varphi \colon \pi_i W_0 \to \pi_i W_1$.

When $i < n$ both homomorphisms, φ and ψ are well-defined. Since $\psi = \varphi^{-1}$, the homomorphism φ is an isomorphism when $i < n$. When $i = n$, we may still deform any representative of $\pi_i W_1$ to W_0, though (possibly) not in a unique way. Therefore, in this case, the homomorphism φ is surjective. The element x is clearly in the kernel of φ. □

3.4 *Examples: Stable homotopy groups π_1^S and π_2^S*

Stable homotopy group π_1^S.

In this section we will use spherical cobordisms (as well as the Compression Theorem) to calculate π_1^S. In fact, we will show that π_1^S is

[6] We assume that the projection of a disc D^i with vector fields $\tau_{k+1}, ..., \tau_{k+j}$ is a base of surgery. To motivate this assumption, suppose that W_0 consists of two round circles of radius 1 on $\mathbb{R}^2$ centered at $(-2,0)$ and $(2,0)$, framed by a vector field τ_1 which is outward normal over one circle and inward normal over the second one. There is a disc D^1 in $\mathbb{R}^3$ together with vector fields τ_1, τ_2 which satisfy all requirements of a base of framed surgery except that the disc D^1 together with the vector field τ_2 do not project to a base of surgery. In this case the disc D^1 together with vector fields τ_1, τ_2 only define a framed immersed cobordism.

[7] Note that the dimension of the cobordism W is $m+1$. Since the core disc D^i is of dimension $n+1$, the belt disc D^j is of dimension $(m+1)-(n+1) = m-n$. The dimension of the belt sphere $S_b = \partial D^j$ is one less than $m-n$.

isomorphic to $\mathbb{Z}_2$.

By the Pontryagin construction, the group π_1^S can be identified with the cobordism group of framed manifolds of dimension 1 in $\mathbb{R}^4$. By the Compression theorem, the latter group is isomorphic to the cobordism group of oriented immersed 1-manifolds L in $\mathbb{R}^2$. We define a map $\psi\colon \pi_1^S \to \mathbb{Z}_2$ by associating with $[L]$ the number of self-intersection points of L mod 2. For example, the curve in Figure 3.18 has 4 self-intersection points, and, therefore, the value of ψ on its cobordism class is zero.

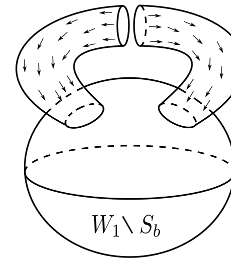


Figure 3.16: A compression of $W_1 \setminus S_b$ to $W_0 \setminus h_i$ for the cobordism in Figure 3.11.

We claim that the homomorphism ψ is well-defined, i.e., if W_0 and W_1 are two cobordant oriented immersed 1-manifolds in $\mathbb{R}^2$, then the values of ψ on the cobordism classes $[W_0]$ and $[W_1]$ are the same. Indeed, suppose that W is an immersed cobordism in $\mathbb{R}^2 \times [0,1]$ between W_0 and W_1. We may perturb slightly the surface W so that the set γ of its self-intersection points is a union of immersed curves. Indeed, a generic immersed surface W in the 3-dimensional space $\mathbb{R}^2 \times [0,1]$ may have double and triple points. In a neighborhood of a double point (respectively, a triple point), in an appropriate coordinate chart the immersed surface W appears as an intersection of two (respectively, three) coordinate planes, see Figure 3.19. Therefore, in both cases, the set of self-intersection points of W in the chart consists of immersed curves.

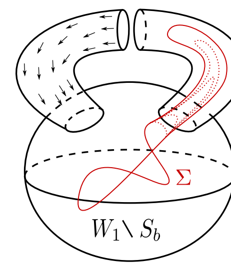


Figure 3.17: A sphere Σ in $W_1 \setminus S_b$ and its deformation to $W_0 \setminus h_i$.

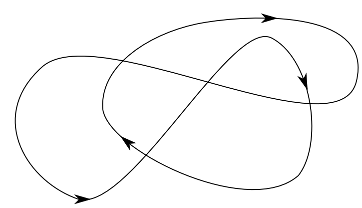


Figure 3.18: A curve with $\psi = 0$ self-intersection points mod 2.

The coordinate plane description of double and triple points of W also shows that the end points of the curves γ lie on W_0 and W_1. In fact, the set $\partial\gamma$ is precisely the set of double points of W_0 and W_1. Since the number of end points of γ is even, we conclude that the parities of the numbers of double points of W_0 and W_1 agree.

Since taking the disjoint union of two oriented immersed curves results in adding the number of double points, the map ψ is a homomorphism. In view of the figure "8" immersed curve, the homomorphism ψ is surjective.

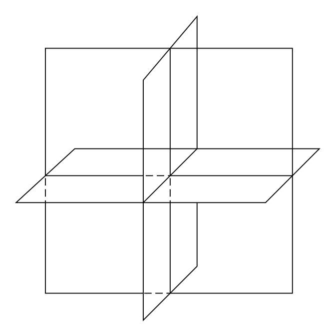


Figure 3.19: The immersed surface W in a neighborhood of a triple point.

Let us show that ψ is injective. To this end, suppose that L is an oriented immersed 1-manifold in $\mathbb{R}^2$ with even number of double points so that $[L]$ belongs to the kernel of the homomorphism ψ. We claim that L is cobordant to an empty manifold. Indeed, suppose L has n double points. Given a double point p of L we may perform two spherical surgeries in a neighborhood of p to split off a figure "8" without increasing the total number of self intersection points, see Figure 3.20. In such a way we may split off n figure "8" curves so that the curve that is left has no self-intersection points, see Figure 3.21.

It is known that a closed curve in $\mathbb{R}^2$ with no self-intersection points bounds a disc. Thus, we have shown that L is cobordant to the disjoint union of n figure "8" curves. Since n is even and a pair of figure "8" curves is cobordant to zero, we conclude that L is cobordant to zero, and, therefore, the homomorphism ψ is an isomoprhism $\pi_1^S \simeq \mathbb{Z}_2$.

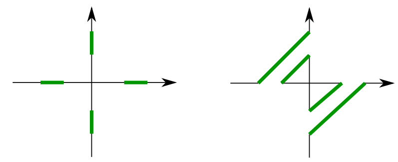


Figure 3.20: Two spherical cobordisms of index 1. Each cobordism of index 1 removes a pair of green segments $S^0 \times D^1$ and then attaches a pair of green segments $D^1 \times S^0$.

Unstable homotopy group $\pi_3 S^2$.

The Pontryagin construction identifies an element in $\pi_3 S^2$ with a cobordism class of closed framed manifold of dimension 1 in $\mathbb{R}^3$, i.e., with a framed link. By the Compression theorem, a framed link admits an isotopy to a framed link with the second (last) normal vector field being vertically up. We note that there exists a unique orientation of the link such that at each point of the link the orienting vector followed by the two vectors of the frame produces a positive basis of $\mathbb{R}^3$.

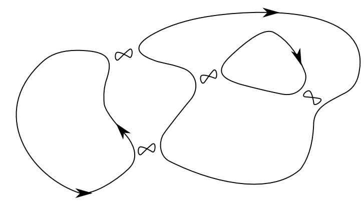


Figure 3.21: A curve obtained from the one in Figure 3.18 after splitting off 4 figure "8" curves.

We may assume that the projection π of the framed link L to the horizontal plane $\mathbb{R}^2 \subset \mathbb{R}^3$ is an immersion with no triple point. To each double point we assign a sign according to the sign convention of Figure 3.22. Define a map $\varphi\colon \pi_3 S^2 \to \mathbb{Z}$ by associating with the framed link the algebraic number of the double points of its projection, i.e., the number of double points counted with signs. We may also define $\varphi\colon \pi_3 S^2 \to \pi_2 S^2$ by associating with the cobordism class of L the cobordism class of the framed manifold in $\mathbb{R}^2$ of double points $D(\pi L)$ of πL, where a double point p of πL is framed by the orientation vector of the upper strand of πL passing through p followed by the orientation vector of the lower strand of πL passing through p.

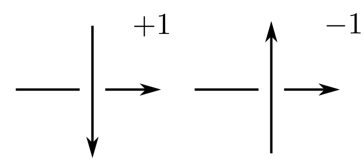


Figure 3.22: The sign convention.

To begin with let us show that the map φ is well-defined[8]. Suppose that W is a cobordism in $\mathbb{R}^3 \times [0,1]$ between two framed links W_0 and W_1 such that the frames of W_0 in $\mathbb{R}^3 \times \{0\}$ and W_1 in $\mathbb{R}^3 \times \{1\}$ are vertically up. Then by the (relative version of the) Compression Theorem we may assume that the second vector field of the frame over W coincides with e_3. Then the projection of W to $\mathbb{R}^2 \times [0,1]$ along the direction e_3 is an immersion, which we still denote by π. Again, we may assume that the set $D(\pi W)$ of points in πW with multiple preimages is a framed manifold of dimension 1 in $\mathbb{R}^2 \times [0,1]$. It defines a framed cobordism of $D(\pi L)$. Therefore $\varphi\colon \pi_3 S^2 \to \pi_2 S^2$ is well-defined.

[8] We need to show that if W_0 and W_1 are two cobordant framed links, then the framed manifolds of double points of πW_0 and πW_1 are also cobordant

The map φ is a homomorphism since the algebraic number of self-intersection points of a union of two projected links is the sum of the algebraic numbers of the self-intersection points of the links. Suppose that a framed link L represents an element in the kernel of φ. Again, by applying a spherical cobordism finitely many times, we may split off from πL a disjoint union of n positive figure "8" curves, and n negative figure "8" curves. Therefore, L is cobordant to an empty link.

Finally, the homomorphism φ is clearly surjective, which completes the proof that $\pi_3 S^2$ is isomorphic to $\mathbb{Z}$.

Stable homotopy group π_2^S

We have seen that every element of π_2^S can be represented by a framed surface in $\mathbb{R}^{2+k}$ with $k > 3$. Choose a framed surface M of minimal genus representing a given element of π_2^S. We claim, then, that M is either a sphere or projective plane.

Indeed, if the genus of the surface M is g, then it can be constructed from a sphere by attaching g handles. Let $\alpha_1, ..., \alpha_g$ denote the belt spheres of the handles, while $\beta_1, ..., \beta_g$ denote closed simple curves in M that have the properties that β_i intersects α_i at a unique point at which β_i and α_i are not tangent and β_i does not intersect any other curves β_j and α_j.

Choose an orientation of each curve γ in the set $\{\alpha_i, \beta_i\}$. The curve γ comes with k normal vector fields inherited from the frame of M, as well as a unique unit vector field tangent to M that at each point $x \in \gamma$ is directed to the right with respect to the orientation of γ. Thus the curve γ itself represents an element $[\gamma]$ in the group $\pi_1^S \simeq \mathbb{Z}_2$. If $[\gamma] = 0$, then there is a framed disc D in $\mathbb{R}^{2+k}$ bounded by $\gamma \in \mathbb{R}^{2+k}$. Since $\dim M + \dim D < \dim \mathbb{R}^{2+k}$, we may assume that the interior of the disc D does not contain points of M. Then, the framed disc D is a core of a spherical surgery on M that decreases the genus of M. Since we assume that M is of minimal genus, each class $[\alpha_i]$ and $[\beta_i]$ should be non-trivial in π_1^S.

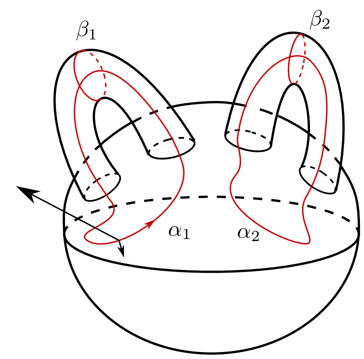


Figure 3.23: The curves α_i, β_i. The curve α_1 comes with k normal vector fields inherited from the frame of M, as well as a unique unit vector field tangent to M directed to the right with respect to a chosen orientation of α_1.

Suppose that M is of genus $g > 1$. Choose a path from a point in α_1 to a point in α_2. Let $\gamma \subset M$ be an embedded closed curve that travels along α_1, passes to α_2 along the chosen path, traverses α_2 and then comes back to α_1 along a path parallel to the chosen one. We may choose γ to be embedded by changing, if necessary, the direction in which γ traverses α_2. Since $[\gamma] = [\alpha_1] \pm [\alpha_2]$ is the zero element in $\pi_1^S \simeq \mathbb{Z}_2$, there is a spherical surgery on M—along the attaching sphere γ—that reduces the genus of M. Thus, we conclude that the genus of M is at most $g = 1$, i.e., the surface M is either a sphere or torus.

Suppose that M is a sphere. We may use the multicompression theorem to compress M to an immersed sphere in $\mathbb{R}^3$. The proof of the Smale paradox shows that any two immersed spheres in M are regularly homotopic. Therefore we may assume that M is an embedded standard sphere in $\mathbb{R}^3$. It clearly bounds a framed disc in $\mathbb{R}^3 \times [0,1]$ and therefore M represents a trivial element in π_2^S.

Suppose now that M is a torus. Again, by the Compression Theorem, we may assume that M is immersed into $\mathbb{R}^3$.

Let's determine the number of different immersions of a torus into $\mathbb{R}^3$ up to regular homotopy. By the Smale-Hirsch theorem, immersing M into $\mathbb{R}^3$ is equivalent to constructing a family of injective homomorphisms $T_xM \to \mathbb{R}^3$ parametrized by points x in M. In view of the canonical basis of T_xM given by the unit vectors directed along the parallel and meridian respectively, we may canonically identify each vector space T_xM with $\mathbb{R}^2$. Thus, a formal immersion, is a map that associates with each $x \in M$ an injective homomorphism $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^3$. Notice that φ extends to an isomorphism $\mathbb{R}^3 \to \mathbb{R}^3$ by $\varphi(e_3) = \varphi(e_1) \times \varphi(e_2)$ where $\times$ stands for the vector product. Finally, the correspondence $x \mapsto \varphi$ defines a map $F\colon M \to GL_3(\mathbb{R})$ with image in the path component of $GL_3(\mathbb{R})$ of matrices with positive determinant. Furthermore, by the Gram-Schmidt orthonormalizing process, we may assume that the image of F is in subgroup SO_3 of orthogonal matrices. However, up to homotopy, there is only one non-constant map of a torus to SO_3.

Exercise 3.7. Show that up to regular homotopy there is only one class of immersions of a torus to $\mathbb{R}^3$ with $[\alpha_1] = [\beta_1] \neq 0$. [9]

We conclude that the group π_2^S is either trivial or $\mathbb{Z}_2$. We will see later that the group π_2^S is $\mathbb{Z}_2$.[10]

3.5 *Further reading*

The study of critical points of a function of n variables was initiated by Morse in **Relations between the critical points of a real function of n independent variables** [Mo25]. Both, the Morse lemma classifying generic critical points and Lemma 3.1 are proved in [Mo25]. Handle decompositions of cobordisms proved to be very useful in the

[9] Suppose that there are two non-homotopic formal immersions $F, G\colon M \to SO_3$. Define a map $FG^{-1}\colon M \to SO_3$ by sending x to $F(x)G^{-1}(x)$. Then FG^{-1} does not deform to the constant map. On the other hand, the map FG^{-1} is a constant map over the meridian and the parallel of the torus. Therefore it admits a lifting with respect to a double covering $S^3 \to SO_3$. Of course, any map from a torus to S^3 can be deformed to a constant map. Thus F deforms to G. It is also easy to see that a formal immersion F with $[\alpha_1] = [\beta_1] \neq 0$ exists.

[10] Alternatively, in order to prove that π_2^S is non-trivial, we can use the Pontryagin invariant that associates with a framed (not necessarily connected) manifold M the number $[\alpha_1]\cdot[\beta_1] + \cdots + [\alpha_g]\cdot[\beta_g]$ in $\mathbb{Z}_2$.

Smale's proof of **The generalized Poincaré conjecture in higher dimensions** [Sm60]. Worth mentioning is also the development of the spherical cobordisms in the paper **A procedure for killing homotopy groups of differentiable manifolds** [Mi61] by Milnor, and in the papers **Modifications and cobounding manifolds** [Wa60] and **A geometric method in differential topology** [Wa62] by Wallace.

A highly recommended book on cobordisms is the classic **Lectures on the h-cobordism theorem** by Milnor [Mi65]. The computation of π_1^S and $\pi_3 S^2$ involving an isomorphism $\pi_3 S^2 \to \pi_2 S^2$ that we presented here are different from how Pontryagin computed these groups. The present computation of π_2^S follows the Pontryagin's line of reasoning.

In presenting a smoothing technique, we followed the line of reasoning by Cairns in **Homeomorphisms between topological manifolds and analytic manifolds** [Ca40] and J. H. Whitehead in **Manifolds with transverse fields in Euclidean space** [Wh61], presented in elementary terms by Pugh in **Smoothing a topological manifold** [Pu02]. Given a topological manifold M of dimension m in $\mathbb{R}^{m+k}$, the smoothing procedure relies on the existence of a tubular neighborhood U of M, which consists of disjoint discs D_x of dimension k. Each disc D_x is linear in the sense that it belongs to a k-dimensional affine subspace of $\mathbb{R}^{m+k}$ and intersects M at a unique point x. If fact, the discs D_x are required to be transverse to M, and, in particular, in a neighborhood of each point the manifold M is the graph of a Lipschitz function. For topological manifolds that are not locally Lipschitz, the field of transverse linear discs D_x does not exist, but there still may exist a field of non-linear discs D_x whose union is a topological vector bundle neighborhood of M. It turns out that if $\dim M \geq 5$, then there is a C^0-isotopy of M to a smooth submanifold in $\mathbb{R}^{m+k}$ if and only if M has a topological vector bundle neighborhood. This is the Kirby-Siebenmann smoothing theorem presented in **Foundational Essays on Topological Manifolds, Smoothings and Triangulations** [KS77].

4

The Whitney trick

4.1 The Whitney weak embedding and immersion theorems

We say that a smooth map $f : M \to \mathbb{R}^n$ is an *immersion* if the differential $d_x f$ is injective at each point x. An immersion f is an *embedding* if the map f is a homeomorphism onto image.

Figure 4.1: Hassler Whitney, 1907-1989

One of the first questions that arose in the Immersion Theory was to determine for a given manifold M the minimal dimension of the Euclidean space that admits an embedding/immersion of M. The first results in this direction are due to Hassler Whitney.

Theorem 4.1 (The Whitney weak embedding theorem)**.** *Every closed manifold of dimension m admits an embedding to $\mathbb{R}^{2m+1}$.*

In order to prove the Whitney weak embedding theorem, we will use the Sard theorem asserting that for any smooth map of a manifold of dimension n into a sphere of dimension at least $n+1$, the image of f is of measure zero. The manifold under consideration is STM.

Exercise 4.2. Let STM denote the subset of the space $\mathbb{R}^{m+k} \times \mathbb{R}^{m+k}$ that consists of pars (x, v) of points $x \in M$ and unit tangent vectors $v \in T_x M$. Show that STM is a manifold.

Proof of the Whitney weak embedding theorem. Let M be a manifold of dimension m in $\mathbb{R}^{m+k}$ and $k > m$. We claim that there is a non-zero vector v in $\mathbb{R}^{m+k}$ such that no line in the direction v is tangent to M. Indeed, forgetting the first component in (x, v) defines a smooth map

$STM \to S^{m+k-1}$ of a manifold of dimension $2m-1$ into the manifold of dimension at least $2m$. Therefore by the Sard theorem, most of the directions v are not tangent to M. Choosing a copy of v at each point of M, defines a normal vector field over M. By rotating the manifold M together with the vector field v we may assume that the vector field v is vertical up.

If $k > m+1$, then as in the proof of the Freudenthal theorem, we may slightly rotate the manifold M in $\mathbb{R}^{m+k}$ so that no line in the direction v intersects M at more than one point. We may still assume that the vector field v is normal. Projecting M along $v = e_k$ to the horizontal hyperspace results in a placement of M into $\mathbb{R}^{m+k-1}$. The argument can be iterated as long as $k > m+1$ and eventually produces an embedding of M into $\mathbb{R}^{2m+1}$. □

Note that in the proof of the Whitney weak embedding theorem, when $k = m+1$ we still place M into $\mathbb{R}^{m+k}$ in such a way that the vector field $v = e_{m+k}$ is normal over M, but we may not be able to guarantee that any vertical line in the direction of v intersects M at a unique point. In this case, the projection of M along e_{m+k} to the horizontal hyperspace is an immersion. Therefore, we proved the Whitney weak immersion theorem.

Theorem 4.3 (The Whitney weak immersion theorem)**.** *Every closed manifold of dimension m admits an immersion to $\mathbb{R}^{2m}$.*

4.2 *The Whitney trick*

The Whitney trick allows one to simplify an immersion of a manifold of dimension m into $\mathbb{R}^{2m}$. To formulate the Whitney construction we will need two new notions: transversality and orientation.

Two immersions $f\colon M \to \mathbb{R}^k$ and $g\colon N \to \mathbb{R}^k$ are said to be *transverse* if for each x in M and y in N with the same image $f(x) = g(y)$, the sum of spaces $f_* T_x M$ and $g_* T_y N$ is the vector space $\mathbb{R}^k$.[1] By the Transversality theorem, any pair of immersions can be approximated by a C^∞-close pair of transverse immersions.

[1] We will use both the notation df and f_* for the differential of f.

Similarly, an immersion f is *self-transverse* if for all x and y in M with $f(x) = f(y)$, the sum of spaces $f_* T_x M$ and $f_* T_y M$ is all $\mathbb{R}^{m+k}$. By the Transversality theorem (which we will not prove here), any immersion

f admits an approximation by a close self-transverse immersion.

A self-transverse immersion is often easier to work with.

Exercise 4.4. A *double point* of an immersion f of a manifold M is a pair of distinct points x and y in M with $f(x) = f(y)$. Show that if the manifold M is closed, then the set of double points (x, y) of a self-transverse immersion f of M into $\mathbb{R}^{2m}$ is finite.[2]

[2] *Hint to Exercise 4.4.* Suppose that $f(x) = f(y)$ for two distinct points x and y. Since f is self-transverse, the affine spaces f_*T_xM and f_*T_yM are transverse. Therefore there are no points x' near x and y' near y with $f(x') = f(y')$. Consequently, the set of double points (x, y) in $M \times M$ is discrete. Compactness of M implies now that the set of double points (x, y) is finite.

Any pair of embeddings can be approximated by a C^∞-close pair of transverse embeddings. Two transverse manifolds of dimensions m and n respectively in $\mathbb{R}^k$ meet along a manifold of dimension $m + n - k$. For example two transverse manifolds of dimension m in $\mathbb{R}^{2m}$ meet along a discrete set of points.

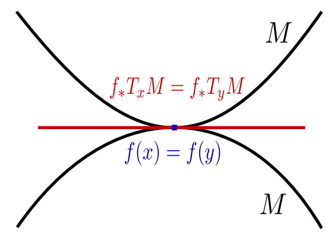


Figure 4.2: Not a self-transverse immersion.

A priori a self-transverse map f of a manifold M of dimension m into $\mathbb{R}^{2m}$ may have triple points, i.e., tuples x, y, z of distinct points in M with the same image. However, for a small closed disc neighborhood D about z, we may slightly modify the immersion f over the interior of D so that it is transverse to the set of double points of f. Such a modification eliminates the triple point. Therefore, we may always assume that an immersion of a manifold of dimension m into $\mathbb{R}^{2m}$ does not have triple points.

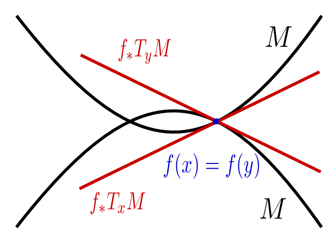


Figure 4.3: A self-transverse immersion.

Definition 4.5. An *orientation* of a manifold M of positive dimension at a point $x \in M$ is an equivalence class of bases of T_xM. Given two bases $\{e_i\}$ and $\{g_i\}$ of T_xM, we have $g_i = \sum a_{ij}e_j$. The bases are equivalent if the determinant of the matrix $\{a_{ij}\}$ is positive. Otherwise we say that $\{e_i\}$ and $\{g_i\}$ define opposite orientations. An *orientation* on M is a continuous choice of an orientation at each point of x. If M is oriented, then a basis of T_xM is said to be *positive* if it agrees with the orientation of M; otherwise it is *negative*.

We also adopt the convention that an orientation of a point is a sign. Thus, there are two orientations on a point: the positive and the negative ones.

A manifold may not admit an orientation; the projective plane is such a manifold. On the other hand, if a manifold admits an orientation, then it admits precisely two orientations.

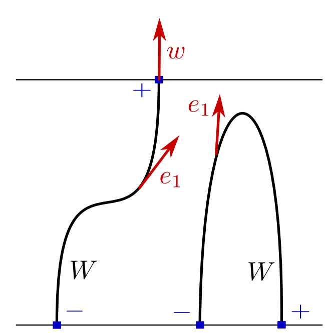


Figure 4.4: An oriented manifold with boundary and the orientation of its boundary.

Definition 4.6. Suppose $W \subset \mathbb{R}^{m+k} \times [0, 1]$ is an oriented manifold with boundary. Then the orientation on W induces an orientation on ∂W: a basis $\{e_1, ..., e_m\}$ at $x \in \partial_i W$ is positive if $\{w, e_1, ..., e_m\}$ is a positive basis at $x \in W$ with respect to the orientation on W, where w is an outward directing vector normal to $\mathbb{R}^{m+k} \times \{i\}$.[3]

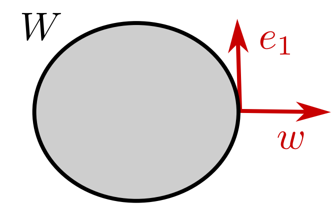


Figure 4.5: The standard orientation on a disc induces counterclockwise orientation on the circle boundary. Note that the induced orientation on the boundary of a manifold does not depend on the way the manifold with boundary is placed in $\mathbb{R}^{m+k} \times [0,1]$.

For example, an orientation on a segment essentially is a choice of an initial point p and a terminal point q. The terminal point q is positive, while the initial point is negative, see Fig. 4.4. The standard orientation of a circle in $\mathbb{R}^2$ is counterclockwise, see Fig. 4.5.

Let M and N be two oriented manifolds of dimensions m and n respectively in $\mathbb{R}^{m+n}$. At their intersection point x choose a positive basis $\{e_i\}$ of T_xM and a positive basis $\{g_i\}$ of T_xN. We say that the intersection point x of M and N is *positive* if $\{e_1, ..., e_m, g_1, ..., g_n\}$ is a positive basis of $\mathbb{R}^{m+n}$; otherwise the intersection point is said to be *negative*. Note that if both dimensions m and n are odd, then an intersection point x of M and N is positive if and only if it is negative as an intersection point of N and M. If m or n is even, then the order of M and N is not essential.

[3] **Note** that the convention on the orientation of the boundary is chosen so that it agrees with the Stocks formula $\int_{\partial W} w = \int_W dw$, with the simplicial boundary formula

$$d[a_0, ..., a_n] = (-1)^i \sum [a_0, ..., \hat{a}_i, ..., a_n],$$

and with the behaviour of fundamental classes under the boundary homomorphism $H^{m+1}(W, \partial W) \to H^m(\partial W)$.

In the presence of two intersection points p and q of opposite signs, Whitney devised an isotopy of M eliminating the pair of intersection points in $M \cap N$. Suppose that there is a simple curve L_M in M from p to q and a simple curve L_N from q to p in N. Suppose that there is a disc D in $\mathbb{R}^{m+n}$ bounded by $L_M \cup L_N$ such that the interior of D is disjoint from $M \cup N$. Then there is an isotopy that pulls a neighborhood of L_M in M along the disc D past L_N. It eliminates the pair p and q of intersection points.

Theorem 4.7 (The Whitney trick)**.** *Let M and N be compact oriented connected transversely intersecting manifolds of dimensions $m, n > 2$ in $\mathbb{R}^{m+n}$. Let p and q be a positive and negative intersection points. Then there exists an ambient isotopy h_t with $t \in [0,1]$ such that $h_0 = \mathrm{id}$ and $h_1 M \cap N = M \cap N - \{p, q\}$.*

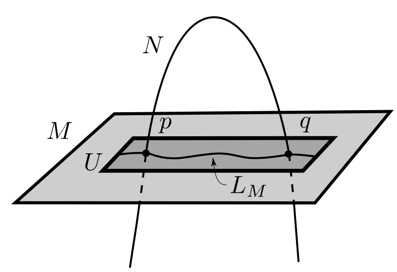


Figure 4.6: The intersection points p and q of opposite signs. There is a curve L_M in M from p to q, and a curve L_N in N from q to p.

Proof. Since the manifold M is connected, there is a curve L_M in M from p to q. Since the dimension of the manifold M is at least 3, we may assume that L_M is an embedded curve in M. Similarly, there is an embedded curve L_N in N from p to q, see Figure 4.6. The union of curves $L_M \cap L_N$ is an embedded closed curve in $\mathbb{R}^{m+n}$, which bounds a disc D of dimension 2. We may assume that the disc D is nowhere tangent to M and N. Choosing D so that its interior is transverse to M and N insures that the interior of D has no points in M and N. [4]

[4] If the interior of the disc D is transverse to, say, M, then the intersection set has dimension $\dim D + \dim M - \dim \mathbb{R}^{m+n} = 2 - n < 0$.

We aim to push a neighborhood U of the curve L_M in the manifold

M along the disc D past L_N. We will first push the manifold M along the disc D randomly to investigate the obstruction to the existence of a Whitney isotopy. Then we will carefully construct the Whitney isotopy from scratch.

The trace of the flow of U under an isotopy along D is a thickening H of D. In order to describe H, extend the curve L_M slightly past its endpoints so that p and q are interior points in L_M, and then pick vector fields $v_1, ..., v_{m-1}$ over extended L_M such that at each point $x \in L_M$ the vectors $v_1(x), ..., v_{m-1}(x)$ form an orthonormal basis of the subspace in T_xM perpendicular to T_xL_M. Then extend the disc D so that D intersects M only along its boundary and $D \cap M = L_M$. Furthermore, $\partial D \setminus L_M$ is a curve in the complement to M and N. Next we extend the vector fields $v_1, ..., v_{m-1}$ to perpendicular vector fields over the disc D. There exists a thickening $H = D \times D^{m-1}_\varepsilon$ of the disc D in $\mathbb{R}^{m+n}$ prescribed by the vectors $v_1, ..., v_{m-1}$ in the sense that for each point $x \in D$, the coordinate vectors $\{0\} \times \{e_i\}$ in the tangent space to $\{x\} \times D^{m-1}_\varepsilon$ coincide with the corresponding vectors v_i at x, and such that the subset $L_M \times D^{m-1}_\varepsilon$ of H coincides with U, see Figure 4.7 and Lemma **??**.

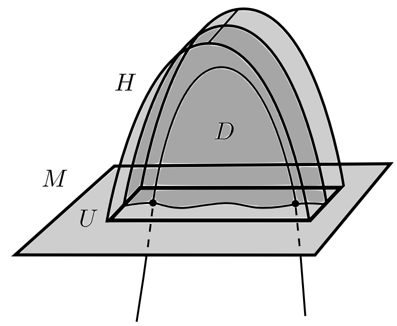


Figure 4.7: The union of curves $L_M \cup L_N$ bounds a disc D. Its thickening is denoted by H.

The boundary of the thickening H of D consists of U and $\partial H \setminus U$. We may push $\overline{U}$ relative its boundary through H to the position $\partial H \setminus U$. We can slightly change the pushing so that it is smooth and defines an isotopy. Then the result M_1 of the isotopy of M is obtained from the manifold M by replacing U with $\partial H \setminus U$ and smoothing the corners. We need to make sure that $M_1 \cap N = M \cap N - \{p, q\}$. Clearly, all intersection points in $M \cap N$ except for p and q remain in $M_1 \cap N$. Furthermore, the points p and q are no more in the intersection $M_1 \cap N$. Thus we only need to make sure that the new part $\partial H \setminus U$ has no points in N. If the thickening $H = D \times D^{m-1}_\varepsilon$ of D is thin enough, meaning that the radius ε of the disc D^{m-1}_ε is small enough, then the only common points of $\partial H \setminus U$ and N could be in a neighborhood of $D \cap N = L_N$. On the other hand, if we can choose the vector fields $v_1, ..., v_{m-1}$ over D so that their restrictions to L_N are perpendicular to N, then clearly $\partial H \cap U$ would have no common points with N in a neighborhood of L_N.

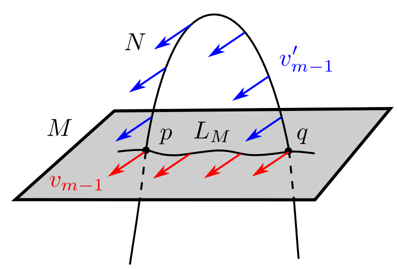


Figure 4.8: The vector field v'_{m-1} may agree with the vector field v_{m-1}.

Let's construct the vector fields v_i with required properties from scratch. To begin with, we define the vector fields v_i over L_M. Next we aim to extend them over L_N in such a way that for each point $x \in L_N$, the vectors $v_1(x), ...v_{m-1}(x)$ form a basis for the subspace of dimension $m-1$ in $T_x\mathbb{R}^{m+n}$ that is perpendicular to the space $T_xN \oplus T_xD$ of dimension $m+1$. We can easily extend vector fields $v_1, ..., v_{m-2}$ by induction, but

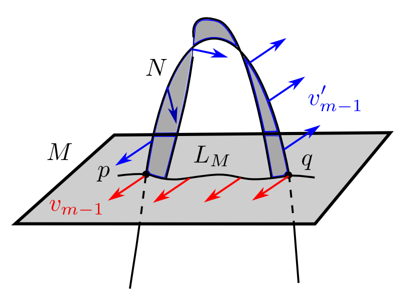


Figure 4.9: The vector field v'_{m-1} may not agree with the vector field v_{m-1}. The dark strip in the Figure is a part of the disc D.

when it comes to extending the last vector field v_{m-1} a bit more care is necessary. Let v'_{m-1} denote the unique unit vector field over L_N perpendicular to N, to D, and to vector fields $v_1, ..., v_{m-2}$ and such that $v'_{m-1}(p) = v_{m-1}(p)$. Then $v'_{m-1}(q) = \pm v_{m-1}(q)$, see Figures 4.8 and 4.9. Since by the hypothesis in the theorem the signs of p and q are opposite, we deduce that $v'_{m-1}(q) = v_{m-1}(q)$, and therefore, we may extend the vector field v_{m-1} over L_N by v'_{m-1}.

Exercise 4.8. Show that $v'_{m-1}(q) = v_{m-1}(q)$.

Finally the vector fields $v_1, ..., v_{m-1}$ can be extended to perpendicular vector fields over D. [5] □

[5] Here is a sketch of a proof that any normal linear independent vector fields $v_1, ..., v_{m-1}$ over $\partial D \subset \mathbb{R}^{m+n}$ admit an extension over D. To begin with we note that if there is an isotopy that brings D to a disc D' and vector fields v_i to vector fields v'_i over $\partial D'$, then it suffices to extend vector fields v'_i over D' since we can then use the inverse isotopy to bring the extended vector fields back to D. Thus we can freely change D and the vector fields v_i by isotopy. In particular, we can apply the inductive argument in the Freudenthal theorem. First, by general position we can extend v_i over D, use the Compression Theorem to straighten v_i up, and compress D to $\mathbb{R}^{m+n-1}$. Then we continue by induction to extend and straighten all other vector fields.

Exercise 4.9. Let M and N be closed oriented connected immersed manifolds of dimensions $m > 2$ and $n > 1$ in $\mathbb{R}^{m+n}$. Suppose that M and N intersect transversally. Let p and q be intersection points in $M \cap N$. Then there is a regular homotopy h_t with $t \in [0, 1]$ of M such that $h_0 = \text{id}$ and $h_1 M \cap N = M \cap N - \{p, q\}$.

4.3 *Strong Whitney embedding theorem*

We have seen that every closed manifold M in $\mathbb{R}^{m+k}$ is isotopic to a manifold in $\mathbb{R}^{2m+1}$. Furthermore, it admits a compression to $\mathbb{R}^{2m}$ into an immersed manifold with finitely many double points. In this section we will show that the manifold M admits an embedding into $\mathbb{R}^{2m}$. We will begin with an immersed manifold M in $\mathbb{R}^{2m}$. Near each of the double points of M we will construct an immersed sphere with a unique double point. Using a surgery we will merge the spheres with M without changing the diffeomorphism type of M. Then by the Whitney trick, each of the original double points can be eliminated with a double point in one of the merged spheres.

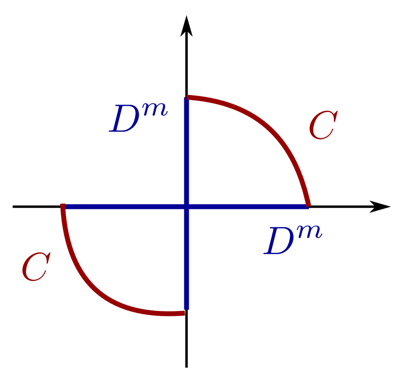


Figure 4.10: The sphere $\mathbf{S}$ of dimension m in $\mathbb{R}^{2m}$ with a unique intersection point is constructed by taking two discs D^m, connecting their boundaries with a cylinder C, and smoothing the corners.

To construct an immersed sphere in $\mathbb{R}^m \times \mathbb{R}^m$, let D_1 and D_2 be the two unit discs in $\mathbb{R}^m \times \{0\}$ and $\{0\} \times \mathbb{R}^m$. There is a rotation h_t, with $t \in [0, 1]$ of $\mathbb{R}^m \times \mathbb{R}^m$ that takes each vector $v \times \{0\}$ into the vector $\{0\} \times v$ and each vector $\{0\} \times v$ into the vector $-v \times \{0\}$. The trace $h_t(\partial D_1)$ of ∂D_1 under this rotation is a cylinder C, which together with D_1 and D_2 forms a sphere with corners in $\mathbb{R}^{2m}$. After smoothing the corners we obtain a desired sphere $\mathbf{S}$ of dimension m immersed into $\mathbb{R}^{2m}$ with a unique double point. When m is even, the sign of the double point is well-defined. We may obtain an immersed sphere with a unique double point of opposite sign by applying an orientation reversing diffeomorphism $\varphi\colon \mathbb{R}^{2m} \to \mathbb{R}^{2m}$. Namely, the sign of the self-intersection point of $\varphi(\mathbf{S})$ is opposite to that of $\mathbf{S}$.

Theorem 4.10 (Strong Whitney embedding theorem)**.** *Every closed connected manifold M of dimension m admits an embedding into $\mathbb{R}^{2m}$.*

Proof. If $m \leq 2$, then we may construct an explicit embedding of M into $\mathbb{R}^{2m}$. Suppose that $m \geq 3$ and choose a self-transverse immersion of M into $\mathbb{R}^{2m}$ with the least possible number of self-intersection points. Let x be a self-intersection point of M.

In the complement to M in $\mathbb{R}^{m+k}$ we place an immersed sphere S with a unique self-intersection point y, see Figure 4.11. Next we choose a path from a point on S to a point on M and perform a 1-surgery on $M \sqcup S$ along the path. The obtained immersed manifold M' is diffeomorphic to M, but it has an additional self-intersection point y.

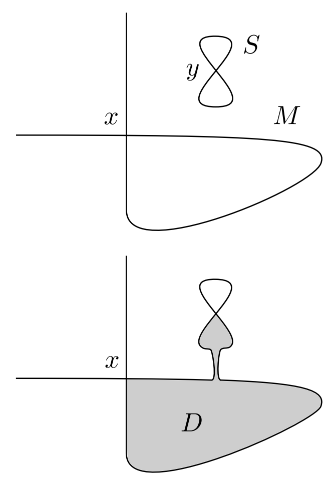


Figure 4.11: Removing a self-intersection point

Near the points x and y the manifold M' consists of two discs which we order arbitrarily. Choose a path L_P on M' which starts from x follows along the first disc and terminates at y along the first disc. Similarly, choose a path L_Q on M' from x which traverses the second disc near x and terminates at y along the second disc. Let P denote a ε-neighborhood of L_M in M' and let Q denote a ε-neighborhood of L_Q in M'. The manifolds P and Q intersect at two points p and q. We may assume that the signs of the intersection points p and q are opposite,[6] and therefore there is a Whitney trick that isotope P into P_1 such that P coincides with P_1 near $\partial\overline{P}$ and P_1 has no common points with Q. Then the immersed manifold $(M \setminus P) \cup P_1$ has fewer self-intersection points than M contrary to the assumption. Thus, M admits an embedding. □

[6] If the signs of the intersection points p and q are the same, then we may change the sign of q by slightly modifying the construction. Namely, if m is odd, then we only need to reorder the two discs of M' near q. If m is even, then we may replace S with $\varphi(S)$ where $\varphi\colon \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ is any orientation reversing diffeomorphism.

4.4 *Strong Whitney immersion theorem*

The strong Whitney immersion theorem improves the least dimension of the Euclidean space into which M admits an immersion.

Theorem 4.11 (The Whitney strong immersion theorem)**.** *Every oriented closed manifold M of dimension m admits an immersion into $\mathbb{R}^{2m-1}$.*

Proof. Again we may assume that $m > 2$. By the strong Whitney embedding theorem, we may assume that M is embedded into $\mathbb{R}^{2m}$. Choose a vector field v over M in $\mathbb{R}^{2m}$ such that at each point $x \in M$, the vector $v(x)$ is perpendicular to T_xM; we allow $v(x)$ to be zero. The end points of v define a copy M' of M. By the Transversality theorem we may assume that M' is transverse to M.

Since the manifolds under consideration are closed, there are only finitely many intersection points in $M \cap M'$; and since the manifolds are oriented, each intersection point comes with a sign. We claim that the algebraic number of intersection points—i.e., the number of intersection points counted with signs—- is 0. Indeed, the immersion $f = F_0$ extends to a regular homotopy $F_t\colon M' \to \mathbb{R}^{2m}$ such that F_1 is a map with image in a far away ball disjoint from M. For example, we may choose F to be the composition of a trivial isotopy for $t \in [0, \varepsilon)$, a parallel translation of M' to a far away ball in the period of time $[\varepsilon, 1-\varepsilon]$, and a trivial isotopy for $t \in (1-\varepsilon, 1]$. The regular homotopy F_t defines a map

$$F\colon M' \times [0,1] \longrightarrow \mathbb{R}^{2m} \times [0,1]$$

by associating $(F_t(x), t)$ with (x, t). Furthermore, slightly perturbing F, we may assume that it is an immersion transverse to $M \times [0,1]$. Then $F^{-1}(M)$ is an oriented manifold of dimension 1 with boundary. Since all the boundary points are the intersection points in $M \cap M'$, their algebraic number is zero.

By the Whitney trick, we may assume now that the immersed manifolds M and M' are disjoint. In other words, we may assume that the perpendicular vector field v over M is nowhere zero. The Compression theorem straightens v vertically up, after which the projection of M to the horizontal hyperplane $\mathbb{R}^{2m-1}$ defines a desired immersion. □

The Whitney strong immersion theorem can be slightly improved.

Theorem 4.12. *Let $f\colon M \to N$ be a smooth map of an oriented closed manifold M of dimension m to an oriented manifold N of dimension $2m-1$. Then M is homotopic to an immersion. If the map f is already an embedding over an open subset $U \subset M$, then the homotopy of f to an immersion can be chosen to be trivial over U, i.e., we may choose a homotopy that modifies f only outside U.*

Proof. We may first postcompose the map f with an inclusion i of N to $N \times \mathbb{R}$. Then, by the Whitney strong embedding theorem, we may modify the obtained map into an embedding $M \to N \times \mathbb{R}$. Next, as in the proof of Theorem 4.11, we find a nowhere zero vector field v perpendicular to M in $N \times \mathbb{R}$.[7] The compression theorem now allows us to compress the embedded manifold $M \subset N \times \mathbb{R}$ to an immersed manifold in N. □

[7] To begin with we construct a generic vector field v over M in $T(N \times \mathbb{R})$ perpendicular to M. If the manifold $N \times \mathbb{R}$ is in $\mathbb{R}^{n+k} \times \mathbb{R}$, then the tips of the vector field v define a manifold M' in $\mathbb{R}^{n+k} \times \mathbb{R}$. It can be projected to a manifold in $N \times \mathbb{R}$ which we also denote by M'. We note that the manifold M' can be translated in $N \times \mathbb{R}$ away from M and therefore the algebraic number of intersection points of M and M' is zero. The Whitney trick allows us to make M and M' disjoint. The manifold M' now defines a new nowhere zero vector field v over M perpendicular to M in $N \times \mathbb{R}$.

4.5 *Solution to Exercises*

Solution to Exercise 4.8. Recall that L_M is a slightly extended curve on M from p to q, L_N is a slightly extended curve on N from p to q. We have constructed vector fields $v_1, ..., v_{m-2}$ over $L_N \cup L_M$ as well as vector fields v_{m-1} over M and v'_{m-1} over L_N. Let v_m be the velocity vector field of L_M. We may assume that $(v_1(x), ..., v_m(x))$ is a positive basis for T_xM for every point $x \in L_M$. Let $\eta_1, ..., \eta_{n-1}$ be vector fields over L_N perpendicular to L_N and tangent to N. Let η_n be the velocity vector field of L_N. We may assume that $(\eta_1(x), ..., \eta_n(x))$ is a positive basis for T_xN for all $x \in L_N$. Finally, let d be a vector field over L_N tangent to D orthogonal to L_N and such that $d(p) = v(m)$. Then $d(q) = -v(m)$.

At each point x in L_N there is a basis for $T_x\mathbb{R}^{m+n}$:

$$\mathbf{b} = (v_1, ..., v_{m-2}, v'_{m-1}, d, \eta_1, ..., \eta_m).$$

Indeed, the vector fields $\eta_1, ..., \eta_m$ span the tangent space T_xN, the vector fields η_m and d span T_xD, while the other vector fields span the vector space in $T_x\mathbb{R}^{m+n}$ perpendicular to T_xN and T_xD. We note that the basis is positive. Indeed, at the point p we have $v'_{m-1}(p) = v_{m-1}(p)$ and $d(p) = v_m$. Thus, the basis $\mathbf{b}$ is composed of the positively oriented basis for T_pM followed by the positively oriented basis for T_pN. Since the intersection point p is positive, this implies that the basis $\mathbf{b}$ is positive.

On the other hand, at the point q we have $d(q) = -v_m(q)$. Since the intersection point q of M and N is negative this implies that $v'_{m-1}(q) = v_{m-1}(q)$. □

4.6 *Further reading*

The Whitney trick theorem originally appeared in the paper **The Self-intersections of a smooth n-manifold in $2n$-space** (in 1944) by Whitney. Theorem 4.1 was proved by Whitney [Wh35] in 1935. There is a nice presentation of the theorem in **Lectures on the h-cobordism theorem** by J. Milnor [Mi65].

5

The Kervaire invariant and signature

In this chapter we continue our study of homotopy classes of pointed maps $S^{m+k} \to S^k$ with $k > m+1$. These are identified with cobordism classes of framed manifolds M of dimension m in $\mathbb{R}^{m+k}$. We aim to simplify the representing manifolds M as much as possible by means of framed surgeries, which, we know, correspond to framed cobordisms.

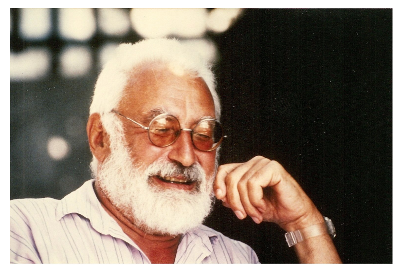


Figure 5.1: Michel Kervaire, 1927–2007

In fact, we will see (Theorem 5.7) that a framed manifold M of dimension $m \geq 5$ can be simplified by induction. To begin with, if the manifold M is not path connected, its distinct path components can be connected by means of framed surgeries of index 1. We may proceed by induction. Namely, when the manifold M is $(i-1)$-connected, where $i < m/2$, each generator x of the homotopy group $\pi_i M$ can be represented by an embedded sphere S. A spherical surgery with attaching sphere S kills the generator x. Thus, finitely many appropriate spherical surgeries of index $i+1$ along generators of $\pi_i M$ result in a framed i-connected manifold. The inductive process results in a manifold M with trivial homotopy groups in degrees $< m/2$.

When $m = 2q+1$ is odd, an additional step for $i = q$ is possible to perform yielding a q-connected manifold M. By Poincare Duality, such a manifold M is actually $(m-1)$-connected, and, in fact, homeomorphic to the sphere S^m (Theorem 5.10).

When $m = 2q$ is even, for each generator $x \in \pi_q M$ we will choose a special representing sphere S in M, called the *Wall representative* of x, and count the algebraic number $\mu(x)$ of self-intersection points of S. A spherical surgery along a sphere representing x is possible if and only if $\mu(x) = 0$ (Theorems 5.3, 5.4). Of course, as in the computation of π_2^S, if the values of μ on two generators α_1 and α_2 are non-zero, we may still attempt to perform a surgery on a sphere representing $\alpha_1 \pm \alpha_2$. In other words, in order to be able to kill all homotopy elements in $\pi_q M$ we only need that certain invariants of μ are trivial rather than $\mu \equiv 0$. The invariants of interest are the *signature* and *Kervaire invariant* of M. When these invariants are trivial, the framed manifold M is cobordant to a framed homotopy sphere. [1]

[1] It is known that the signature of a framed manifold is always zero. Thus, we will see that a framed manifold M of dimension $2q$ with q even is always cobordant to a framed homotopy sphere, while a framed manifold of dimension $2q$ with q odd is cobordant to a framed homotopy sphere if and only if its Kervaire invariant is zero.

5.1 *The Wall representative and the invariant μ*

To construct Wall representatives we will essentially use the Multicompression theorem (see Theorem 99.4), which asserts that all but one frame vectors of a framed manifold can be straightened up by means of an ambient isotopy.

Theorem 5.1. *Suppose that the manifold M of dimension m in $\mathbb{R}^{m+k}$ is equipped with $n < k$ linearly independent perpendicular vector fields $v_1, ..., v_n$. Then there is an ambient isotopy F_t with $t \in [0,1]$ that straightens the vectors $v_1, ..., v_n$ up in the sense that $F_0 = \mathrm{id}$ while $dF_1(v_i) = e_i$ for $i = 1, ..., n$.*

Now, let M be a manifold of dimension[2] $m \geq 5$ in $\mathbb{R}^{m+k}$, with k sufficiently big, equipped with a frame $\tau_1, ..., \tau_k$, and let x be an element in the homotopy group $\pi_q M$ in the range $q \leq m/2$. Our goal is to determine when there is a spherical surgery along an attaching sphere representing x. Such a surgery would reduce the homotopy group $\pi_q M$.

[2] We will discuss in Remark 5.5 the cases $m \leq 4$.

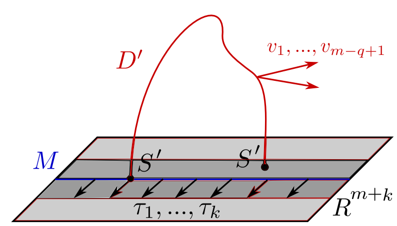


Figure 5.2: The (red) disc D' bounding (black) sphere S'. The (blue) manifold M is equipped with vector fields $\tau_1, ..., \tau_k$, which define a tubular neighborhood (dark grey) $M \times D^k$ of M. The disc D' is equipped with perpendicular vector fields $v_1, ..., v_{m-q+1}$.

In view of the k-frame $\tau_1, ..., \tau_k$ over the manifold M, we may use the exponential map to identify a tubular neighborhood of M with $M \times D^k$. We note that under the exponential map, the vector fields $\tau_1, ..., \tau_k$ over $M \times \{0\}$ are identified with the coordinate vector fields in the direction D^k. In particular, we may extend $\tau_1, ..., \tau_k$ over the tubular neighborhood by the coordinate vector fields in the direction D^k, see Figure 5.2.

By the Whitney embedding theorem, there is a an embedded sphere S' in $M \times D^k$ representing the element x in $\pi_q M \approx \pi_q(M \times D^k)$. [3] We will identify the space $\mathbb{R}^{m+k}$ in which $M \times D^k$ is located with the horizontal subspace $\mathbb{R}^{m+k} \times \{0\}$ in $\mathbb{R}^{m+k+1}$. Then there exists a unique up to isotopy disc D' in $\mathbb{R}^{m+k} \times [0,1]$ of dimension $q+1$ that bounds S', see Figure 5.2. Choose an orthonormal frame $v = \{v_i\}$ of D'; by Exercise 5.2 the choice of the frame of D' is essentially unique.[4]

[3] We can start with a not necessarily embedded sphere S' representing x in M and then perturb it in a tubular neighborhood of M so that it is embedded. Here we essentially use the assumption that k is sufficiently big.

[4] Since the dimension of D' is $q+1$, and it lies in $\mathbb{R}^{m+k+1}$, the number of frame vector fields over D' is $m-q+k$.

__Exercise__ **5.2.** Show that every orthonormal frame $v' = \{v'_i\}$ over D' can be continuously deformed to the frame $v = \{v_i\}$ through orthonormal frames. [5]

Next, we will attempt to modify the sphere S' and the disc D' as well as its frame v so that the framed disc D' is a base of framed surgery and S' is an attaching sphere in M. To this end, we need to deform S' back to the manifold M, and we need that $\tau_i = v_i$ for the first k vector fields v_i of the frame of the disc D'.

Since the vector fields $v_1, ..., v_k$ restricted to S' are perpendicular and linearly independent, the Multicompression Theorem guarantees the existence of an ambient isotopy of S' in $\mathbb{R}^{m+k}$ with support in a small neighborhood of S' that slightly perturbs S' and brings each vector v_i to τ_i for $i = 1, ..., k$. The ambient isotopy F_t of S' in $\mathbb{R}^{m+k}$ extends to an ambient isotopy of $\mathbb{R}^{m+k+1}$ and carries D' to a disc $F_1 D'$ bounded by the sphere $F_1 S'$.

There is a regular homotopy of the sphere $F_1(S')$ that compresses it to its projection $S \subset M$ along the vertical disc fibers of the tubular neighborhood of M. We may extend the regular homotopy of $F_1(S')$ over the disc $F_1 D'$ and obtain a disc $D \subset \mathbb{R}^{m+k} \times [0, 1]$ bounded by S.[6] Without loss of generality we may assume that the sphere S in M is immersed and self-transverse. It still represents the homotopy class x in $\pi_q M$, and has the property that the frame fields $\tau_1, ..., \tau_k$ of M over S extend to normal vector fields $v_1 = \tau_1, \quad \cdots, \quad v_k = \tau_k$ over D.

We will say that the inclusion of the sphere S into M is the *Wall representative* of x. the Wall representative of a given class x is defined uniquely up to regular homotopy. Note, when S is embedded, it has a frame $v_1 = \tau_1, ..., v_k = \tau_k, v_{k+1}, ..., v_{m-q+k}$ which extends over D.[7]

When the dimension q of the sphere S is even, every self-intersection point of S has a well-defined sign. On the other hand, when q is odd, the sign of a self-intersection point is not well-defined and therefore any pair of self-intersection points of S can be canceled by the Whitney trick. Define $\mu(x)$ to be the algebraic number of self-intersection points of the Wall representative S of x. It is an integer if q is even, and an element in $\mathbb{Z}_2$ if q is odd.

Whether q is even or odd, the invariant $\mu(x)$ may be non-trivial only if $m = 2q$. Indeed, we assumed that $m \geq 2q$. On the other hand, if $m < 2q$, then any generic immersed sphere S in M is actually embedded.

[5] *Hint to Exercise 5.2* At a point x in D', the frame v' is obtained from the frame v by a rotation $\varphi(x)$. The correspondence $x \mapsto \varphi(x)$ is a continuous map $D' \to SO(n)$ into the group of rotations, where n is the number of vectors in the frame v. The contraction of φ to the constant map that maps each point of D' to $1 \in SO(n)$ defines a deformation of v' to v. More precisely, if $\varphi_t : D' \to SO(n)$ is a homotopy of $\varphi_0 = \varphi$ to the constant map φ_1 to $1 \in SO(n)$, then $\{\varphi_t v_i\}$ is the homotopy of $\{v'_i\}$ to $\{v_i\}$.

[6] A particular choice of a disc D is not relevant, since any immersed disc bounded by S is regularly homotopic to any other immersed disc bounded by S whenever the dimension k is sufficiently big. Furthermore, the regular homotopy can be chosen relative to S.

[7] Note that the frame of S consists of two types of vectors: the vectors $v_1, .., v_k$ that are perpendicular not only to S, but also to M, and the vectors $v_{k+1}, ..., v_{m-q+k}$ that are perpendicular to S and tangent to M.

Theorem 5.3. *The class $\mu(x)$ is well-defined.*

To prove Theorem 5.3, we will assume that there are two Wall representatives S_0 and S_1 of the class x, and construct a *regular homotopy* S_t of S_0 to S_1 in M.[8] We will observe that throughout the regular homotopy the self-intersection points of the surface S_t appear and disappear in pairs. Furthermore, when the signs of the self-intersection points are well-defined the two points in each appearing and disappearing pair are of different signs. This implies that the algebraic number $\mu(x)$ of self-intersection points of S_0 is the same as that of S_1.

[8] A *regular homotopy* from S_0 to S_1 is a smooth deformation through immersions, i.e., it is a smooth family S_t of immersions with $t \in [0,1]$.

Proof. Since $\mu(x) = 0$ unless $m = 2q$, we may assume that $m = 2q$. For $i = 0, 1$, let S_i be a Wall representative of x equipped with a frame v^i in $\mathbb{R}^{m+k}$. Slightly perturb each of the spheres with their framings by isotopy in a tubular neighborhood of M so that each sphere S_i is embedded. In fact, in view of the exponential map we may identify the tubular neighborhood of M with $M \times D^k$ and choose the isotopy in such a way that each point $x \in S_i$ flows along the disc $\{x\} \times D^k$. Choose a homotopy of S_0 to S_1 in a tubular neighborhood of M. By the Whitney embedding theorem, we may assume that its trace is a manifold-with-boundary $\mathbf{S}$ in $\mathbb{R}^{m+k} \times [0,1]$ with $\partial_i \mathbf{S} = S_i$, see Figure 5.4. Let D be a disc in $\mathbb{R}^{m+k} \times [1,2]$ bounding S_1. Note that the frame v^1 over S_1 extends to a frame over D. Similarly, the frame v^0 over S_0 extends over the union $\mathbf{S} \cup D$. By Exercise 5.2, we may assume that v^0 agrees with v^1 over D.[9] Consequently, there is a frame v_j over $\mathbf{S}$ extending v^0 and v^1.

[9] More explicitly, the frames v^0 and v^1 over D define a map $f\colon D \to \mathrm{SO}(n)$ which associates with x the rotation from $v^0(x)$ to $v^1(x)$. A deformation of the map f to the constant map onto $1 \in \mathrm{SO}(n)$ defines a deformation of the frame v^0 to v^1 over D. Such a deformation extends to a deformation of v^0 over all $\mathbf{S} \cup D$ with support in the union of D and a collar neighborhood of $\partial_1 \mathbf{S}$ in $\mathbf{S}$.

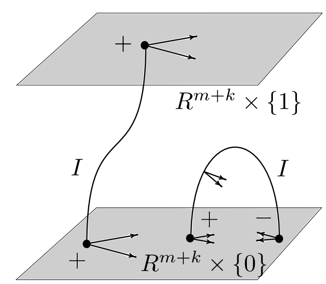


Figure 5.3: Signs of self-intersection points.

Now we may use the vector fields $v_1, ..., v_k$ to compress $\mathbf{S}$ into $M \times [0,1]$ by means of the Multicompression Theorem. Namely, we may extend the vector fields $\tau_1, ..., \tau_k$ over the tubular neighborhood $(M \times D^k) \times [0,1]$ by translation, and regard the extended vector fields as the basis vector fields in the direction D^k of the neighborhood. We may assume that $v_1, ..., v_k$ coincide with $\tau_1, ..., \tau_k$ over a collar neighborhood of $\partial\mathbf{S}$ in $\mathbf{S}$. By the Relative Multicompression Theorem, there exists an ambient isotopy of $\mathbf{S}$ in the tubular neighborhood of M that brings $v_1, ..., v_k$ to $\tau_1, ..., \tau_k$ everywhere over $\mathbf{S}$. Furthermore, we may assume that the isotopy is trivial over a collar neighborhood of $\partial\mathbf{S}$. Hence, after applying the isotopy to $\mathbf{S}$, we may compress $\mathbf{S}$ to M and obtain an immersion $\mathbf{S} \to M \times [0,1]$ that takes its boundary to the union of S_0 and S_1.

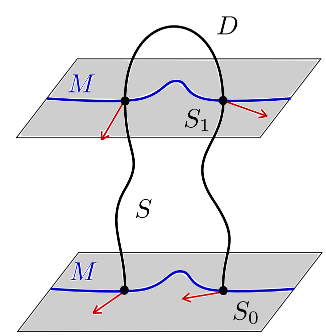


Figure 5.4: The trace S of a homotopy and a disc D.

Let X denote the set of self-intersection points of the compressed manifold $\mathbf{S}$ in $M \times [0,1]$. Then X is a manifold-with-boundary of dimension 1—i.e., a union of segments and circles—and $X_i = \partial_i X$ coincides with the set of self-intersection points of S_i in $M \times \{i\}$. We claim that X defines a framed cobordism between the manifolds of self-intersection points of S_0 and S_1. Indeed, let x be any point in X. Then it is at the intersection of two sheets of X. Let $v_{k+1}, ..., v_{k+q}$ denote the frame vectors at x of $\mathbf{S}$ in $M \times [0,1]$ over one of the sheets, and $v'_{k+1}, ..., v'_{k+q}$ be the frame vectors at x over the other sheet. Then the vectors $v_{k+1}, ..., v_{k+q}, v'_{k+1}, ..., v'_{k+q}$ define a frame over X in $M \times [0,1]$. Recall that the signs of self-intersection points of S_0 and S_1 are defined by means of similar frames. Therefore, the manifold X is a framed cobordism between the self-intersection points X_0 of S_0 and X_1 of S_1, and, in particular, the cobordism class $\mu(x)$ of the framed manifolds $[S_0] = [S_1]$ is well-defined. □

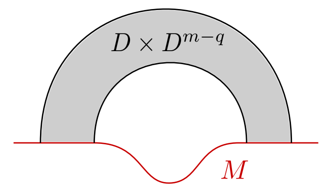


Figure 5.5: The set $D \times D^{m-q}$ defines a normal spherical surgery on M.

We note that given a spherical surgery on a manifold M, the attaching sphere S represents an element $x \in \pi_q M$ with $\mu(x) = 0$. Indeed, the base of framed surgery consists of a framed disc bounded by S, and therefore, the sphere S in this case is the Wall representative of x with no self-intersection points. The converse statement is also true.

Theorem 5.4. *Let x be an element of the homotopy group $\pi_q M$ of a manifold of dimension $m \geq 5$ with $q \leq m/2$. If $\mu(x) = 0$, then there is a spherical surgery along an attaching sphere S that represents x.*

Proof. If $\mu(x) = 0$, then all self-intersection points of the Wall representative S of x can be cancelled by regular homotopy of S in M. Indeed, if $m > 2q$, then it suffices to use a regular homotopy that places S into a self-transverse position. If $m = 2q$ and q is even, then $\mu(x) = 0$ implies that each positive self-intersection point of S can be paired with a negative self-intersection point; such a pair can be cancelled by the Whitney trick. Finally, if $m = 2q$ and q is odd, then the sign of each self-intersection point can be changed by the change of the order of intersecting sheats of S near the self-intersection point. Again $\mu(x) = 0$ in $\mathbb{Z}_2$ implies that we can pair the self-intersection points and choose the orders of intersecting sheets of S near the self-intersection points so that each pair can be cancelled by the Whitney trick. Thus, if $\mu(x) = 0$, then we may assume that the Wall representative S is an embedded manifold.

Let $\{\tau_1, ..., \tau_k\}$ be the frame over the manifold M in $\mathbb{R}^{m+k} \times \{0\}$. Since S is the Wall representative of x, there is a disc D in $\mathbb{R}^{m+k} \times [0,1]$ bounded by S and framed by vector fields $v_1, ..., v_{m-q+k}$ such that over S each vector field v_i restricts to τ_i for $i = 1, ..., k$. Then D is a base of framed surgery over S. $\square$

Remark 5.5. We have already considered the case $m = 1$ and $m = 2$, see section 3.4. In the case $m = 2$ there is indeed an obstruction $\mu(x)$ to performing a surgery along x. We identified that obstruction with an element in $\pi_1^S \simeq \mathbb{Z}_2$. In the case $m = 3$ (and $q \leq 1$), the obstruction $\mu(x)$ is trivial, since a generic sphere of dimension q in a 3-dimensional manifold is embedded. Therefore, the conclusion of Theorem 5.4 is also true in the case $m = 3$. In the case $m = 4$ and $q = 1$, we can certainly perform a surgery along an attaching sphere representing any class $x \in \pi_1 M$. However, if $m = 4$ and $q = 2$, the Whitney trick is not available in general.

In the rest of the section we will show that when the dimension of the manifold M is $2q + 1 \geq 5$, then every embedded sphere of dimension q in M is a Wall representative. In fact, we will proof a slightly stronger result. Namely, let $M \subset \mathbb{R}^{m+k}$ be a framed simply connected manifold of dimension $m \geq 5$ with a frame $\tau_1, ..., \tau_k$.

Theorem 5.6. *Suppose that $m \geq 2q$ and $m \geq 5$. Let λ be an element in $\pi_q(M)$. Suppose that $\mu(\lambda) = 0$. Then any embedded sphere $S \subset M$ representing λ is a Wall representative of λ.*

Proof. Since $\mu(\lambda) = 0$, there exists an embedded Wall representative S_λ of λ. Choose a homotopy $S^q \times [0,1] \to M \times [0,1]$ between the inclusions of a Wall representative S_λ and S into M. It is a map of a manifold of dimension $q+1$ to a manifold of dimension at least $2(q+1) - 1$, and it is an embedding near the boundary. Therefore, by the Whitney strong immersion theorem, we may modify it away from the boundary into an immersion. Let $\mathbf{S}$ denote the image of this immersion in $\mathbf{M} = M \times [0,1]$. In $\mathbb{R}^{m+k} \times [0,1] \times \mathbb{R}$ there is an immersed disc $\mathbf{D} = D^{q+1} \times [0,1]$ such that the disc $D^{q+1} \times \{0\}$ is the core D_λ of a framed surgery along S_λ, the disc $D^{q+1} \times \{1\}$ is a disc in $\mathbb{R}^{m+k} \times \{1\} \times \mathbb{R}$, while $\partial D^{q+1} \times [0,1]$ coincides with $\mathbf{S}$, see Figure 5.6

A base of framed surgery along S_λ consists of the disc D_λ as well as perpendicular vector fields $v_1, ..., v_{m-q+k}$ over D such that the vector fields v_i coincide with τ_i for $i \leq k$ over S_λ. We may extend vector fields τ_i over $\mathbf{M} = M \times [0,1]$ by translation in the direction $[0,1]$. Since S_λ is a deformation retract of $\mathbf{S}$ we may extend the vector fields $v_i | S_\lambda$ to perpendicular vector fields over $\mathbf{S}$ in $\mathbb{R}^{m+k} \times [0,1]$ in such a way that $v_i = \tau_i$ for $i \leq k$. Since $\mathbf{S} \cup D_\lambda$ is a deformation retract of $\mathbf{D}$, we may extend the vector fields v_i over $\mathbf{D}$. It remains to observe that the disc

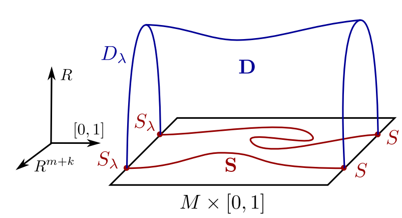


Figure 5.6: The immersed disc $\mathbf{S}$ in $\mathbf{M}$ as well as the disc $\mathbf{D}$.

$D^{q+1} \times \{1\}$ together with vector fields v_i over it is a base of framed surgery along S. □

In particular, suppose that $m \geq 5$. When $m = 2q+1$, a framed surgery is possible along any embedded sphere S of dimension q in M. When $m = 2q$, an immersed sphere S of dimension q in M is a Wall representative of $[S]$ if and only if it is regular homotopic to an embedding and $\mu([S]) = 0$. [10]

[10] When $m = 2q$ and q is even, then $\lambda \cdot \lambda = 2\mu(\lambda)$, and therefore a framed surgery along an embedded sphere S exists if and only if its normal bundle is trivial. The same is true when $m = 2q$ with q odd, but $q \neq 1, 3, 7$, e.g., see [Ko07, Proposition X.2.1].

5.2 *Homotopy spheres*

Theorem 5.7. *A framed manifold M is cobordant to a $(q-1)$-connected manifold if M is of dimension $2q$ or $2q+1$.*[11]

[11] A manifold M is $(q-1)$-*connected* if $\pi_i M = 0$ for all $i \leq q-1$.

Proof. In low dimensions, the Wall representative S of $x \in \pi_i M$ can be chosen embedded, and therefore the obstruction $\mu(x)$ to the existence of a spherical surgery along S is trivial. By Lemma 11.4,[12] a surgery along S kills a subgroup in $\pi_i M$ containing x. Thus we may make M $(q-1)$-connected by induction in dimension i starting with $i = 1$. For each dimension we only need to perform finitely many surgeries since for an $i-1$ connected manifold M the i-th homotopy group is finitely generated (it is isomorphic to $H_i M$.) □

[12] Let W be a spherical cobordism between W_0 and W_1 of index $n+1 \leq m/2$, where m is the dimension of the manifolds W_0 and W_1. Suppose that the attaching sphere of the corresponding surgery represents an element $x \in \pi_n W_0$. Then $\pi_i W_0 = \pi_i W_1$ for $i < n$, while $\pi_n W_1$ is a factor group of $\pi_n W_0$ by a subgroup containing x.

For example, when the framed manifold M is of dimension 2 or 3, Theorem 5.7 asserts that M can be made path connected, while a manifold M of dimension 4 or 5 can be made simply connected.

To summurize, we have shown that a framed manifold M of dimension $2q$ or $2q+1$ is cobordant to a framed $q-1$ connected manifold. We will show now that a q-connected manifold is necessarily a homotopy sphere.

Definition 5.8. A closed manifold M of dimension m is said to be a *homotopy sphere* if it is homotopy equivalent to the standard sphere.[13]

[13] We say that a topological space X is homotopy equivalent to a topological space Y if there are continuous maps $f: X \to Y$ and $g: Y \to X$ such that $f \circ g$ and $g \circ f$ are homotopic to identity maps.

Exercise 5.9. Show that an oriented closed manifold M of dimension m is a homotopy sphere if and only if $\pi_i M = 0$ for $i < m$.[14]

[14] *Hint:* By the Whitehead theorem, two connected manifolds (or, even CW-complexes) X and Y are homotopy equivalent if there is a map $f: X \to Y$ that induces an isomorphism f_* of homotopy groups in all degrees. The map $f: M \to S^m$ collapsing the complement to an open disc in M induces an isomorphism of homology groups in all degrees. Therefore, by the relative Hurewicz theorem, it induces an isomorphism of homotopy groups in all degrees.

It turns out that every homotopy sphere of dimension m is actually homeomorphic to the standard sphere S^m. This statement is a deep theorem that is often referred to as the topological generalized Poincaré conjecture. It was proved by Smale (for $m \geq 5$), Freedman (for $m = 4$), and Perelman (for $m = 3$).

Theorem 5.10. *A closed q-connected manifold of dimension $m = 2q$ or $m = 2q + 1$ is a homotopy sphere.*

Sketch of a proof. Since M is q-connected, its homology groups are trivial in dimensions $\leq q$. Consequently, by the Poincare duality,[15] its cohomology groups are trivial in dimensions $\geq m - q$ except for those in dimension m. By the Universal Coefficient Theorem,[16] this implies that the homology groups of M are trivial in dimensions $\geq m - q$ except for those in dimension m. There is a map of M to S^m of degree 1, which, by the Whitehead theorem, is a homotopy equivalence. □

[15] **The Poincare Duality:** Let M be an oriented closed manifold of dimension m. Then $H^k(M) \simeq H_{m-k}(M)$.

[16] **The Universal Coefficient Theorem:** $H^k(X) \simeq \text{Torsion}\, H_{k-1}(X) \oplus \text{Free}\, H_k(X)$.

Thus the obstruction to the existence of a cobordism of a framed manifold to a homotopy sphere may only occur when we attempt to surge out homotopy classes in the middle dimension.

Suppose that a framed manifold M is $(q-1)$-connected. We will see promptly that if M is of odd dimension $2q+1$, then it is actually cobordant to a q-connected manifold. By contrast, if M is of even dimension $2q$, then it may not be cobordant to a framed q-connected manifold.[17] There are obstructions which we briefly describe next.

[17] Recall, for example, that any orientable framed surface is cobordant either to a framed torus or a framed sphere, but a framed torus may not be cobordant to a framed sphere. In analysis of framed surfaces F, we worked with algebraic numbers of intersections of circles α_i, β_i representing elements in $\pi_1 F$.

Let $\mathbb{K}$ be a field, and V denote the vector space $\pi_q M \otimes \mathbb{K}$. An element in V is a linear combination of homotopy classes x of maps $S^q \to M$. The self-intersection numbers $\mu(x)$ that we have already defined give rise to a so-called self-intersection form μ on V. It is a *quadratic form* associated with the *bilinear intersection form*.

The mentioned obstructions are defined in terms of invariants of quadratic forms, which we will review in section 5.3. Note that when q is even, the self-intersection numbers $\mu(x)$ are integers, while for q odd, the self-intersection numbers are only defined mod 2. For this reason, we need invariants of quadratic forms over $\mathbb{K} = \mathbb{R}$ (or $\mathbb{K} = \mathbb{Q}$) as well as over $\mathbb{K} = \mathbb{Z}_2$.

5.3 Bilinear and quadratic forms

In this section we will give definitions of bilinear and quadratic forms over a field $\mathbb{K}$, and review their invariants.

A *bilinear form* on a vector space V is a map $\varphi\colon V \times V \to \mathbb{K}$ such that $\varphi(x, y)$ becomes a linear function when any of the two parameters is fixed. We will often write the value $\varphi(x, y)$ of a bilinear form on two vectors as $x \cdot y$. A bilinear form is *symmetric* if $x \cdot y = y \cdot x$, *symplectic* if $x \cdot x = 0$, and *skew-symmetric* if $x \cdot y = -y \cdot x$. The form is *non-degenerate* if for every linear function f on V there are unique vectors v and w such that $f(y) = v \cdot y$ for every vector y, and $f(x) = x \cdot w$ for every vector x. [18]

[18] In other words, a form is non-degenerate if the homomorphisms $V \to \mathrm{Hom}(V, \mathbb{K})$ defined by $x \mapsto \varphi(x, -)$ and $x \mapsto \varphi(-, x)$ are isomorphisms of vector spaces over $\mathbb{K}$. We note that the homomorphism $x \mapsto \varphi(x, -)$ is invertible if and only if $x \mapsto \varphi(-, x)$ is.

A *quadratic form* $V \to \mathbb{K}$ is a function q that satisfies $q(\alpha x) = \alpha^2 q(x)$ for all vectors x and scalars α, and such that the function

$$(x, y) = q(x + y) - q(x) - q(y)$$

of two arguments is a bilinear form on V; it is necessarily symmetric.[19] Every bilinear form φ restricts to a quadratic form $q(x) = \varphi(x, x)$. However, the restricted quadratic form q is associated with

$$(x, y) = q(x + y) - q(x) - q(y) = \varphi(x, y) + \varphi(y, x),$$

not with φ. In particular, if we begin with a symmetric bilinear form φ, then the bilinear form $(,)$ associated with the restricted quadratic form satisfies $(x, x) = 2\varphi(x, x) = 2q(x)$.

[19] Over the field of real numbers, it is tempting to define the associated bilinear form by $(x, y) = \frac{1}{2}[q(x + y) - q(x) - q(y)]$. Then $q(x) = (x, x)$. However, in general, division by 2 in $\mathbb{K}$ may not be available. That is why we define the associated bilinear form without the factor $1/2$.

When division by 2 in $\mathbb{K}$ is possible, the restricted quadratic form $q(x) = \varphi(x, x)$ completely recovers the bilinear symmetric form φ.[20] In fact, in this case every bilinear symmetric form can be obtained from a unique quadratic form.

[20] Indeed, if we know all the values $x \cdot x$, then the value $x \cdot y$ can be found by the formula

$$2x \cdot y = (x + y) \cdot (x + y) - x \cdot x - y \cdot y.$$

Quadratic forms over $\mathbb{R}$ or $\mathbb{Q}$. Recall that with each quadratic form q there is an associated bilinear symmetric form φ.

Theorem 5.11 (Isomorphism types of symmetric forms)**.** *Let φ be a symmetric non-degenerate bilinear form on a finite dimensional vector space V over a field $\mathbb{K}$ in which 2 is invertible. Then there is an orthogonal basis for V.*[21]

[21] i.e., there is a basis $v_1, ..., v_n$ in V such that $v_i \cdot v_j = 0$ for $i \neq j$.

Proof. Pick any two vectors v and w with $\varphi(v, w) \neq 0$. If $\varphi(v, v) \neq 0$ or $\varphi(w, w) \neq 0$, then one of these vectors can be chosen to be the first basis vector e_1. Otherwise, put $e_1 = v + w$ and note that $\varphi(v + w, v + w) = 2\varphi(v, w)$, which is non-zero provided that 2 is invertible. The linear function $x \mapsto \varphi(e_1, x)$ is non-zero, so its kernel is a vector space of dimension $\dim V - 1$ with a non-degenerate symmetric bilinear form. Thus an orthogonal basis can be found by induction in $\dim V$. □

Over the field of real numbers $\mathbb{K} = \mathbb{R}$, we may further scale vectors in an orthogonal basis of V to obtain a new basis $\{e_i\}$ such that $\varphi(e_i, e_i) = \pm 1$. Using the new basis, the values of the form φ are especially easy to compute:

$$(v_1 e_1 + \cdots + v_n e_n) \cdot (w_1 e_1 + \cdots + w_n e_n) = \pm v_1 w_1 \pm \cdots \pm v_n w_n.$$

We say that n is the *rank* of the form φ. The *signature* of the form φ is the difference between the number of basis vectors e_i with $e_i \cdot e_i = 1$ and the number of basis vectors e_j with $e_j \cdot e_j = -1$. We note that the rank and the signature of the form φ do not depend on the choice of the basis $\{e_i\}$.[22]

[22] This fact is known as the Sylvester's law of inertia.

Corollary 5.12. *A symmetric non-degenerate space of finite dimension over $\mathbb{R}$ is determined by its rank and signature. The same is true for quadratic forms over $\mathbb{R}$.*

Since quadratic forms determine symmetric non-degenerate forms, Theorem 5.11 classifies not only bilinear but also corresponding quadratic forms. It is important that $\operatorname{char} \mathbb{K} \neq 2$ here.

Quadratic forms over $\mathbb{Z}_2$. We say that a basis $\{a_i, b_i\}$ is a symplectic basis for a vector space V over a field $\mathbb{K}$ with symplectic non-degenerate form, if $a_i \cdot b_i = 1$ for all i and all other products $a_i \cdot a_j$ and $a_i \cdot b_j$ and $b_i \cdot b_j$ are trivial.

Theorem 5.13 (Isomorphism types of symplectic forms.)**.** *For every symplectic non-degenerate space V of finite dimension, there is a symplectic basis $\{a_i, b_i\}$ in the vector space V. Thus, a symplectic non-degenerate form is determined by its rank.*

Proof. Choose a_1 to be an arbitrary vector. There exists a vector b_1 such that $a_1 \cdot b_1 = 1$.[23] The space V_a orthogonal to a_1 is of dimension $\dim V - 1$ and contains a_1. Similarly, the space V_b orthogonal to b_1 is of dimension $\dim V - 1$ and contains b_1. Thus the orthogonal complement to $\langle a_1, b_1 \rangle$ is the space $V_a \cap V_b$ of dimension $\dim V - 2$. The restriction of the symplectic form to the orthogonal complement

[23] To show the existence of b_1, pick any linear function f on V with $f(a_1) = 1$. Since φ is non-degenerate, there exists a vector b_1 such that

$$\varphi(a_1, b_1) = f(a_1) = 1.$$

to $\langle a_1, b_1\rangle$ is again a symplectic non-degenerate form. Therefore, by induction, it has a symplectic basis $a_2, b_2, ..., a_n, b_n$. Then $a_1, b_1, ..., a_n, b_n$ is a symplectic basis for V. □

A *quadratic form* on a vector space V over $\mathbb{Z}_2$ is a function $q\colon V \to \mathbb{Z}_2$ such that the associated form

$$(x,y) = q(x) + q(y) + q(x+y)$$

is bilinear. Note that over $\mathbb{Z}_2$ we do not need to require in the definition of a quadratic form that $q(\alpha x) = \alpha^2 q(x)$.[24] The form q is called a *quadratic non-degenerate form* if the associated bilinear form is a non-degenerate form (it is necessarily symplectic).

[24] Indeed, if $\alpha = 1$, then the condition $q(\alpha x) = \alpha^2 q(x)$ is clearly vacuous. On the other hand, if $\alpha \neq 1$, i.e., if $\alpha = 0$, then $q(0 \cdot x) = 0 \cdot q(x)$ follows from

$$0 = (0,0) = q(0) + q(0) + q(0,0).$$

If $V_1, ..., V_k$ are vector spaces with quadratic forms $q_1, ..., q_k$ respectively, then on the vector space $V_1 \oplus \cdots \oplus V_k$, there is a well-defined quadratic form $q_1 \oplus \cdots \oplus q_k$ given by

$$q_1 \oplus \cdots \oplus q_k(x_1 \oplus \cdots \oplus x_k) = q_1(x_1) + \cdots + q_k(x_k).$$

Since $q(x+y)$ is the sum of (x,y), $q(x)$ and $q(y)$, in symplectic basis the quadratic form is determined by its values on the basis vectors. For $i, j = 0, 1$, let us denote by q_{ij} the form on a symplectic space over $\mathbb{Z}_2$ with symplectic basis a, b defined by $q_{ij}(a) = i$ and $q_{ij}(b) = j$. Then $q_{0,0}$ is isomorphic to $q_{1,0}$ and $q_{0,1}$, and is not isomorphic to $q_{1,1}$.[25] Thus, every non-degenerate quadratic form over $\mathbb{Z}_2$ of rank 2 is isomorphic either to $q_{0,0}$ or $q_{1,1}$. Also $q_{1,1} \oplus q_{1,1}$ is isomorphic to $q_{0,0} \oplus q_{0,0}$.[26]

[25] In the vector space $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ with syplectic basis $\{a, b\}$ there are only three vectors a, b and $a+b$. Since $(a,b) = 1$, among the three values $q(a), q(b)$ and $q(a+b)$ either there is one appearance of 1 in which case the form is isomorphic to $q_{0,0}$, or there are three appearances of 1 in which case the form is isomorphic to $q_{1,1}$.

[26] Indeed, when we choose another basis

$$a_1' = a_1 + a_2, \quad a_2' = a_1 + b_1 + a_2 + b_2$$
$$b_1' = b_1 + a_2, \quad b_2' = a_1 + b_1 + b_2$$

the form $q_{11} \oplus q_{11}$ becomes $q_{0,0} \oplus q_{0,0}$.

Let now V be a vector space with a non-degenerate quadratic form q of arbitrary rank. Choose a symplectic basis $\{a_1, b_1, ..., a_n, b_n\}$ for V. The number

$$\mathrm{Arf}(q) = q(a_1)q(b_1) + \cdots + q(a_n)q(b_n)$$

in $\mathbb{Z}_2$ is called the Arf-invariant of the form q; it does not depend on the choice of the symplectic basis for V. To prove the invariance of the Arf-invariant, note that q is the orthogonal sum $q_1 \oplus \cdots \oplus q_n$ where q_i is the restriction of q to the vector subspace $\langle a_i, b_i\rangle$ of V. Then each q_i is a non-degenerate form; indeed, its associated bilinear form is the symplectic form of rank 2 with symplectic basis $\{a_i, b_i\}$. We have seen that each non-degenerate form of rank 2 is isomorphic to $q_{0,0}$ or $q_{1,1}$. Thus, we deduce that q is isomorphic to $mq_{0,0} + nq_{1,1}$. Clearly, the Arf-invariant of q is 0 if n is even, and 1 otherwise. It remains to observe that if n is odd, then most of the values of the quadratic form $mq_{0,0} + nq_{1,1}$ are 1, while if n is even, then most of the values are 0. Therefore if the form $mq_{0,0} + nq_{1,1}$ is isomorphic to a form $m'q_{0,0} + n'q_{1,1}$, then the parities of n and n' are the same.

5.4 Main theorems

In the case $m = 2q + 1$ the obstruction μ is zero by dimensional reason. We will show that in this case, indeed, it is possible to surge out all the classes in dimension q.

Theorem 5.14. *If M is a closed framed manifold of dimension m in $\mathbb{R}^{k+m}$ with odd $m \geq 5$ and $k \gg m$, then M is cobordant to a homotopy sphere.*

In the case of $(q-1)$-connected manifolds M of even dimension $2q > 4$ there are obstructions to the existence of a framed cobordism of M to a homotopy sphere, which we describe next. Let $\mathbb{K}$ be a field. We are mostly interested in the case where $\mathbb{K}$ is the field of real numbers $\mathbb{R}$ (or rational numbers $\mathbb{Q}$) and the case where $\mathbb{K}$ is the field $\mathbb{Z}_2$. Let V denote the vector space $\pi_q M \otimes \mathbb{K}$. For example, when $\mathbb{K} = \mathbb{R}$, the basis of the vector space $V = \mathbb{R}^n$ consists of generators $[x]$ of the free part of $\pi_q M$.

There is a bilinear form $V \otimes V \to \mathbb{K}$ on the vector space V called the *intersection form*. To a pair of basis vectors $[x]$ and $[y]$ it puts into correspondence the number $[x] \cdot [y]$ of intersection points of x and y counted with signs. For general vectors x, y in V, the value $x \cdot y$ of the bilinear form is computed by linearity. We have also encountered a quadratic form μ that assigns to a basis vector $[x]$ the self-intersection number of the Wall representative x.

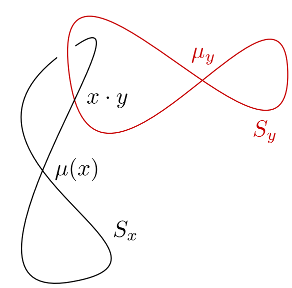


Figure 5.7: Intersections and self-interesections of S_x and S_y.

The quadratic form μ is associated with the intersection form. Indeed, let S_x and S_y be Wall representatives in M of classes x and y. The spheres S_x and S_y can be joined by a thin tube whose interior is disjoint from S_x and S_y; the resulting sphere S_{x+y} is the Wall representative of $x + y$, see Figures 5.7 and 5.8. The set of self-intersection points of S_{x+y} consists of the self-intersection poings of S_x, the self-intersection points of S_y, and the intersection points of S_x and S_y. In other words,

$$\mu(x + y) = \mu(x) + \mu(y) + x \cdot y.$$

Thus, indeed, the map μ is a quadratic form associated with the intersection form.

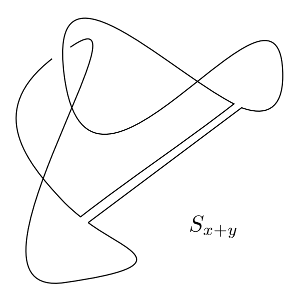


Figure 5.8: Self-interesctions of S_{x+y}.

Remark 5.15. Since the intersection form is associated with the self-intersection form, we have $2\mu(x) = x \cdot x$. This equality could be verified geometrically. Indeed, the product $x \cdot x$ is the algebraic number of intersection points of an immersed framed sphere S representing x and a sphere S' in M obtained from S by a slight isotopy in the direction of, say, the last frame vector. Each self-intersection point in $\mu(x)$ corresponds to two intersection points in $x \cdot x$; hence $2\mu(x) = x \cdot x$, see Figure 5.9

Suppose now that the dimension m of the framed manifold M is $2q$ with q even. In this case the intersection form is well-defined over integers, and we choose the ground filed $\mathbb{K}$ to be the field of rational numbers. In other words, $V = \pi_q M \otimes \mathbb{Q}$. The vector space V is the direct sum of the subspaces V_- and V_+ over which μ is negative and positive definite respectively. We have seen that the isomorphism type of a quadratic non-degenerate form is determined by its rank $= \dim V$ and the signature $\sigma_{\mathbb{Q}} = \dim V_+ - \dim V_-$. The rank of the form μ may change under a normal cobordism, but the signature $\sigma_{\mathbb{Q}}$ does not.

Theorem 5.16. *If M is a closed framed manifold of dimension m in $\mathbb{R}^{k+m}$ with $m = 4s > 4$ and $k \gg m$, then M is cobordant to a homotopy sphere if and only if $\sigma_{\mathbb{Q}} = 0$.* [27]

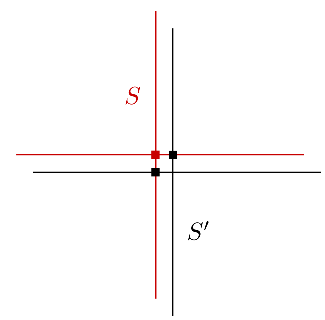


Figure 5.9: Every self-intersection point (red) in $\mu(x)$ corresponds to two intersection points (black) in $x \cdot x$.

[27] It is known that the signature of a framed manifold is always zero.

Suppose now that $m = 2q$ with q odd. Then the sign of a double point of a sphere S^q in M is not well-defined. In other words, the quadratic form μ is only well-defined over $\mathbb{Z}_2$, which we choose to be our ground field $\mathbb{K}$.

The *Kervaire invariant* of the framed manifold M is defined to be the Arf-invariant of the quadratic form μ.

Theorem 5.17. *Let M be a closed framed manifold in $\mathbb{R}^{k+m}$ of dimension $2q$ with q odd. Then M is cobordant to a homotopy sphere if and only if the Kervaire invariant of M is trivial.*

Exercise 5.18 (Pontryagin)**.** Prove that $\pi_2^S \simeq \mathbb{Z}_2$.

5.5 Further reading

The Kervaire invariant of framed surfaces was first used by *Pontrjagin* in **Homotopy classification of the mappings of an $(n+2)$-dimensional sphere on an n-dimensional one** to calculate $\pi_{n+2}S^n$ [Po50]. A detailed geometric solution to the problem appeared in his book **Smooth manifolds and their applications in homotopy theory** [Po85].

In 1960 *Kervaire* introduced the invariant in **A manifold which does not admit any differentiable structure** [Ke60] for 10-dimensional manifolds to detect a triangulated manifold which does not admit a smooth structure. In order to construct the invariant Arf, Kervaire used the loop space $\Omega = \Omega S^6$ over the sphere of dimension 6. Such a space is homotopy equivalent to a CW-complex with one cell of each dimension divisible by 5. Let e_k denote the cohomology class generating the $5k$-th cohomology group of ΩS^6. Since $\pi_9(\Omega)$ is trivial, it can be shown that for any triangulated manifold M of dimension 10, and any class $x \in H^5 M$, there is a map $f \colon M \to \Omega$ such that $f^* e_1 = x$. This allowed Kervaire to construct an operation $\varphi \colon H^5(M; \mathbb{Z}_2) \to \mathbb{Z}_2$; given a class x, find a function f such that $f^* e_1$ is a class whose reduction modulo 2 is x, reduce the class $f^* e_2$ to a class mod 2, and then integrate it over M to obtain an element in $\mathbb{Z}_2$. Kervaire showed that the operation φ is a well-defined quadratic form associated with the intersection form on M. Furthermore, the Arf invariant Arf (φ) is trivial for the quadratic form φ on every smooth manifold of dimension 10. On the other hand, Kervaire presented a triangulated manifold M whose Arf invariant is clearly 1. Kervaire concluded that M does not admit a smooth structure.

In **Groups of Homotopy spheres I** [KM63] *Kervaire and Milnor* interpreted the Kervaire invariant differently. Namely, let M be a framed $q-1$ connected manifold of dimension $2q$ with q-odd. Every element x in $\pi_q M \otimes \mathbb{Z}_2$ can be represented by an embedded sphere. There is a function $\psi \colon \pi_q M \otimes \mathbb{Z}_2 \to \mathbb{Z}_2$ that detects a twisting of the normal bundle in M of the embedded sphere representing x, i.e., $\psi(x) = 0$ if and only if the normal bundle of a sphere representing x is trivial. The function ψ is a quadratic form associated with the intersection form on M. Kervaire and Milnor defined the Kervaire invariant of the manifold M to be the Arf invariant of the quadratic form ψ.

We adopted a slightly different approach. Instead of representing x by an embedded sphere and detecting a twisting of its normal bundle, we choose a Wall representative of x and count the number of self-intersection points of the Wall representative. With such a definition the relation of the quadratic form to the intersection form is clear.

Spherical modifications of framed manifolds below the middle dimension were studied by *Milnor* in an unpublished note **Differentiable manifolds which are homotopy spheres** [Mi59]. In a published form the technique appeared in **A procedure for killing homotopy groups of differentiable manifolds** [Mi61]. At the same time a non-framed version of spherical modifications below the middle dimension appeared in **Modifications and cobounding manifolds** [Wa60] by *Wallace*. The main theorem for odd dimensional manifolds (asserting that a framed closed manifold of dimension $2q + 1 \geq 5$ is cobordant to a homotopy sphere) was independently proved by *Wall* in **Killing the middle homotopy groups of odd dimensional manifolds** [Wall62], and by *Kervaire and Milnor* in **Groups of Homotopy spheres I** [KM63]. In the same paper, Kervaire and Milnor established the main theorems for manifolds of dimension $2q$ both when q is odd and even. The expected paper **Groups of Homotopy spheres II** never appeared.

It turned out that in most of the dimensions there are no framed manifolds with non-trivial Kervaire invariant. In fact, *Browder* [Bra69] showed that only manifolds of dimension $2^k - 2$ may have non-trivial Kervaire invariant. Recently *Hill, Hopkins and Ravenel* [HHR16] showed that in high dimensions ($2^k - 2 > 126$) manifolds with Kervaire invariant one do not exist. It is still not known if such a manifold exists in dimension 126; in lower dimensions $2^k - 2$ such manifolds have been constructed.

6

Surgery on framed manifolds

We have seen that it is easy to surge out homotopy classes of a closed framed manifold M of dimension $m = 2q$ or $2q + 1$ in dimensions below q. In particular, every closed framed manifold is cobordant to a framed $(q-1)$-connected manifold M. Furthermore, if the manifold M is actually q-connected, then it is homeomorphic to a sphere, in which case we say that M is a homotopy sphere.

When $m = 2q$ is even, the manifold M may not be cobordant to a homotopy sphere. In §6.1 we will show that when q is even, the signature $\sigma(M)$ of a manifold M is a well-defined invariant, while when q is odd, the Kervaire invariant $\mathrm{Arf}(M)$ is a well-defined invariant of a cobordism. In particular, if $\sigma(M) \neq 0$ when q is even, or $\mathrm{Arf}(M) \neq 0$ when q is odd, then M is not cobordant to a homotopy sphere.

On the other hand, we will show that when q is even, the invariants $\sigma(M)$ and $\mathrm{Arf}(M)$ are the only obstructions to the existence of a cobordism of M to a homotopy sphere. In fact, since the manifold M is $(q-1)$-connected its q-th homotopy group is free. If the obstructing invariants are trivial, then we may always find a generator λ of $\pi_q M$ with an embedded Wall representative S_λ. Then a surgery along S_λ results in a framed $(q-1)$-connected manifold M' with a reduced homotopy group $\pi_q M' \approx \pi_q M / \langle \lambda \rangle$.

The case $m = 2q + 1$ turns out to be more complicated. In this case any embedded sphere S representing a class $\lambda \in \pi_q M$ is a Wall representative of λ. When λ is a generator of a free summand of $\pi_q M$ a surgery along S reduces the homotopy group of $\pi_q M$, as in the case of $m = 2q$. By performing surgery along embedded Wall representatives of generators of free summands of $\pi_q M$, we may assume that $\pi_q M$ is finite.

A surgery along a generator λ of finite order results in a manifold M' with a class λ' in $\pi_q M'$ such that $\pi_q M/\langle\lambda\rangle \approx \pi_q M'/\langle\lambda'\rangle$. If q is even, the order of the generator λ' is infinite, and therefore we may always first perform a surgery along S_λ to "replace" λ with λ' and then reduce λ'. If q is odd, and the class λ is of finite order s, then the class λ' may be of some finite order t. In this case we need to consider several cases to insure that the order of $\pi_q M$ can always be reduces by a sequence of appropriate surgeries. As a result, it follows that when m is odd, the manifold M is always cobordant to a homotopy sphere.

6.1 *Invariance of signature and Kervaire invariant*

In this section we will show that the signature and Kervaire invariant of a manifold M do not change under cobordism. The proof relies on the existence of a nice basis for the homotopy group $\pi_q M$ of a bounding manifold M.

Let M be a closed framed $q-1$ connected manifold of dimension $2q$ bounding a $q-1$-connected. In particular, the group $\pi_q M$ is a free abelian group. Suppose that M bounds a compact framed manifold W. Let $i\colon M \to W$ denote the inclusion of the boundary, and $K \subset \pi_q M$ the kernel of the induced homomorphism i_* in homotopy groups.

Lemma 6.1. *We have $x \cdot y = 0$ for any elements x, y in K.*

Proof. The classes x and y are represented by embedded transverse spheres S_x and S_y in M, see Figure 6.1. The number $x \cdot y$ is the algebraic number of intersection points of S_x and S_y. Since x and y belong to the kernel of the homomorphism i_*, the spheres S_x and S_y bound discs D_x and D_y in W. We may assume that D_x is transverse to D_y. Then the curve $D_x \cap D_y$ defines an oriented cobordism to zero of the intersection manifold $S_x \cap S_y$. Thus, $x \cdot y = 0$. □

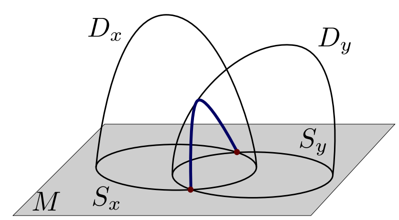


Figure 6.1: The (grey) horizontal manifold M bounds a manifold W (upper half space). Spheres S_x and S_y in M intersect along a (red) manifold. The discs D_x and D_y in W bound S_x and S_y respectively. Their (blue) intersection bounds the (red) intersection between S_x and S_y.

Lemma 6.2. *We have* $\operatorname{rank} K = \frac{1}{2} \operatorname{rank} \pi_q M$.

Proof. Choose a basis $e_1, ..., e_n$ for K. By Lemma 6.1, the intersection product of vectors in K is trivial. On the other hand, since the intersection form over $\pi_q M$ is non-degenerate, by induction there are elements $e_1^*, ..., e_n^*$ in $\pi_q M$ such that $e_i^*(e_j) = \delta_{ij}$ and $e_i^* \cdot e_j^* = 0$ for all i and j, where δ_{ij} is the Kronecker symbol.[1] The vectors $e_1, ..., e_n^*$ are linearly independent.[2] Furthermore, we claim that $\{e_i, e_i^*\}$ is a basis for

[1] Suppose that $e_1^*, ..., e_{k-1}^*$ have been constructed. Consider a linear function $f\colon \pi_q M \to \mathbb{Z}$ such that $f(e_k) = 1$ and f is trivial on all vectors v with $v \cdot e_k = 0$. Then there is a vector e_k^* such that $f(w) = w \cdot e_k^*$ for all w.

[2] If $\sum \alpha_i e_i + \beta_i e_i^* = 0$, then taking the product with e_j gives $\beta_j = 0$, and taking the product with e_j^* gives $\alpha_j = 0$.

$\pi_q M$. Indeed, if not, then there is a vector $e \neq 0$ perpendicular to each e_i and e_j^*.[3] Since e is not in the kernel K, the element $i_* e$ is non-trivial in $\pi_q W$. Since the intersection form

$$\pi_q W/Tor \otimes \pi_{q+1}(W, M)/Tor \longrightarrow \mathbb{Z}$$

is non-degenerate with respect to the first factor, we deduce that there is an element g in $\pi_{q+1}(W, M)$ such that $i_* e \cdot g = 1$. Then $e \cdot \partial g = 1$. Since ∂g is in K, this contradicts our assumption that e is perpendicular to K. Thus, indeed, the vectors $\{e_i, e_i^*\}$ span $\pi_q M$. □

[3] There is a non-zero function $f : \pi_q M \to \mathbb{Z}$ which is trivial on all e_i and e_j. Since the intersection pairing is non-degenerate, there is a vector e such that $f(w) = e \cdot w$ for all w.

Theorem 6.3. *The signature and Kervaire invariant do not change under cobordism.*

Proof. It suffices to show that if a $q-1$ connected closed framed manifold M is a boundary, then its mentioned invariants are trivial. [4] On the other hand, if M is bounded by a framed manifold W, we may assume that W is $q-1$ connected as there is no obstruction to framed surgery of indices below the middle range.

[4] For example, suppose that we have proven that the signature of a bounding manifold is trivial. Let us show that $\sigma(M_0) = \sigma(M_1)$ whenever M_0 is cobordant to M_1. Since M_0 is cobordant to M_1, the disjoint union of M_0 and a copy $-M_1$ of M_1 with reverse orientation is a bounding manifold. Therefore

$$0 = \sigma(M_0 \sqcup (-M_1)) = \sigma(M_0) - \sigma(M_1).$$

If $\dim M = 2q$ with q even, then by Lemma 6.2, the vector space $\pi_q M \otimes \mathbb{R}$ contains a subspace $K \otimes \mathbb{R}$ of half rank over which the non-degenerate intersection form vanishes. Thus, the signature of M is zero.

Let M be a manifold of dimension $2q$ with q odd in $\mathbb{R}^k$ bounded by a framed manifold $W \subset \mathbb{R}^k \times [0, 1]$; we will denote the frame over W by $\tau_1, ..., \tau_k$. We may choose a symplectic basis $\{a_i, b_i\}$ of $\pi_q M \otimes \mathbb{Z}_2$ such that $\{a_i\}$ is a basis for $K \otimes \mathbb{Z}_2$[5]. We claim $\mu(a) = 0$ for all $a \in K \otimes \mathbb{Z}_2$. Indeed, by the Whitney embedding theorem, we may choose an embedded sphere S representing a. Since $i_*(a) = 0$, there is an immersed disc D in W bounded by S. At each point $x \in D$, the perpendicular space $T_x^\perp D$ of D is a direct sum $U_x \oplus V_x$ of orthogonal subspaces, where U_x is the orthogonal complement to $T_x D$ in $T_x W$, and V_x is the perpendicular space $T_x^\perp W$. The vector spaces V_x are framed by $\tau_1, ..., \tau_k$. Choose an arbitrary frame for U_x continuously depending on x for all $x \in D$.[6] We may slightly perturb D in $\mathbb{R}^{k+1}$ together with its frame $\tau_1, ..., \tau_{k+q}$ so that it becomes a base for framed surgery. The existence of D, then, implies that S is an embedded Wall representative of a, and therefore $\mu(a) = 0$. Finally, since $\mu(a_i) = 0$, the Kervaire invariant $\sum \mu(a_i)\mu(b_i)$ is trivial. □

[5] Note that since $\pi_q M$ is a free abelian group, it is isomorphic to a direct sum of copies of $\mathbb{Z}$. We have constructed its basis $e_1, ..., e_n^*$. The group $\pi_q M \otimes \mathbb{Z}_2$ is obtained from $\pi_q M$ by replacing each summand $\mathbb{Z}$ with a copy of $\mathbb{Z}_2$. There is a mod 2 reduction homomorphism $\pi_q M \to \pi_q M \otimes \mathbb{Z}_2$. It takes the basis $\{e_i, e_i^*\}$ to a symplectic basis $\{a_i, b_i\}$.

[6] Every plane field over a disc can be framed.

We note that *a priori* the signature and Kervaire invariants are only defined for $(q-1)$-connected manifolds of dimension $m = 2q$. Theorem 6.3 implies, however, that the definition of these invariants can be extended over arbitrary closed framed manifolds. Indeed, given an arbitrary closed framed manifold M_0, it is cobordant to a framed $(q-1)$-connected manifold M_1. We may define the invariants of M_0 to be the same as those of M_1. To show that the invariants of M_0 are well-defined, suppose that M_0 is cobordant to two different framed $(q-1)$-connected manifolds M_1 and M_1'. By Theorem 6.3, the invariants of M_1 and M_1' are the same. Thus the signature and Kervaire invariant are well-defined for arbitrary closed framed manifolds.

Furthermore, in the definition of signature $\sigma(M)$ and in the proof of invariance of signature in Theorem 6.3 we do not use that the manifold M, or cobordisms, are framed. In other words, the signature is well-defined for an arbitrary oriented closed manifold of dimension $m = 2q$ for q even, and it is invariant under (non-framed) cobordisms.

6.2 *The main theorems for even dimensional manifolds*

Effect of surgery on homotopy groups

Let M be a $q-1$-connected manifold of dimension $2q \geq 5$. In particular, the group $\pi_q M$ is free abelian. A framed spherical surgery on M consists of removing a closed neighborhood $h_q = S^q \times D^q$ of a sphere S_λ, and then attaching to the remaining manifold M_0 a thickening $h_q' = D^{q+1} \times S^{q-1}$ of a sphere $S_{\lambda'}$. The resulting manifold will be denoted by M'. We will denote the disc $\{*\} \times D^q$ in the neighborhood h_q by D_λ. Let λ and λ' denote the classes $[S_\lambda]$ and $[S_{\lambda'}]$ in $\pi_q M$ and $\pi_q M'$ respectively. Suppose that λ is a generator of an infinite cyclic summand of $\pi_q M$.

Lemma 6.4. *The meridian ∂D_λ in ∂h_q is null homotopic in M_0.*

Proof. Since the intersection pairing on homotopy groups $\pi_q M$ is non-degenerate, there is an element $\mu \in \pi_q M$ such that $\mu \cdot \lambda = 1$. Using the Whitney trick, we may represent μ by a sphere S_μ which intersects S_λ transversally at a unique point. Furthermore, we may assume that near the intersection point S_μ coincides with D_λ. Then $S_\mu \setminus D_\lambda$ is a disc in M_0 bounding ∂D_λ. □

Exercise 6.5. The homotopy group $\pi_k(M, M_0)$ is trivial for $k < q$. For $k = q$ it is isomorphic to $\mathbb{Z}$ with a generator given by the class of D_λ.

For $k = q+1$ it is isomorphic to $\mathbb{Z}_2$ with a generator represented by the map $D \to D_\lambda$ which is a cone over the Hopf map $\partial D \to \partial D_\lambda$. [7] The group $\pi_k(M', M_0)$ is trivial for $k \leq q$, and $\pi_{q+1}(M', M_0) \approx \mathbb{Z}$. [8]

[7] The group $\pi_{q+1}(M, M_0)$ is isomorphic to $\pi_{q+1}(S^q \times D^q, S^q \times S^{q-1})$, while the latter fits an exact sequence

$$\pi_{q+1}(S^q \times D^q) \xrightarrow{j_*} \pi_{q+1}(S^q \times D^q, S^q \times S^{q-1})$$
$$\to \pi_q(S^q \times S^{q-1}) \xrightarrow{i_*} \pi_q(S^q \times D^q).$$

The homomorphism j_* is trivial since the homomorphism

$$\pi_{q+1}(S^q \times S^{q-1}) \to \pi_{q+1}(S^q \times D^q)$$

is surjective. On the other, hand, the kernel of the homomorphism i_* is $\mathbb{Z}_2$ generated by the Hopf map to $\{*\} \times S^{q-1}$.

[8] The group $\pi_k(M', M_0)$ is isomorphic to the k-th homotopy group of the pair $(h'_q, \partial h'_q)$. We note that the homomorphism j_* from $\pi_*(\partial h'_q)$ to $\pi_* h'_q$ is surjective with kernel isomorphic to $\pi_*(S^q \times \{pt\})$. Therefore $\pi_k(h'_q, \partial h'_q) \approx \pi_{k-1}(S^q \times \{pt\})$.

Lemma 6.6. *The manifold M_0 is $q-1$ connected.*

Proof. Let f be an embedding representing an element $\pi_k M_0$ for $k < q$. Since the manifold M is $q-1$ connected, the map f extends to an immersion $F\colon D \to M$ transverse to S_λ. If $k < q-1$, then $F(D)$ avoids the sphere S_λ, and therefore we may assume the image of F is in M_0. Thus, the element $[f]$ is trivial in $\pi_k M_0$. Suppose now that $k = q-1$. Then $F(D)$ is an immersed manifold in M intersecting S_λ at finitely many points. We may assume that near each intersection point $F(D)$ coincides with D_λ. By replacing each copy of D_λ in $F(D)$ with a copy of $S_\mu \setminus D_\lambda$ constructed in the proof of Lemma 6.4 , we obtain an immersed disc in M_0 bounding $f(S^k)$. □

Lemma 6.7. *We have $\pi_q M \approx \pi_q M_0 \oplus \mathbb{Z}$.*

Proof. Since the pairing in M is non-degenerate, in the homotopy exact sequence

$$\pi_{q+1}(M, M_0) \xrightarrow{\partial} \pi_q M_0 \xrightarrow{i_*} \pi_q M \xrightarrow{j_*} \pi_q(M, M_0)$$

the homomorphism j_* is surjective onto a group isomorphic to $\mathbb{Z}$. On the other hand, the homomorphism ∂ is trivial. Indeed, by Exercise 6.5, the source of ∂ is represented by the cone over the Hopf map $\partial D \to \partial D_\lambda$. Thus, the image of ∂ is generated by the class of the Hopf map to ∂D_λ. Since ∂D_λ bounds a disc in M_0, we conclude that the image of ∂ is trivial. □

Lemma 6.8. *We have* rank $\pi_q M > \pi_q M'$.

Proof. Since $\pi_q(M', M_0) = 0$, there is an exact sequence

$$\to \pi_{q+1}(M', M_0) \to \pi_q M_0 \to \pi_q M' \to 0.$$

Therefore rank $\pi_q M = \operatorname{rank} \pi_q M_0 + 1 \geq \operatorname{rank} \pi_q M' + 1$. □

Remark 6.9. Though we do not need it at the moment, the effect of the surgery can be described more explicitly. Namely, since the manifold M is $(q-1)$-connected of dimension $2q$, its homotopy group $\pi_q M$ is free abelian. Suppose that there is a basis $\{a_1, ..., a_k, b, c\}$ of the group $\pi_q M$ such that b is orthogonal to all a_i and b, c is orthogonal to all

a_i and c and $b \cdot c = 1$. Since $2q \geq 6$, we may choose representatives of the basis classes such that the geometric numbers of intersection points agrees with the algebraic number of intersection points of the representatives. Suppose also that $\mu(b) = 0$. By Lemma 6.7, the group $\pi_q M$ is a free abelian group generated by $\{a_1, ..., a_k, b\}$, while by the exact sequence in Lemma 6.8, the group $\pi_q M'$ is free abelian generated by $\{a_1, ..., a_k\}$.

The case $m = 2q$ with q even.

Theorem 6.10. *An oriented closed manifold of dimension $4q > 4$ is normally cobordant to a homotopy sphere if and only if $\sigma(M) = 0$.*

Proof. We only need to prove that if $\sigma = 0$, then there is a sequence of normal spherical surgeries killing all homotopy groups of M. We may assume that M is $q - 1$ connected. This implies that $\pi_q M$ has no torsion. Hence there exists a class $x \in \pi_q(M)$ such that $x \cdot x = 0$.[9] We can assume that x is a generator of a free factor of $\pi_q(M)$. Since $2\mu(x) = x \cdot x = 0$, there exists a framed surgery along a sphere S_x representing x. Since x is generator of a free factor of $\pi_q M$, by Lemma 6.8 a framed surgery along S_x reduces the rank of $\pi_q M$. Therefore, in finitely many steps, we obtain a homotopy sphere. □

[9] **Theorem** *If $\sigma = 0$, then there exists an integral class x with $x \cdot x = 0$.* Diagonalize the form over $\mathbb{Q}$ and find a rational class u with $u \cdot u = 0$. Its multiple $x = ku$ is integral for some integer k. On the other hand, we have $x \cdot x = k^2 u \cdot u = 0$.

The case $m = 2q$ with q odd.

Now let us turn to the case of dimension $m = 2q$ with q odd.

Theorem 6.11. *Every $q - 1$ connected framed closed manifold M of dimension $2q \geq 5$ with q odd is cobordant to a homotopy sphere if and only if the Kervaire invariant of M is trivial.*

Proof. We only need to prove that if the Kervaire invariant of M is trivial, then M is cobordant to a homotopy sphere. We may assume that in its framed cobordism class the manifold M has the minimal number of generators of the torsion free group $\pi_q M$. Choose a symplectic basis $\{a_i, b_i\}$ of $\pi_q(M; \mathbb{Z}_2)$ every element in which is represented by a sphere S^q such that $[S^q]$ is a generator of an infinite cyclic summand of $\pi_q M$. If $\mu(a_i) = 0$ for some i, then we may perform a surgery on M along the representing sphere to kill an infinite cyclic summand of $\pi_q M$, which contradicts the choice of M. Thus, $\mu(a_i) = 1$ for all i, and similarly, $\mu(b_j) = 1$ for all j. If $\pi_q M$ is at least of rank 4, then we can replace the symplectic basis elements a_i, b_i, a_j, b_j by

$a_i' = a_i + a_j, b_i' = b_i, a_j' = b_j - b_i, b_j' = a_i$ so that $\mu(a_i') = 0$.[10] Again a surgery along a_i' leads to a contradiction with the choice of M. Therefore the rank of $\pi_q M$ is at most 1. However, in this case the Kervaire invariant $\mu(a_1)\mu(b_1)$ is 1 which contradicts the assumption that the Kervaire invariant of M is trivial. Thus, the group $\pi_q M$ is trivial, and therefore M is a homotopy sphere. □

[10] We have

$$\mu(a_i') = a_i \cdot a_j - \mu(a_i) - \mu(a_j) \equiv 0.$$

6.3 *The main theorems for odd dimensional manifolds*

Effect of surgery on homotopy groups.

Assume the $q-1$ connected manifold M is of dimension $m = 2q+1$, and $m \geq 5$. In particular, we may define the homotopy classes in $\pi_i M$ to be free (not pointed) maps $S^i \to M$ up to free homotopy.

Recall that a spherical surgery in dimension q consists of removing a handle $h_q = S^q \times D^{q+1}$ from M, and, to the remaining manifold M_0, attaching a new handle $h_q' = D^{q+1} \times S^q$. The core of the handle h_q is a sphere S_λ of dimension q representing a homotopy class λ of M. We will denote the belt disc by D_λ. Similarly, the core of the handle h_q' is a sphere $S_{\lambda'}$ of dimension q representing a class λ' of the resulting manifold M' of the surgery. The belt disc is denoted by D_λ'.

<u>**Exercise**</u> **6.12.** The group $\pi_i(M, M_0)$ is trivial for $i < q+1$ and it is isomorphic to $\mathbb{Z}$ for $i = q+1$. The latter group is generated by the class $[D_\lambda]$ of the inclusion of the disc D_λ in h_q whose boundary is the meridian of ∂h_q, see Figure 6.2.

We note that under the boundary homomorphism the generator $[D_\lambda] = [1]$ in the relative homotopy group $\pi_{q+1}(M, M_0)$ maps to the class $\partial(1)$ in $\pi_q M_0$ represented by a meridian of the torus ∂h_q. The meridian of the torus ∂h_q coincides with a longitude of the torus $\partial h_q'$, and therefore the homomorphism $i\colon \pi_q M_0 \to \pi_q M'$ induced by the inclusion takes the meridian class $\partial(1)$ to the class λ'. Since $\partial(1)$ and λ' are represented by the same meridian, it is tempting to use the same notation for these two classes. However, for calculations below it will be important to distinguish $\partial(1)$ from λ'. For this reason we will keep using a somewhat strange notation $\partial(1)$ for the meridian class in M_0. Similarly, we have $i' \circ \partial'(1) = \lambda$ where i' and ∂' are counterparts of i and ∂ for a surgery along S_λ'.

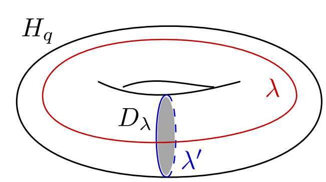


Figure 6.2: A handle h_q, its meridian λ' and the parallel λ.

Proposition 6.13. *We have $\pi_q M/\langle\lambda\rangle \approx \pi_q M'/\langle\lambda'\rangle$.*

Proof. The inclusion of M_0 into M induces a surjective homomorphism $\pi_q M_0 \to \pi_q M$.[11] Its kernel is generated by $\partial(1)$. Similarly, the homomorphism $\pi_q M_0 \to \pi_q M'$ is surjective with kernel $\langle \partial'(1) \rangle$. [12] □

[11] There is a long exact sequence

$$\to \pi_{q+1}(M, M_0) \to \pi_q M_0 \to$$
$$\to \pi_q M \to \pi_q(M, M_0) \to$$

where the group $\pi_q(M, M_0)$ is trivial by Exercise 6.12.

[12] More precisely, $\pi_q M \approx \pi_q M_0 / \langle \partial(1) \rangle$. Similarly, $\pi_q M' \approx \pi_q M_0 / \langle \partial'(1) \rangle$. Therefore, the inclusions $M_0 \to M$ and $M_0 \to M'$ induce isomorphisms $\pi_q M / \langle \lambda \rangle \simeq \pi_q M_0 / \langle \partial(1), \partial'(1) \rangle \simeq \pi_q M' / \langle \lambda' \rangle$.

Corollary 6.14. *By a sequence of spherical surgeries on M we may get a $(q-1)$-connected closed framed manifold M' such that $\pi_q M'$ is a torsion group.*

Proof. Let λ be a generator of an infinite cyclic summand $\mathbb{Z}$ of $\pi_q M$, then $\pi_q M' \simeq \pi_q M / \mathbb{Z}$. Indeed, since λ is a generator of an infinite cyclic summand, by the Poincare duality, there is a class μ such that $\lambda \cdot \mu = 1$. By the Whitney trick, we may represent μ by a sphere S_μ which intersects S_λ at a unique point. Without loss of generality, the sphere S_μ intersects h_q along D_λ. Consequently, the sphere $S_{\lambda'}$ is null-homotopic in M' as it is homotopic to the boundary of $S_\mu \setminus D_\lambda$. Now, by Proposition 6.13, we have $\pi_q M' \approx \pi_q M / \langle \lambda \rangle$. We may repeat the argument to eliminate all infinite cyclic summands in $\pi_q M$. □

Exercise 6.15. Show that the above arguments holds for homology with coefficients in $\mathbb{Z}_p$. In particular, if an element λ is non-trivial in $H_q(M; \mathbb{Z}_p) = \pi_q M \otimes \mathbb{Z}_p$, then rank $\pi_q M' \otimes \mathbb{Z}_p <$ rank $\pi_q M \otimes \mathbb{Z}_p$.[13]

[13] Note that when λ is a non-trivial element in $\pi_q \otimes \mathbb{Z}_p \approx \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p$, the class λ is a generator of a summand in $\pi_q \otimes \mathbb{Z}_p$. Therefore, the rank of $\pi_q M' \otimes \mathbb{Z}_p \approx \pi_q M \otimes \mathbb{Z}_p / \langle \lambda \rangle$ is less than the rank of $\pi_q M \otimes \mathbb{Z}_p$. However, the order of $\pi_q M$ may still be the same as the order of $\pi_q M'$. Indeed, we may for example have, $\pi_q M = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ while $\pi_q M' = \mathbb{Z}_4$.

A base of (not framed) spherical surgery consists of a disc D embedded into $\mathbb{R}^{m+k} \times [0,1]$ with boundary $\partial D = S_\lambda$ together with $q+1$ perpendicular vector fields $\tau_{k+1}, ..., \tau_{k+q+1}$ over D that restrict to perpendicular vector fields over S_λ in M. The base of surgery along S_λ is not unique as we may choose another set $\tau'_{k+1}, ..., \tau'_{k+q+1}$ of vector fields over D. Let $w \colon S^q \to SO_{q+1}$ be the difference function; at a point x in $S^q = S_\lambda$, we set $w(x)$ to be the rotation that brings each vector $\tau_i(x)$ to $\tau'_i(x)$. Conversely, given a difference function $w \colon S^q \to SO_{q+1}$ we may define new vector fields $\tau'_i = w\tau_i$ over S_λ and then extend them (uniquely up to homotopy) over D.[14] The difference function w can be visualized by its action on the thickening $h_q = S^q \times D^{q+1}$ of the attaching sphere S_λ. It acts by taking a point (x, y) to $(x, w(x))$, see Figure 6.3. In particular, it takes the meridian $\partial(1) \in \pi_q(h_q)$ to itself, and the longitude $\partial'(1)$ to $\partial'(1) + \alpha\partial(1)$.

[14] The extension is unique up to homotopy since we are extending only $q+1$ vector fields in perpendicular vector spaces of dimension $\gg q$.

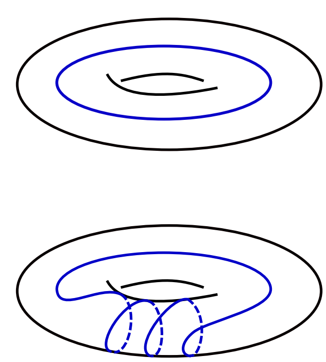


Figure 6.3: The solid torus $S^q \times D^{q+1}$, the longitude $\mathfrak{l} = \{(x, \tau_{k+1})\}$ on the top, and its twist $\{(x, w(x)\tau_{k+1})\}$ on the bottom.

Exercise 6.16. Suppose that $(D; \tau_{k+1}, ..., \tau_{k+q+1})$ extends to a base of framed surgery. Show that if $(D; w\tau_{k+1}, ..., w\tau_{k+q+1})$ also extends to a base of framed surgery, then α is even. Conversely, show that for every even integer α there exists a difference function w such that $(D; w\tau_{k+1}, ..., w\tau_{k+q+1})$ extends to a base of framed surgery, and $w(\partial'(1))$ is $\partial'(1) + \alpha\partial(1)$.

The case $m = 2q+1$ with q even.

We need to carefully investigate the result of a spherical surgery along an attaching sphere representing a torsion class λ. We will need a preliminary lemma.

Lemma 6.17. *If λ is of finite order in $\pi_q M$, then the class $\lambda' \in \pi_q M'$ is of infinite order.*

Proof. Any sphere of dimension $q+1$ in M algebraically intersects S_λ zero times.[15] Hence $S_{\lambda'}$ represents a class $\partial(1)$ of infinite order in $\pi_q M_0$.[16] Suppose that λ' is of finite order $t' > 0$. Then $t'\partial(1)$ is in $\ker i' = \operatorname{Im} \partial'$.[17] Hence, the class $x = t'\partial(1) + s'\partial'(1)$ in ∂h_q becomes trivial in $\pi_q M_0$ for some integer s'. By Lemma 6.1, we have $x \cdot x = 0$ in ∂h_q, and therefore $s' = 0$. [18] Consequently, $t'\partial(1) = 0$. Since $\partial(1)$ of infinite order, we conclude that $t' = 0$ which contradicts its definition. Thus, λ' is of infinite order. □

Theorem 6.18. *A closed connected framed manifold M of dimension $2q+1$ with q even is cobordant to a homotopy sphere.*

Proof. We may assume that M is $q-1$ connected, and $\pi_q M$ is a torsion group. Choose a Wall representative for a generator λ of $\pi_q M$, and perform a spherical surgery to obtain M'. Then $\pi_q M/\langle\lambda\rangle \approx \pi_q M'\langle\lambda'\rangle$. Next, eliminate the infinite cyclic element λ' by a spherical surgery along a Wall sphere representing λ'. As a result, we reduce the number of cyclic summands of $\pi_q M$. In finitely many steps, we eliminate all torsion classes of $\pi_q M$. The resulting q-connected closed oriented manifold is a homotopy sphere. □

The case $m = 2q+1$ with q odd.

Now assume that q is odd. We will prove Theorem 6.19.

Theorem 6.19. *Let M be a framed closed manifold of dimension $m = 2q+1$ with q odd. Then M is cobordant to a homotopy sphere.*

We may assume that M is $(q-1)$-connected, and $\pi_q M$ is finite as all elements of infinite order can be eliminated by surgery.

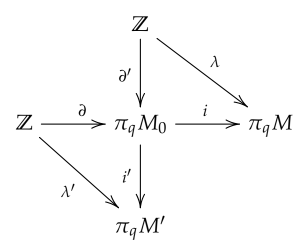
Let λ be a class in $\pi_q M$. If it is of finite order s, then $\partial(1)$ is of infinite order,[19]. Then the class $s\partial(1)$ is in $\ker i = \operatorname{im} \partial$, i.e., as in the proof of Lemma 6.17, a class $x = s\partial'(1) + t\partial(1)$ is zero in M_0 for some t. If

[15] Indeed, suppose that λ is of finite order s. Then $s(x, \lambda) = (x, s\lambda) = 0$ for any $x \in \pi_{q+1}M$. In particular, the homomorphism $j_* : \pi_{q+1}M \to \pi_{q+1}(M, M_0) \approx \mathbb{Z}$ which takes x to (x, λ) is trivial.

[16] The class $\partial(1)$ is of infinite order:

$$\pi_{q+1}M \xrightarrow{\lambda\cdot} \pi_{q+1}(M, M_0) \xrightarrow{\partial} \pi_q M_0.$$

[17] There is a commutative diagram:



The generator of the left most group $\mathbb{Z}$ is $[D^\lambda]$. Its boundary is the meridian class in the torus boundary of M_0. When included to M' it becomes λ'. In other words $i' \circ \partial(1) = \lambda'$. Similarly, $i \circ \partial'(1) = \lambda$.

[18] Note that $\partial(1) \cdot \partial(q)$ and $\partial'(1) \cdot \partial'(1)$ are 0 in $\pi_q \partial h_q$, while $\partial'(1) \cdot \partial(1) = -\partial(1) \cdot \partial'(1)$ since q is even.

[19] The argument is similar to one we had before. Namely, the homomorphism

$$\pi_{q+1}M \to \pi_{q+1}(M, M_0) \approx \mathbb{Z}$$

takes any class x to a multiple $\alpha[D_\lambda]$ of the generator of $\pi_{q+1}(M, M_0)$, where $\alpha = x \cdot \lambda$. Since λ is finite, we have $x \cdot \lambda = 0$. Therefore the above homomorphism is trivial. From the long exact sequence

$$\pi_{q+1}M \to \pi_{q+1}(M, M_0) \xrightarrow{\partial} \pi_q M_0$$

we deduce that the homomorphism ∂ is injective.

$t = 0$, then λ' is of infinite order,[20] and therefore $\pi_q M$ can be reduced by, first, trading λ for λ', and then cancelling λ'. Suppose that $t \neq 0$.

[20] If $t = 0$, then $s\partial'(1) = 0$, and therefore $\ker i' = \operatorname{im} \partial'$ is of finite order. In particular, $\ker i'$ is disjoint from $\langle \partial(1) \rangle$. On the other hand, if $s'\lambda' = 0$, then $\ker i' \cap \langle \partial(1) \rangle$ contains $s'\partial(1)$. Thus, λ' is of infinite order.

Theorem 6.20. *The order of λ' is $\pm t$.*

Proof. The order $t' > 0$ of λ' divides t as the embedding of M_0 into M' turns the equality $x = 0$ into $t\lambda' = 0$. Now, interchanging the roles of λ and λ' we get that $t'\partial(1) + s'\partial'(1) = 0$ for some integer s' such that the order s of λ divides s'. In other words, we get two equations

$$s\partial'(1) + t\partial(1) = 0,$$

$$s'\partial'(1) + t'\partial(1) = 0,$$

in $\pi_q M_0$. If $s' = 0$, then $t'\partial(1) = 0$, which contradicts $t' > 0$ since $\partial(1)$ is of infinite order. If $s' \neq 0$, then subtracting $l = s'/s$ copies of the first equation from the second equation, we get $t' = lt$. Thus $t = \pm t'$ where t' is the order of λ'. □

Recall that in ∂h_q, the sphere $\partial(1)$ is the meridian, and $\partial'(1)$ is the longitude. So if one framed surgery along S_λ produces a relation $s\partial'(1) + t\partial(1) = 0$ in $\pi_q M_0$, then there is a twisted framed surgery which produces a relation $s[\partial'(1) + \alpha\partial(1)] + t\partial(1) = 0$, or equivalently $s\partial'(1) + (t - s\alpha)\partial(1) = 0$, where α is any even integer, see Exercise 6.16. Applying the homomorphism i'_* leads to a relation $(t - s\alpha)\lambda' = 0$ in M'. Thus, unless t is divisible by s, we may choose an even integer α so that the order of λ' is strictly less than s. When t is divisible by s, we may choose a surgery so that the order of λ' is s. In particular, we may perform a framed surgery along any embedded sphere in M of dimension q in such a way that the order of $\pi_q M$ does not increase.

Now let us return to the proof of Theorem 6.19. Let p be the largest prime which divides the order of $\pi_q M$. We may perform a series of surgeries without increasing the order of $\pi_q M$ to kill $\pi_q M \otimes \mathbb{Z}_p$, see Exercise 6.15. Then $\pi_q M$ is not divisible by p anymore. Therefore such a series of surgeries decreases the order of $\pi_q M$. We may continue by induction to kill $\pi_q M$. When $\pi_q M$ is trivial, the manifold M is a homotopy sphere.

6.4 Solutions to exercises

Solution to Exercise 6.12. We need to show that the group $\pi_{m+1}(M, M_0)$ is isomorphic to $\mathbb{Z}$, and that it is generated by the class $[D_\lambda]$ of the inclusion of the disc D_λ in h_q whose boundary is the meridian of ∂h_q.

Since M is $q-1$ connected, the same is true for M_0, and therefore, the pair (M, M_0) is $q-1$ connected. Indeed, removing a tubular neighborhood of a sphere of dimension q from M results in a $q-1$ connected manifold M_0; generically any homotopy in M of a sphere of dimension $\leq q-1$ in M_0 misses the removed sphere by general position. Thus, M_0 is $q-1$ connected. By the long exact sequence, the pair (M, M_0) is also $q-1$ connected. In fact, $\pi_q(M, M_0) \approx \pi_q(M/M_0)$ for $i \leq 2q-2$.[21]

[21] The following theorem appears for example as Proposition 4.28 in [Ha02].

Theorem 6.21. *If a CW pair (X, A) is r-connected and A is s-connected, with $r, s \geq 0$, then the map $\pi_i(X, A) \to \pi_i(X/A)$ induced by the quotient map $X \to X/A$ is an isomorphism for $i \leq r+s$ and a surjection for $i = r+s+1$.*

Since the CW-complex M/M_0 is the same as the CW-complex $h_q/\partial h_q$, we conclude that in degrees $\leq 2q-2$, the homotopy groups of the pairs (M, M_0) and $(h_q, \partial h_q)$ are the same. On the other hand, we claim that $\pi_*(h_q, \partial h_q)$ is trivial for $* \leq q$ and is isomorphic to $\mathbb{Z}$ for $* = q+1$. To prove this, use the homotopy exact sequence of the pair $(h_q, \partial h_q)$:

$$\to \pi_{i+1}(h_q, \partial h_q) \to \pi_i(\partial h_q) \to \pi_i(h_q) \to .$$

Note that $\pi_i(\partial h_q) \to \pi_i(h_q)$ is surjective, and therefore

$$\pi_{i+1}(h_q, \partial h_q) \approx \pi_i(S^q \times S^q)/\pi_i(S^q \times \{0\}).$$

Consequently, the group $\pi_q(M, M_0)$ is isomorphic to $\mathbb{Z}$. □

Solution I to Exericse 6.15. We will give an argument using homology groups. Since the manifold M is $(q-1)$-connected, the groups $\pi_q M$ and $H_q M$ are isomorphic. By the Universal Coefficient Theorem, then the group $H_q(M; \mathbb{Z}_p)$ is also isomorphic to $\pi_q M \otimes \mathbb{Z}_p$. Similarly, we there is an isomorphism $H_q(M'; \mathbb{Z}_p) \approx \pi_q M' \otimes \mathbb{Z}_p$. We will denote the class $\lambda \otimes 1$ simply by λ, and similarly the class $\lambda' \otimes 1$ by λ'. By the Poincare duality theorem, the pairing

$$H_q(M; \mathbb{Z}_p) \otimes H_{q+1}(M; \mathbb{Z}_p) \longrightarrow \mathbb{Z}_p$$

is non-degenerate. In particular, since the class λ is non-trivial, there exists a class μ such that $\lambda \cdot \mu = 1$. We may represent the class μ by a chain $D_\lambda + C$ with coefficients in $\mathbb{Z}_p$, where C is a linear combination of cells in M_0. Then ∂C is a chain representing the class λ'. Therefore λ' is trivial in $H_q(M'; \mathbb{Z}_p)$. Finally, we have isomorphisms

$$H_q(M; \mathbb{Z}_p)/\langle \lambda \rangle \approx H_q(M'; \mathbb{Z}_p)/\langle \lambda' \rangle = H_q(M'; \mathbb{Z}_p),$$

which completes the proof. □

Solution II to Exercise 6.15. The argument in terms of homology groups is similar to one for homotopy groups. Indeed, the homology group

$H_{q+1}(M, M_0; \mathbb{Z}_p)$ is infinite cyclic generated by a class $[D_\lambda]$. Consider the long exact sequence

$$H_{q+1}(M;\mathbb{Z}_p) \xrightarrow{\lambda\cdot} H_q(M, M_0;\mathbb{Z}_p) \xrightarrow{\partial} H_q(M_0;\mathbb{Z}_p) \xrightarrow{i_*} H_q(M;\mathbb{Z}_p) \to 0$$

The kernel of i_* is generated by $\lambda' = \partial[D_\lambda]$. We deduce that

$$H_q(M;\mathbb{Z}_p)/\langle\lambda\rangle \approx H_q(M_0;\mathbb{Z}_p)/\langle\lambda,\lambda'\rangle \approx H_q(M';\mathbb{Z}_p)/\langle\lambda'\rangle.$$

If λ is a generator of $H_q(M;\mathbb{Z}_p)$, i.e., if λ is a non-trivial element, then the homomorphism $\lambda\cdot$ is surjective. Consequently, the map ∂ is trivial. Since $\lambda' = \partial[D_\lambda]$ is trivial, we deduce that $H_q(M';\mathbb{Z}_p) \approx H_q(M;\mathbb{Z}_p)/\langle\lambda\rangle$. By the Universal Coefficient Formula and the Hurewicz isomorphism, we have $H_q(M;\mathbb{Z}_p) = \pi_q M \otimes \mathbb{Z}_p$. □

Solution to Exercise 6.16. Suppose that $(D;\tau)$ is a base of framed surgery along $\partial D = S_\lambda$ where τ is the set $\tau_1, ..., \tau_{k+q+1}$ of perpendicular vector fields. The base of framed surgery consists of a base of surgery $(D; \tau_{k+1}, ..., \tau_{k+q+1})$ as well as additional perpendicular vector fields $\tau_1, ..., \tau_{k+1}$ over D that agree over $\partial D = S_\lambda$ with the frame vector fields of the manifold M in $\mathbb{R}^{m+k} \times \{0\}$. Given another base of framed surgery $(D; \tau'_1, ..., \tau'_{k+q+1})$, we may form the difference function $\bar\omega\colon S^q \to \mathrm{SO}_{k+q+1}$ by letting $\bar\omega(x)$ be the rotation that brings each vector field τ_i to τ'_i for every point $x \in S^q = S_\lambda$. Note that if we let x vary over all points $x \in D$, then we get an extension of $\bar\omega$ over a disc, which implies that $\bar\omega$ is null-homotopic. Conversely, every null-homotopic map $\bar\omega\colon S^q \to \mathrm{SO}_{k+q+1}$ can be used to construct a new base of framed surgery out of $(D;\tau)$, by defining new vector fields $\tau'_i = \bar\omega\tau$ over S_λ and then extending them over D.

Suppose now that we are given a base of framed surgery (D,τ) as well as a new base of surgery corresponding to a difference function $w\colon S^q \to \mathrm{SO}_{q+1}$. As we have seen, the new base of surgery can be lifted to a base of framed surgery only if the composition $\bar\omega$ of w and the inclusion $\mathrm{SO}_{q+1} \to \mathrm{SO}_{k+q+1}$ is null-homotopic.

Exercise 6.22. Prove that $[w]$ is in the kernel of the homomorphism $\pi_q\mathrm{SO}_{q+1} \xrightarrow{i} \pi_q\mathrm{SO}_{k+q+1}$ induced by the inclusion if and only if it is in the kernel of the homomorphism $\pi_q\mathrm{SO}_{q+1} \to \pi_q\mathrm{SO}_{q+2}$ induced by the inclusion. [22]

[22] Consider the map $\mathrm{SO}_{r+1} \to S^r$ which takes a rotation φ to the point $\varphi(e_1)$ on the sphere S^r. The fiber of this fibration is SO_r. Therefore, there is a long exact sequence

$$\pi_{r-1}(S^r) \to \pi_{r-2}(\mathrm{SO}_r) \to \pi_{r-2}(\mathrm{SO}_{r+1})$$

which implies, by induction, that the homomorphism $\pi_q\mathrm{SO}_{q+2} \to \pi_q\mathrm{SO}_{k+q+1}$ induced by the inclusion is an isomorphism.

The map i in Exercise 6.22 appears in the long exact sequence

$$\pi_{q+1}S^{q+1} \xrightarrow{\partial} \pi_q\mathrm{SO}_{q+1} \xrightarrow{i} \pi_q\,\mathrm{SO}_{q+2}.$$

of a fibration. Let $\mathbf{1}$ denote the generator of $\pi_{q+2}S^{q+2}$. Since the kernel of i coincides with the image of ∂, we have proved the following lemma.

Lemma 6.23. *The base of surgery with difference function w can be lifted to a base of framed surgery if and only if $[w] \in \pi_q \mathrm{SO}_{q+1}$ is a multiple of $\partial(\mathbf{1})$.*

The boundary of the solid torus neighborhood $S^q \times D^{q+1}$ of S_λ is diffeomorphic to the torus $S^q \times S^q$ with meridian $\mathfrak{m}$ and longitude $\mathfrak{l}$, see Figure 6.3. We will identify the longitude with the set $\{(x, \tau_{k+1})\}$ where x ranges over S^q, and where τ_{k+1} is a frame vector in the base of framed surgery $(D; \tau)$.

Lemma 6.24. *Suppose that $[w] = \partial(\mathbf{1})$. In particular w is the difference function for two bases of surgery (D, τ) and (D, τ') which lifts to a difference function for two bases of framed surgery. Then $w(\mathfrak{m}) = \mathfrak{m}$ and $w(\mathfrak{l}) = \mathfrak{l} + 2\mathfrak{m}$.*

Let us study the map $\bar{w}$. Consider its restriction, denoted by the same symbol, $\bar{w}\colon S^q \times S^q \longrightarrow S^q \times S^q$. Let's compute its action on homology groups. The meridian $\mathfrak{m}$ is clearly preserved, but the longitude $\mathfrak{l}$ is not. Recall that w belongs to the kernel of the homomorphism of homotopy groups of the inclusion

$$\mathrm{SO}_{q+1} \longrightarrow \mathrm{SO}_{q+2} \longrightarrow \cdots \longrightarrow \mathrm{SO}_{q+1+k}.$$

In the sequence of inclusions, each map is the fiber map of a fibration over S^{q+i}. Thus all this inclusions except for the first one are $q+1$-connected. Thus, w belongs to the kernel of the inclusion i in the diagram

$$\pi_{q+1}S^{q+1} \xrightarrow{\partial} \pi_q \mathrm{SO}_{q+1} \xrightarrow{i} \pi_q \mathrm{SO}_{q+2}.$$

Consequently, w belongs to the image of ∂. Let $\mathbf{1}$ be the generator of $\pi_{q+2}S^{q+2} = \mathbb{Z}$. Then w is a multiple of $\partial(\mathbf{1})$. We are interested in seeing how SO_{q+1} acts on S^{q+1}. There is a map p from SO_{q+1} to S^{q+1} that associates to a rotation r the vector $r(e_1)$. In fact, there is a commutative diagram: [23]

$$\begin{array}{ccccc}
\pi_{q+1}S^{q+1} & \xrightarrow{\partial} & \pi_q \mathrm{SO}_{q+1} & \longrightarrow & \pi_q \mathrm{SO}_{q+2} \\
\downarrow{=} & & \downarrow{p_*} & & \downarrow \\
\pi_{q+1}S^{q+1} & \longrightarrow & \pi_q S^q & \longrightarrow & \pi_q V_2(\mathbb{R}^{q+2})
\end{array}$$

[23] There is a pair of fibrations:

$$\begin{array}{ccccc}
\mathrm{SO}_{q+1} & \longrightarrow & \mathrm{SO}_{q+2} & \longrightarrow & S^{q+1} \\
\downarrow{p} & & \downarrow & & \downarrow \\
S^q & \longrightarrow & V_2(\mathbb{R}^{q+2}) & \longrightarrow & S^{q+1}
\end{array}$$

We are interested in $p_*\partial(1)$, and by the commutativity of the diagram we see that it is multiplication by 2. Thus $\bar{w}(\mathfrak{l}) = \mathfrak{l} + 2\mathfrak{m}$. In other words, $\mathfrak{m}' = \mathfrak{m}$ and $\mathfrak{l}' = \mathfrak{l} + 2\mathfrak{m}$. $\square$

7

Characteristic classes of vector bundles

7.1 Vector bundles over topological spaces

Fix a non-negative integer $n \leq m+k$. Let f be a function on a topological subspace X in $\mathbb{R}^{m+k}$ that associates with each point x in X an affine n-space $E_x \subset \mathbb{R}^{m+k}$ containing the point x. Of course, each E_x is then a vector space with origin at x. We will assume that f is *continuous* in the sense that over an open neighborhood $U \subset \mathbb{R}^{m+k}$ of every point $x \in X$ there are n continuous vector fields $v_1, ..., v_n$ such that the affine space E_y is the span of the vectors $v_1(y), ..., v_n(y)$ for every point y in $X \cap U$.

We may think of the function f from a fairly different perspective. To begin with, the union E of the affine subspaces E_x is a subspace of $\mathbb{R}^{m+k} \times \mathbb{R}^{m+k}$ of pairs (x, v) of points $x \in X$ and vectors $v \in E_x$. Forgetting the second component in (x, v) defines a continuous map $\pi : E \to X$, called a *vector bundle*. The space E is called the *total space*, X the *base space*, and π the *projection* of the vector bundle.[1] The vector spaces $E_x = \pi^{-1}(x)$ are called the *fibers*, while the dimension n of fibers E_x is said to be the *dimension* of the vector bundle.

[1] It is also common to say that the triple $\xi = (E, X, \pi)$ is a vector bundle, or even that E is a vector bundle.

We have already seen some important examples of vector bundles. Say, the tangent bundle TM of a manifold M is a vector bundle. Its total space $E = TM$ consists of pairs (x, v) of points in M and vectors v tangent to M at x. Another important example of a vector bundle is the perpendicular vector bundle $T^{\perp}M$. Its total space consists of pairs (x, v) of points x in M and vectors v perpendicular to T_xM at x. Given a topological space X in the horizontal subspace $\mathbb{R}^{m+k}$ of $\mathbb{R}^{m+k} \times \mathbb{R}^n$, the space ε^n of pairs (x, v) of points $x \in X$ and vertical vectors v is the total space of another, so-called *trivial*, vector bundle over X of dimension n.

Many operations on vector spaces extend to operations on vector bundles. For example, if E and F are vector bundles over a topological space X in $\mathbb{R}^{m+k}$ such that the fibers E_x and F_x share no non-zero vectors for each $x \in X$, then $E \oplus F$ is the vector bundle over X that consists of the family of vector spaces $E_x \oplus F_x$.

A *map* $f: E \to F$ of a vector bundle over a topological space X into a vector bundle over a topological space Y is a continuous fiberwise preserving map[2] of topological spaces such that $f|E_x$ is a linear homomorphism $E_x \to E_y$ of vector spaces for $y = y(x)$ and each x. We say that f is *injective, surjective*, or an *isomorphism* if each $f|E_x$ is.

[2] A map $f: E \to F$ of vector bundles is *fiberwise preserving* if for every point $x \in X$ there is a point $y \in Y$ such that f sends the fiber E_x to the fiber F_y.

Let $f: X \to Y$ be a map of a topological space $X \in \mathbb{R}^m$ to a topological space $Y \in \mathbb{R}^n$. We may place X in $\mathbb{R}^m \times \mathbb{R}^n$ by sending x to $(x, f(x))$. Then the map f is the projection of X to Y in the sense that it is given by $(x, y) \mapsto y$. Given a vector bundle F over Y, the so-called *pullback* vector bundle $E = f^*F$ is the vector bundle that consists of the family $\{E_x\}$ of spaces $E_x = \{x\} \times F_{f(x)}$ parametrized by $x \in X$.

7.2 Characteristic classes of vector bundles

Let us recall that a CW complex X is constructed by induction by means of its skeleta. The the 0-th skeleton $X^{(0)}$ of X consists of a discrete set of points. When the k-th skeleton $X^{(k)}$ has been constructed, one chooses attaching maps $\varphi_\alpha: \partial D^{k+1}_\alpha \to X^{(k)}$, and defines $X^{(k+1)}$ to be the union of $X^{(k)}$ and the discs D^{k+1}_α subject to the identification $x = \varphi_\alpha(x)$ for each $x \in \partial D^{k+1}_\alpha$. We say that $X^{(k+1)}$ is obtained from $X^{(k)}$ by attaching the discs ∂D^{k+1}_α along their boundaries by means of attaching maps φ_α. The CW complex X is the union of all its skeleta.

Every smooth manifold has a structure of a CW-complex. A CW-structure on a manifold M allows us to define cellular cohomology of M with coefficients in R. We will be interested in the cases where R is either $\mathbb{Z}$, $\mathbb{Z}_2$ or $\mathbb{Q}$. Let us briefly review the definition. A k-th cochain on M is a function that associates with each cell D^k_α a value in R. The set of cochains forms a free group $C^k(M)$ under the operation of taking the sum of functions. There is a so-called coboundary homomorphism $\delta_k: C^k(M) \to C^{k+1}(M)$. The cochains in the group $\operatorname{Im} \delta_{k-1}$ are said to be *coboundaries*, while the cochains in the group $\operatorname{Ker} \delta_k$ are *cocycles*. The k-th cohomology group of M is defined to be the factor group $\ker \delta_k / \operatorname{im} \delta_{k-1}$. In other words, a cohomology class is a cocycle defined up to coboundaries.

7.2.1 The Euler class

Given an oriented vector bundle $\xi = (E, M, \pi)$ of dimension n over a manifold of dimension m, the Euler class $e(\xi)$ is a cohomology class in $H^n(M)$ with coefficients in $\mathbb{Z}$. It is the obstruction to constructing a unit vector field v over M such that for each $x \in M$ the vector $v(x)$ is in the plane E_x.

When possible, the vector field is constructed by induction over the subsequent skeleta of M. To begin with, there is a vector field v over the discrete set of points M^0. Suppose that the vector field has been constructed over the $(k-1)$-st skeleton of M. To extend it over the next skeleton, consider a cell D^k_α of dimension k. Note that since the boundary of D^k_α is in the $(k-1)$-st skeleton, the vector field v has been constructed over ∂D^k_α.

We note that the vector bundle ξ over the disc D^k_α is trivial, i.e., the total space of ξ over D^k_α is $D^k_\alpha \times \mathbb{R}^n$ while the vector bundle projection is one onto the first factor. Therefore any vector field of ξ over D^k_α can be identified with a function $D^k_\alpha \to \mathbb{R}^n$. Similarly, a unit vector field is a function $D^k_\alpha \to S^{n-1}$. In particular, the unit vector field v extends from ∂D^k_α over the disc D^k_α if and only if the corresponding function $\partial D^k_\alpha \to S^{n-1}$ extends to a function $D^k_\alpha \to S^{n-1}$.

If $k < n-1$, then any map of the sphere ∂D^k_α to S^{n-1} extends to a map of the disc D^k_α. Therefore, there is no obstruction to constructing the unit vector field v over the $(n-1)$-st skeleton of M. When $k = n$, the map $\partial D^n_\alpha \to S^{n-1}$ extends over the disc D^n_α if and only its degree d_α is 0. We define the obstruction cochain $e \in C^n(M)$ by associating with each cell D^n_α the degree d_α. It follows that the obstruction cochain e is actually a cocycle, representing a cohomology class $e(\xi)$. The cohomology class $e(\xi)$ neither depends on the CW-structure on M, nor on the choice of the vector field v over the $(n-1)$-skeleton of M.[3]

<u>**Exercise**</u> **7.1.** Suppose that the vector bundle ξ is the tangent bundle of the manifold M. Show that under an appropriate correspondence $H^m(M) \approx \mathbb{Z}$ the Euler class $e(\xi)$ corresponds to the Euler characteristics $\chi(M) = e(\xi)[M]$ of the manifold M.[4]

[3] For proofs of the mentioned properties of the Euler class, see references in the section Further Reading.

[4] **Hint for Exercise 7.1.** Indeed, choose a vector field v in TM that is trivial at a unique point x. Then, by a definition of the Euler characteristics $\chi(M)$ of M, the index of the singular point x of v is the number $\chi(M)$. On the other hand, we may choose a CW-structure on M so that the point x is the center of an m-cell. By the definition of the index of a singular point of a vector field, the cochain e associates with all m-cells of M the value 0, except for the m-cell containing x, with which e associates the value $\chi(M)$.

7.2.2 Stiefel-Whitney classes

Let $\xi = (E, M, \pi)$ be a vector bundle of dimension n over a manifold of dimension m. We will define a cohomology class $w_j(\xi)$ on M with coefficients in $\mathbb{Z}_2$ which is an invariant describing the vector bundle ξ. If we put $n - k = j - 1$, then $w_j(\xi)$ is the obstruction to the existence of k vector fields over the j-th skeleton.

More precisely, there exist k orthonormal vector fields of ξ over the $(n-k)$-skeleton $M^{(n-k)}$ of M. Indeed, to begin with, the vector bundle ξ admits a unit vector field v_1 over $M^{(n-1)}$. Let $\xi/\langle v_1 \rangle$ be the vector bundle of dimension $n-1$ over $M^{(n-1)}$ that consists of vectors in ξ orthogonal to v. The vector bundle $\xi/\langle v_1 \rangle$ admits a unit vector field v_2 over $M^{(n-2)}$. Continuing by induction, we construct k orthonormal vector fields $v_1, v_2, ..., v_k$ of ξ over $M^{(j-1)}$.[5]

[5] Recall that $j - 1 = n - k$.

Let D^j_α be a cell of dimension j in M. Its boundary is in the skeleton $M^{(n-k)}$, over which there are k orthonormal vector fields. In fact, the vector fields $v_1, ..., v_{k-1}$ has been constructed over $M^{(j)}$, and, in particular, over D^j_α. We are to determine an obstruction to extending over D^j_α the vector field v_k in the vector bundle $\xi/\langle v_1, ..., v_{k-1} \rangle$. We may identify the vector field v_k over the boundary ∂D^j_α with a function $f_\alpha : \partial D^j_\alpha \to S^{j-1}$. By definition, the cochain $w_j(\xi)$ associates with D^j_α the degree of f_α mod 2. It can be shown that the cochain $w_j(\xi)$ is a cocycle which neither depends on the CW-structure of M nor on the choices of vector fields. Therefore, the cochain $w_j(\xi)$ represents a cohomology class, which we also denote by $w_j(\xi)$.

When ξ is the tangent bundle of the manifold M, we simply write $w_j(M)$ instead of $w_j(TM)$. It is also convenient to write $w(\xi)$ for the formal linear combination $1 + w_1(\xi) + w_2(\xi) + \cdots$.

Example 7.2. The class $w_1(M)$ is the obstruction to the existence of m linearly independent tangent vector fields over the 1-skeleton of M. In other words, $w_1(M) = 0$ if and only if the manifold M is orientable.

Example 7.3. The class $w_m(M)$ is an obstruction to constructing a nowhere zero tangent vector field over M. It follows that $w_m(M) = 0$ if and only if the Euler characteristics of M is even.[6]

[6] Compare the definition of $e(M)$ with that of $w_m(M)$ to see that $w_m(M)$ is the mod 2 reduction of the Euler class.

It is remarkable that Stiefel-Whitney classes can completely be described by their four axioms:

Axiom I: For each vector bundle $E \to X$, there are cohomology classes $w_i(E) \in H^i(X;\mathbb{Z}_2)$ such that $w_0(E) = 1$ and $w_i(E) = 0$ for all i greater than the dimension of the fiber bundle E.

Axiom II: Let $E \to X$ and $F \to Y$ be two vector bundles. Suppose that for a continuous map $f: X \to Y$ there is a fiberwise isomorphism $E \to F$ of vector bundles sending each vector space E_x isomorphically to $F_{f(x)}$. Then $w(E) = f^*w(F)$.[7]

Axiom III: Formally, given two vector bundles E and F over a topological space X, there is an equality $w(E \oplus F) = w(E)w(F)$.

Axiom IV: For the non-trivial vector bundle E of dimension 1 over the circle S^1, the class $w_1(E)$ is non-trivial.

Example 7.4. Let us recall that $\mathbb{R}P^n$ is the quotient of the sphere S^n of unit vectors x in $\mathbb{R}^{n+1}$ under the identification $x = -x$. Its tangent bundle $T\mathbb{R}P^n$ consists of equivalence classes of pairs of vectors (x, y) where $x \in S^n$ and $y \in \mathbb{R}^{n+1}$ is orthogonal to x, subject to the identification of (x, y) and $(-x, -y)$. Similarly, the canonical vector bundle γ consists of equivalence classes of pairs (x, α) where $x \in S^n$ and $\alpha \in \mathbb{R}$, subject to the identification $(x, \alpha) = (-x, -\alpha)$, where the vector space γ_x over the point $[x]$ in $\mathbb{R}P^n$ consists of vectors αx at x that are identified with vectors $(-\alpha)(-x)$ at $-x$. Note that we may also represent the trivial bundle ϵ over $\mathbb{R}P^n$ by the space of pairs (x, α) subject to the identification $(x, \alpha) \sim (-x, \alpha)$ where the vector space ε_x consists of vectors αx at x that are identified with vectors $\alpha(-x)$ at $-x$. The trivial vector bundle ε has a nowhere zero vector field $x \mapsto (x, 1)$.

The Axioms I, II and IV actually imply that for the canonical vector bundle γ over $\mathbb{R}P^n$ the total Stiefel-Whitney class $w(\gamma)$ is $1 + a$, where a is the generator of the first cohomology group of $\mathbb{R}P^n$.[8] On the other hand, the vector bundle $T\mathbb{R}P^n \oplus \varepsilon$ over $\mathbb{R}P^n$ is isomorphic to $(n+1)\gamma = \gamma \oplus \cdots \oplus \gamma$.[9] In particular, by the Axioms I and III, the total Stiefel-Whitney class of $\mathbb{R}P^n$ is

$$1 + w_1 + w_2 + \cdots = (1 + a)^{n+1}.$$

For example, since $(1 + x)^3 = 1 + 3x + 3x^2 + x^3$, the Stiefel-Whitney classes of $\mathbb{R}P^2$ are $w_1 = x$ and $w_2 = x^2$.[10]

[7] In particular, the Axiom II asserts that when $f: X \to Y$ is an inclusion, the class $w(E)$ is the restriction of the class $w(F)$ to X. Axiom II can also be stated as $w(f^*F) = f^*w(F)$.

[8] Indeed, by the Axiom I the class $w(\gamma)$ is of the form $1 + \lambda a$ for some $\lambda \in \mathbb{Z}_2$. By Axiom II, the class $w(\gamma)$ restricts over $\mathbb{R}P^1 \subset \mathbb{R}P^n$ to the total Stiefel-Whitney class of the non-trivial line bundle. Therefore, $\lambda = 1$ by Axiom IV.

[9] The vector bundle $(n+1)\gamma$ consists of equivalence classes of pairs (x, y) where $x \in S^n$ and $y \in \mathbb{R}^{n+1}$ subject to the identification $(x, y) = (-x, -y)$. Clearly, $T\mathbb{R}P^n \subset (n+1)\gamma$, while the quotient of the two bundles is the trivial line bundle that consists of pairs (x, y) with $y = \lambda x$ for $\lambda \in \mathbb{R}$ subject to the identification $(x, \lambda x) = (-x, \lambda(-x))$.

[10] Note that $w_1 \neq 0$ means that $\mathbb{R}P^2$ is non-orientable, while $w_2 \neq 0$ means that the Euler characteristics of $\mathbb{R}P^2$ is odd. In the calculation we implicitly used that $3 \equiv 1 \bmod 2$ and that x^3 is a class in the zero cohomology group $H^3(\mathbb{R}P^2)$.

7.2.3 *Chern classes*

A complex structure on a vector bundle ξ over a manifold M is a choice of a fiberwise homomorphism $F: \xi \to \xi$ that satisfies the condition that $F \circ F = -\mathrm{Id}$. The homomorphism F plays the role of a multiplication by the complex number i so that every fiber of ξ is isomorphic to C^n where n is the complex dimension of fibers of ξ. We will occasionally write iv for $F(v)$.

For a complex vector bundle ξ, i.e., a vector bundle with a fixed complex structure, we may define cohomology classes $c_j(\xi)$ as obstructions to extending $k = m - j + 1$ complex linearly independent vector fields from the $(2j-1)$-st skeleton of M to its $2j$-th skeleton. In contrast to the Stiefel-Whitney classes w_j, the Chern classes $c_j(\xi)$ are cohomology classes of degree $2j$ with coefficients in $\mathbb{Z}$. Notice also that in the definition of the Chern classes the linear independence of vectors is considered to be over complex numbers.

Example 7.5. The class $c_n(\xi)$ is the obstruction to the existence of a nowhere zero vector field in ξ. Therefore $c_n(\xi)$ is the Euler class of the complex vector bundle ξ.

Example 7.6. When ξ is a complex vector bundle, the coefficient reduction $\mathbb{Z} \to \mathbb{Z}_2$ reduces the total Chern class $1 + c_1 + c_2 + \cdots$ to the total Stiefel-Whitney class $1 + w_1 + w_2 + \cdots$. [11]

[11] Note that since Chern classes are even dimensional classes, the identity implies that all odd dimensional Stiefel-Whitney classes of a complex vector bundle are trivial. To prove the statement of Example 7.6, observe that the existence of k linearly independent complex vector fields $v_1, \ldots, v_k$ implies the existence of $2k$ linearly independent real vector fields $v_1, iv_1, \ldots, v_k, iv_k$.

As in the case of Stiefel-Whitney classes, there is an important formula for the Chern classes of the complex projective space $\mathbb{C}P^n$:

$$1 + c_1 + c_2 + \cdots = (1+e)^{n+1}$$

where e is an appropriate generator of $H^2(\mathbb{C}P^n) \approx \mathbb{Z}$.

7.2.4 *Pontryagin classes*

When ξ is a real vector bundle over M, we may use the fiberwise tensor product with $\mathbb{C}$ to construct a complex vector bundle $\xi \otimes \mathbb{C}$ over M. By definition the k-th Pontryagin class of ξ is the cohomology class $p_k(\xi) = (-1)^k c_{2k}(\xi \otimes \mathbb{C})$ of degree $4k$ with coefficients in $\mathbb{Z}$.

If the vector bundle ξ is already complex, then we may calculate both the Chern and Pontryagin classes of ξ. There is an identity:

$$1 - p_1 + p_2 - \cdots = (1 - c_1 + c_2 - \cdots)(1 + c_1 + c_2 + \cdots).$$

This identity implies that the Pontryagin classes of $\mathbb{C}P^n$ satisfy the formula

$$1 + p_1 + p_2 + \cdots = (1 + e^2)^{n+1}.$$

Finally, the coefficient homomorphism $\mathbb{Z} \to \mathbb{Q}$ turns Pontryagin classes to rational Pontryagin classes $p_k(\xi) \in H^{4k}(M; \mathbb{Q})$.

7.3 *The Thom theorem on orientable cobordism groups*

The set of cobordism classes of oriented closed manifolds of dimension m forms a group Ω_m with the disjoint union operation. Its zero element is represented by the class of the empty manifold, while the inverse of the class of a manifold M is the class of the manifold M taken with the opposite orientation. Note that the class of the empty manifold is represented by any oriented closed manifold of dimension m bounding a manifold of dimension $m + 1$.

Exercise 7.7. Show that $\Omega_1 = \Omega_2 = 0$.

In fact, the groups Ω_m for $m \geq 0$ form a ring Ω. In terms of representatives, the operation is given by taking the product of manifolds. The ring $\Omega \otimes \mathbb{Q}$ is completely known.

Theorem 7.8. *The ring $\Omega \otimes \mathbb{Q}$ is the polynomial ring $\mathbb{Q}[\mathbb{C}P^2, \mathbb{C}P^4, \mathbb{C}P^6, ...]$.*

For example, the group $\Omega_4 \otimes \mathbb{Q}$ is generated by the cobordism classes of manifolds $\mathbb{C}P^2 \times \mathbb{C}P^2$ and $\mathbb{C}P^4$, while the group $\Omega_5 \times \mathbb{Q}$ is trivial. Furthermore, the cobordism class in $\Omega_{4m} \otimes \mathbb{R}$ of a manifold M is completely determined by the Pontryagin numbers of M, i.e, the numbers $p_{i_1} \cdots p_{i_k}[M]$ where p_{i_j} are Pontryagin classes with $i_1 + \cdots i_k = m$.

Theorem 7.9 (Thom)**.** *Two closed orientable manifolds represent the same class in $\Omega \otimes \mathbb{Q}$ if and only if all their Pontryagin numbers are the same.*

Theorem 7.9 implies that any invariant of cobordism classes can be expressed in terms of Pontryagin numbers. We have seen that the signature of an oriented closed $4m$-manifold does not change under cobordisms. Hence, it is determined by the Pontryagin classes, i.e., $\sigma(M) = L_m(p_1, ..., p_m)[M]$ for some polynomial L_m. Using the multiplicative properties of L_m and the fact that the signature $\sigma = 1$ on the generators $\mathbb{C}P^{i_1} \times \cdots \times \mathbb{C}P^{i_k}$ of $\Omega_* \otimes \mathbb{R}$, Hirzebruch computed all polynomials L_m.

Exercise 7.10. Prove that $L_1 = p_1/3$ and $L_2 = (7p_2 - p_1^2)/45$. [12]

7.4 *Example: Pontryagin classes of almost framed manifolds*

In what follows we will need the following example of Kervaire-Milnor, which the reader may skip in the first reading.

Let ξ be a real vector bundle of dimension n over a manifold M of dimension $4q < n$. Suppose that ξ admits n orthonormal vector fields over the closure M_0 of the complement to a closed disc $D \subset M$. Then ξ can be sued by means of a clutching function $f_\xi : \partial D \to \mathrm{SO}_n$ from $D \times \mathbb{R}^n$ and $M_0 \times \mathbb{R}^n$ by identifying $(x, v) \in D \times \mathbb{R}^n$ with $(x, f_\xi(v)) \in M_0 \times \mathbb{R}^n$. The map f_ξ itself represents an element $[f_\xi]$ in the homotopy group $\pi_{4q-1}\mathrm{SO}_n = \mathbb{Z}$.

Since there are already n independent vector fields over M_0, the Pontryagin classes $p_1, ..., p_{q-1}$ of the vector bundle ξ are trivial. Let us calculate the class $p_q(\xi)$. We will need information about the homotopy groups of the unitary group U_k. To begin with, $\pi_i U_k = \pi_i(U_{k+1})$ when $i < 2k$.[13] In other words, the homotopy groups of U_k stabilize as k tends to infinity. The stable groups are denoted by $\pi_i(U)$. It is known that $\pi_i(U)$ is trivial for i even, and it is $\mathbb{Z}$ otherwise. In the border case $i = 2k$, we have $\pi_i U_k = \mathbb{Z}_{k!}$. The homotopy groups of $\mathrm{U}_k/\mathrm{SO}_k$ also stabilize, and for $j > 2$ the groups $\pi_j(\mathrm{U}/\mathrm{SO}) = \pi_{j-2}\mathrm{SO}$ depend only on the residue class of j mod 8. Namely, for the residue classes $0, ..., 7$ the homotopy groups are $0, \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0$.

Following the definition of p_q, we replace the real vector space ξ with its complex counterpart $\xi \otimes \mathbb{C}$, and observe that $\xi \otimes \mathbb{C}$ is built from $D \times \mathbb{C}^n$ and $M_0 \times \mathbb{C}^n$ by means of a clutching function $f_{\xi\otimes\mathbb{C}}$, which represents an element $[f_{\xi\otimes\mathbb{C}}]$ in $\pi_{4q-1}U_n$.[14] In fact the clutching function $f_{\xi\otimes\mathbb{C}}$ is the composition of the clutching function $f_\xi : S^{4q-1} \to \mathrm{SO}_n$ and the inclusion $\mathrm{SO}_n \to U_n$. By definition, $p_q(\xi) = (-1)^q c_{2q}(\xi \otimes \mathbb{C})$. Now, following the definition of c_{2q}, we assume that $n - 2q$ vector fields have been constructed over M, and an additional vector field v have been constructed over M_0, and, in particular, over ∂D. The cochain c_{2q} may associate a non-trivial value only with D. Over D the value of c_{2q} is the obstruction to extending v from ∂D over D. Furthermore, $\xi \otimes \mathbb{C}$ restricts over D to a trivial vector bundle $D \times \mathbb{C}^n$, and we may assume that the $(n - 2q)$ vector fields over D are the coordinate vector fields in $D \times \mathbb{C}^{n-2q}$. Thus, we need to extend v in $D \times \mathbb{C}^{2q}$.

[12] **Hint for Exercise 7.10.** Let $p(M) = 1 + p_1(M) + p_2(M) + \cdots$ be the formal sum of Pontryagin classes of a manifold M. Recall that $p(\mathbb{C}P^n) = (1 + e^2)^{n+1}$ where e is the generator of $H^2\mathbb{C}P^n$. We have

$$p(\mathbb{C}P^2) = (1 + e^2)^3 = 1 + 3e^2,$$

$$p(\mathbb{C}P^4) = (1 + e^2)^5 = 1 + 5e^2 + 10e^4.$$

Since there is no torsion,

$$p(\mathbb{C}P^2 \times \mathbb{C}P^2) = p(\mathbb{C}P^2)p(\mathbb{C}P^2) =$$

$$= (1 + e_1^2)^3(1 + e_2^2)^3$$

$$1 + (3e_1 + 3e_2) + (3e_1^4 + 9e_1^2e_2^2 + 3e_2^4).$$

Thus, one deduce the table of values:

	$\mathbb{C}P^2 \times \mathbb{C}P^2$	$\mathbb{C}P^4$
p_1^2	$18e_1^2e_2^2$	$25e^4$
p_2	$9e_z^2e_2^2$	$10e^4$

Evaluate $L_2 = a_1p_1^2 + a_2p_2$ on $\mathbb{C}P^4$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$ to find a_1 and a_2.

[13] Indeed, there is a fibration $\pi : U_{k+1} \to S^{2k+1}$ that associates with a unitary transformation h the image $h(e_1)$ of the first coordinate vector. The fiber of the fibration is the unitary transformation U_k of the remaining coordinate vectors. Thus, from the exact sequence of the fibration we deduce that $\pi_i U_k = \pi_i U_{k+1}$ for $i < 2k$.

[14] It is easy to relate the number $[f_\xi]$ with the number $[f_{\xi\otimes\mathbb{C}}]$ Indeed, consider the exact sequence of the fibration $U \to U/SO$ with fiber SO:

$$\pi_{4q}\mathrm{U}/\mathrm{SO} \to \pi_{4q-1}\mathrm{SO} \to \pi_{4q-1}\mathrm{U} \to \pi_{4q-1}\mathrm{U}/\mathrm{SO}.$$

The first three groups are $0 \to \mathbb{Z} \to \mathbb{Z}$, while the last group is either 0 when q is even, or $\mathbb{Z}_2$ when q is odd. Thus, the homomorphism i_* induced by the inclusion $i : \mathrm{SO} \to \mathrm{U}$ takes $[1] \in \pi_{4q-1}\mathrm{SO}$ to $[a_q]$, where a_q is 1 when q is even, and a_q is 2 when q is odd. Finally, since $f_{\xi\otimes\mathbb{C}} = i \circ f_\xi$, we deduce that $[f_{\xi\otimes\mathbb{C}}] = a_q[f_\xi]$.

Consequently, the obstruction is the degree of the composition

$$\partial D \xrightarrow{f_{\xi\otimes\mathbb{C}}} U_{2q} \xrightarrow{\pi} S^{4q-1},$$

where the second map π associates with a transformation h the image $h(e_1)$ of the first coordinate vector. From the long exact sequence of the fibration π, it follows that the degree of the composition is $(2q-1)!$.[15] Thus, the q-th Pontryagin number of ξ is $p_q(\xi)[M] = (2q-1)![f_{\xi\otimes\mathbb{C}}]$, where $[f_{\xi\otimes\mathbb{C}}]$ is an integer in $\pi_{4q-1}U_{2q} = \mathbb{Z}$. Finally, let us express $p_q(x)$ in terms of the clutching function f_ξ.[16]

Theorem 7.11. *Let M be a manifold of dimension $4q$, and ξ a vector bundle of dimension $n > 4q$ over M that admits n orthonormal vector fields over the complement to a point. Then $p_q(\xi)[M] = a_q(2q-1)! \cdot [f_\xi]$, where $a_q = 2$ when q is odd, and $a_q = 1$ otherwise.*

[15] Indeed, consider the long exact sequence

$$\pi_{4q-1}U_{2q} \to \pi_{4q-1}S^{4q-1} \to \pi_{4q-2}U_{2q-1} \to \pi_{4q-2}U_{2q}$$

of the fibration $\pi\colon U_{2q} \to S^{4q-1}$. It translates to

$$\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_{(2q-1)!} \to 0,$$

which proves the claim.

[16] We only need to use use the equality $[f_{\xi\otimes\mathbb{C}}] = a_q[f_\xi]$.

8

Exotic spheres: discovery of exotic spheres S^7

In fifties Milnor surprised mathematical community with a remarkable discovery of exotic spheres. These are smooth manifolds that are homeomorphic but not diffeomorphic to the standard sphere S^7. In this chapter we will go over the Milnor's surprising discovery.

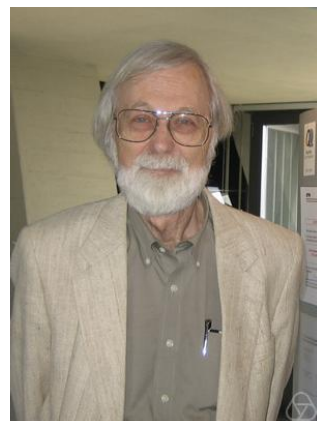


Figure 8.1: John Milnor

Though the geometric construction of exotic spheres is quite explicit, we will still need to perform calculations of algebraic invariants—such as the signature of a manifold—for which we will need new algebraic techniques. To this end, we will introduce the notion of the vector bundle (§7.1), characteristic classes of vector bundles (§7.2), and state without a proof the Thom theorem on the cobordism group of orientable manifolds (§7.3).

The reader familiar with characteristic classes may skip the first three introductory sections and go directly to section 8.1 where we give geometric interpretations of maps $\mathrm{SU}(2) \to \mathrm{SO}(3)$ and $\mathrm{Spin}(4) \to \mathrm{SO}(4)$. Milnor used these two maps to construct a number of exotic spheres of dimension 7. We cover Milnor's construction in detail in the last two sections of the chapter. The presented line of reasoning is the same as in the original paper by Milnor, though the emphasis in the present exposition is to explicit geometric constructions rather than to algebraic proofs.

Later we will see other constructions of exotic spheres which might be simpler than the origin construction by Milnor.

8.1 Maps of SU(2) and Spin(4) to orthogonal groups

We may represent the quaternions $1, i, j$ and k respectively by the matrices

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}, \quad \begin{bmatrix}0 & i\\ i & 0\end{bmatrix}, \quad \begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}.$$

In particular, the quaternion $a + bi + cj + dk$ is represented by the matrix

$$\begin{bmatrix}a + id & -b + ic\\ b + ic & a - id\end{bmatrix}.$$

One of the nice properties of this map is that the unit sphere $S^3 = \{a^2 + b^2 + c^2 + d^2 = 1\}$ in the space of quaternions becomes the unitary group SU(2).[1]

[1] To prove the claim, notice that the determinant of the matrix representing the quaternion $a + bi + cj + dk$ is precisely $\{a^2 + b^2 + c^2 + d^2 = 1\}$.

8.1.1 The map SU(2) → SO(3)

To define the map $\mathrm{SU}(2) \to \mathrm{SO}(3)$, let us identify the space $\mathbb{R}^3_{b,c,d}$ with the space of so-called pure quaternions $bi + cj + dk$. Then a quaternion $A \in \mathrm{SU}(2)$ of length 1 defines a transformation of $\mathbb{R}^3$ by the rule $X \mapsto AXA^{-1}$. This transformation is actually a rotation.

Exercise 8.1. Show that the transformation $X \mapsto AXA^{-1}$ is a rotation. Show that it preserves the vector (b, c, d), and therefore it is a rotation in the plane perpendicular to (b, c, d). Show that the angle θ of rotation is determined by $a = \cos(\frac{1}{2}\theta)$, where the rotation is performed in the counterclockwise direction when viewed from the tip of the vector (b, c, d). [2]

[2] We will prove in the next subsection that even a more general transformation is a rotation. On the other hand, for the vector $(b, c, d) = A - a$ we have $A(A - a)A^{-1} = AAA^{-1} - A(-a)A^{-1} = A - a$, which means that the rotation defined by A preserves the vector $A - a$, and therefore, the transformation of $X \mapsto AXA^{-1}$ is a rotation in the plane perpendicular to (b, c, d).

8.1.2 The map Spin(4) → SO(4)

We are interested in the group $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$ acting on the space $\mathbb{R}^4$ of 2×2-matrices with basis $\langle 1, i, j, k\rangle$. The action of an element (A, B) on a matrix $X \in \mathbb{R}^4$ is $X \mapsto AXB$. Notice that an element (A, B) is the product of pairs (A, I) and (I, B). Furthermore, in appropriate basis with coordinates $\{a, b, c, d\}$, the matrix $B \in \mathrm{SU}(2)$ is diagonal with elements $e^{i\theta}$ and $e^{-i\theta}$ on the diagonal. With respect to this basis, the action of (I, B) is easily described.[3] It is a rotation through the angle θ in the planes (a, d) and (b, c). The action of (A, I) is similar.[4] If the basis is chosen so that A is diagonal with elements $e^{i\varphi}$

[3] The transformation $X \mapsto IXB$ is given by

$$\begin{bmatrix}a + id & *\\ b + ic & *\end{bmatrix} \mapsto$$
$$\begin{bmatrix}a + id & *\\ b + ic & *\end{bmatrix}\begin{bmatrix}e^{i\theta} & 0\\ 0 & e^{-i\theta}\end{bmatrix} =$$
$$\begin{bmatrix}(a + id)e^{i\theta} & *\\ (b + ic)e^{i\theta} & *\end{bmatrix}$$

[4] The transformation $X \mapsto AXI$ is given by

$$\begin{bmatrix}a + id & *\\ b + ic & *\end{bmatrix} \mapsto$$
$$\begin{bmatrix}e^{i\theta} & 0\\ 0 & e^{-i\theta}\end{bmatrix}\begin{bmatrix}a + id & *\\ b + ic & *\end{bmatrix} =$$
$$\begin{bmatrix}(a + id)e^{i\theta} & *\\ (b + ic)e^{-i\theta} & *\end{bmatrix}$$

and $e^{-i\varphi}$ on the diagonal, then (A, I) rotates the plane (a, d) through the angle φ, and rotates the plane (b, c) through the angle $-\varphi$.

In other words, each of the two pairs (A, I) and (I, A) define rotations in two orthogonal planes, say P_1 and P_2, in $\mathbb{R}^4$. Furthermore, since the two planes are the same for the rotation determined by any element (A^h, A^i), in order to determine P_1 and P_2 we may assume that $h = 1$ and $i = -1$. The rotation (A, A^{-1}) rotates P_1 through the angle 2θ and fixes any vector in the plane P_2. On the other hand, when the action of (A, A^{-1}) is restricted to $\mathbb{R}^3$, we have already found that it is a rotation in the plane orthogonal to the vector (b, c, d), where $A = (a, b, c, d)$. Thus, in general the plane P_1 is the plane in $\mathbb{R}^3$ orthogonal to the imaginary part of A, while P_2 is its orthogonal complement.

Let us now briefly calculate the angle of rotation θ of (I, A). In the diagonalizing basis, the diagonal elements of $A \in \mathrm{SU}(2)$ are $e^{i\theta}$ and $e^{-i\theta}$. In particular, its characteristic polynomial is $\lambda^2 - \lambda(e^{i\theta} + e^{-i\theta}) + 1$. On the other hand, in the standard basis the characteristic polynomial is $\lambda^2 - 2a\lambda + 1$, which implies that $a = \cos\theta$.

It is worth mentioning that the map $\mathrm{Spin}(4) \to \mathrm{SO}(4)$ is a homomorphism with kernel that consists of two elements: (I, I) and $(I, -I)$.

8.2 Milnor's fiber bundles

Milnor introduced a family of maps $f_{ij} : \mathrm{SU}(2) \to \mathrm{SO}_4$. The action of $f_{ij}(A)$ on the space $\mathbb{R}^4$ is that of (A^h, A^j). In other words, as before, the quaternion A defines two orthogonal planes P_1 and P_2 in $\mathbb{R}^4$, while the element $f_{ij}(A)$ rotates the plane P_1 through the angle $(h + j)\theta$, and rotates the plane P_2 through the angle $(j - h)\theta$.

We may use the map f_{hj} as a clutching function to sue a vector bundle ξ_{hj} of dimension 4 over S^4. To this end, we only need to identify the meridional sphere $S^3 \subset S^4$ with quaternions A of length 1. We choose the identification that extends to the identification of the upper hemisphere D^4_+ of S^4 with quaternions of length at most 1 given by the standard projection $D^4_+ \subset \mathbb{R}^5 \to \mathbb{R}^4$.

There are interesting relations between the clutching functions $f_{hj} : S^3 \to \mathrm{SO}(4)$. Notice that given two clutching functions f_{hj} and $f_{h'j'}$ we may take their sums as elements of the homotopy group $\pi_3\mathrm{SO}(4)$. The resulting clutching function defines the fiberwise connected sum $\xi_{hj} \sharp \xi_{h'j'}$

of vector bundles over S^4. On the other hand, the two clutching functions can be multiplied pointwise by the product in $SO(4)$ to produce a new clutching function $x \mapsto f_{hj}(x) f_{h'j'}(x)$.

We claim that the two operations on $\pi_3 SO(4)$ are the same and therefore lead to the same fiberwise connected sum. To prove the claim, choose a homotopy representative of the clutching function $S^3 \to SO(4)$ for the vector bundle ξ_{hj} so that it is trivial over the upper hemisphere of S^3, and for $\xi_{h'j'}$ so that it is trivial over the lower hemisphere of S^3. Then the two operations on $\pi_3 SO(4)$ are clearly the same.

Let M_{hj} denote the set of all vectors in ξ_{hj} of length 1. It is a smooth manifold of dimension 7. The projection of ξ_{hj} restricts to a proper submersion of the manifold M_{hj} to S^4. The fiber of the restricted submersion is S^3 as it consists of vectors of length one in the fiber $\mathbb{R}^4$ of the vector bundle ξ_{hj}.

Theorem 8.2. *The manifold M_{hj} is homeomorphic to a sphere if and only if the Euler number of the vector bundle ξ_{hj} is ± 1.*

Proof. Let S^3 denote the fiber of the projection $M_{hj} \to S^4$ over the north pole of S^4. There is a long exact sequence of homotopy groups[5]

$$\cdots \longrightarrow \pi_n(S^3) \xrightarrow{i} \pi_n(M_{hj}) \xrightarrow{j} \pi_n(M_{hj}, S^3) \xrightarrow{\partial} \pi_{n-1} S^3 \longrightarrow \cdots$$

Since the projection $\pi\colon M_{hj} \to S^4$ is a proper submersion, the group $\pi_n(M_{hj}, S^3)$ can be identified with the homotopy group $\pi_n(S^4)$. The identification associates with a map $f\colon D^n \to M_{hj}$ that sends ∂D^n to the fiber S^3 a new map $\pi \circ f\colon D^n \to S^4$ that sends the boundary of D^n to the north pole of S^4. Substituting $\pi_n S^4$ for the homotopy groups of the pair results in a so-called long exact sequence of the fibration π:

$$\cdots \longrightarrow \pi_n(S^3) \xrightarrow{i} \pi_n(M_{hj}) \xrightarrow{j} \pi_n(S^4) \xrightarrow{\partial} \pi_{n-1} S^3 \longrightarrow \cdots$$

Since $\pi_i S^3 = \pi_i S^4 = 0$ for $i = 1, 2$, we deduce that M_{hj} is at least 2-connected. Consider now the homomorphism $\partial\colon \pi_4 S^4 \to \pi_3 S^3$. Both, the source and target groups are isomorphic to $\mathbb{Z}$. We claim that by choosing appropriate orientations of the spheres, we may assume that the homomorphism ∂ is given by multiplication by the Euler number of ξ_{hj}. Suppose for a moment that we have proven the claim. Then, the manifold M_{hj} is homeomorphic to a sphere if and only if it is 3-connected, which is equivalent to the requirement that ∂ is an isomorphism. The latter is equivalent to the assertion that the Euler number of ξ_{hj} is ± 1.

[5] All homomorphisms in the exact sequence have a clear geometric meaning. The homomorphism i postcomposes a map $S^n \to S^3$ with the inclusion $S^3 \subset M_{hj}$ to produce a map $S^n \to M_{hj}$. The homomorphism j represents a map $S^n \to M_{hj}$ as a map $D^n \to M_{hj}$ that sends the boundary of D^n to S^3. Finally, the homomorphism ∂ restricts such a map to the boundary of D^n and produces a map $\partial D^n \to S^3$.

Finally, let us prove the claim that the homomorphism $\partial\colon \pi_4 S^4 \to \pi_3 S^3$ is given by multiplication by $e(\xi_{hj})$. Recall that the standard generator in $\pi_4 S^4$ is given by the identity map of S^4. Under the identification of the group $\pi_4 S^4$ with the group $\pi_4(M_{hj}, S^3)$, the generator of $\pi_4 S^4$ corresponds to a map $v\colon D^4 \to M_{hj}$ which sends the boundary of D^4 to S^3. We recall that $\partial(\mathrm{id}_{S^4})$ is precisely the degree of the map $v|\partial D^4$. We want to relate this number with $e(\xi_{hj})$.

Let $S^4 = D_+ \cup D_-$ denote the decomposition of the sphere into the southern and northern hemispheres. The map v restricted to $D_- \subset D^4$, associates with each $x \in D_-$ a vector of length 1 in the fiber of ξ_{hj} over the point x. In particular, the map v defines a vector field in ξ_{hj} over the boundary $\partial D_+ = \partial D_-$. Since the fiber bundle ξ_{hj} over D_+ is trivial, the vector field v over ∂D_+ can be identified with a map $\partial D_+ \to S^3$. On one hand side, by contracting D_+ to the north pole of S^4, we can identify this map with $v|\partial D^4$. On the other hand side, by definition, the degree of the map $\partial D_+ \to S^3$ is the obstruction to constructing a unit vector field in ξ_{hj}, i.e., it is the Euler number of ξ_{hj}. $\square$

8.3 The quaternionic Hopf fibration

We regard the sphere S^7 to be the unit sphere in the quaternion space $\mathbb{H}^2$. The projective quaternionic space $\mathbb{H}P^1$ is defined to be the space of equivalence classes $[x : y]$, where the pairs (x, y) and $(\lambda x, \lambda y)$ represent the same class for any quaternion λ. The quaternion Hopf fibration is the map $S^7 \to \mathbb{H}P^1$ defined by $(x, y) \mapsto [x : y]$.

The projective quaternionic space $\mathbb{H}P^1$ is diffeomorphic to the sphere S^4. Indeed, it is compact as the image of a compact space S^7 and away from one point $[1 : 0]$ it is covered by a chart $[x : 1]$ diffeomorphic to $\mathbb{H}$. Thus $\mathbb{H}P^1$ is a one point compactification of $\mathbb{H}$. The fiber is S^3. Thus, the quaternionic Hopf fibration is a map $S^7 \to S^4$ with fiber S^3.

The quaternionic Hopf fibration can also be described slightly differently. Let $\mathbb{H}P^2$ denote the projective quaternionic space of dimension 2 with coordinates $[x : y : z]$. Remove a unit disc $|x|^2 + |y|^2 < 1$ from the third coordinate chart $[x : y : 1]$. Then the fibration $[x : y : z] \to [x : y]$ restricts on the boundary sphere $\{|x|^2 + |y|^2 = 1, z = 1\}$ to the quaternionic Hopf fibration. It is actually convenient to consider a slightly larger fibration from $\mathbb{H}P^2 \setminus [0 : 0 : 1]$ over $\mathbb{H}P^1$ with fiber $\mathbb{H}$. We will refer to this fibration as to the quaternionic disc fibration.

Let's describe the clutching function for the Hopf fibration. The target sphere $\mathbb{H}P^1$ consists of two discs: the disc $|x| \leq 1$ in the first chart $[x:1]$, and the disc $|y| \leq 1$ in the second chart $[1:y]$. The transition map identifies $[x:1]$ with $[1:y]$, where $y = x^{-1}$. Over the first disc, the quaternionic disc fibration consists of the points $[x:1:z]$ with $|x| \leq 1$. We may trivialize this $\mathbb{H}$-fibration by a map $[x:1:z] \mapsto z$. Similarly, we may trivialize the quaternionic disc fibration over the second disc by a map $[1:y:z] \mapsto z$. Finally, the clutching function is the identity

$$[x:1:z] = [1:x^{-1}:x^{-1}z] = [1:y:yz],$$

where, recall, the transition map from the boundary of the first disc to the boundary of the second disc identifies $[x:1]$ with $[1:y]$ when $y = x^{-1}$. Thus, the clutching function $S^3 \to \mathrm{SO}(4)$ over a point $y \in S^3$ is given by $z \mapsto yz$. This is the Milnor's fiber bundle $\xi_{1,0}$.

8.4 *Invariants of Milnor's fibrations*

To begin with let's calculate invariants of the fiber bundle ξ_{10}. We have seen that the manifold M_{10} of vectors in ξ_{10} of length 1 is a smooth sphere S^7, and that the fiber of the projection $S^7 \to S^4$ is S^3. [6] By Theorem 8.2, the Euler number of ξ_{10} is ± 1.

[6] We have seen that $\xi_{10} \to S^4$ is the projection of $\mathbb{H}P^2 \setminus [0:0:1]$ to the projective space $\mathbb{H}P^1$ given by $[x:y:z] \to [x:y]$. If we remove from $\mathbb{H}P^2$ the open disc D^8 of points $[x:y:1]$ with $|x|^2+|y|^2 < 1$ then we obtain a manifold with boundary $\partial D^8 = S^7$, and the projection $S^7 \to \mathbb{H}P^1$ given by $(x,y) \to [x:y]$. The manifold S^7 is precisely M_{10}.

To compute the Pontryagin class $p_1(\xi_{1,0})$, recall that the quaternion space $\mathbb{H}$ is actually a complex space, and the quaternion disc bundle is a complex bundle. For complex bundles, there is an identity

$$1 - p_1 + p_2 - \cdots = (1 - c_1 + c_2 - \cdots)(1 + c_1 + c_2 + \cdots).$$

In particular, we have $p_1 = c_1^2 - 2c_2$. On one hand side, the class c_1 of any vector bundle over 3-connected sphere S^4 is trivial. On the other hand, by definition the class c_2 of a complex vector bundle of real dimension 4 equals the Euler class. Thus, $p_1(\xi_{1,0}) = -2e(\xi_{1,0}) = \pm 2$.

Theorem 8.3. *The first Pontryagin number of $\xi_{h,0}$ is $\pm 2h$.*

Proof. The Pontryagin numbers are additive with respect to taking the connected sum of fiber bundles. Therefore, for any positive h we have

$$p_1(\xi_{h,0})[S^4] = p_1(\#_h \xi_{1,0})[S^4] = \pm 2h.$$

On the other hand, if h is negative then the equality

$$p_1(\xi_{h,0} \# \xi_{-h+1,0})[S^4] = p_1(\xi_{1,0})[S^4]$$

implies that the claim of the theorem holds both for positive and negative h. □

Our next step is to extend Theorem 8.3 to the case of more general Milnor fibrations $\xi_{h,j}$. To this end, let us construct a new vector bundle $\bar{\xi}_{h,0}$ with the property that there is a fiberwise orientation reversing isomorphisim $\xi_{h,0} \to \bar{\xi}_{h,0}$ covering the identity map of the common base space S^4. To this end, let us fix an orientation reversing isomorphism $\mathbb{R}^4$, say the homomorphism $r\colon (a,b,c,d) \mapsto (-a,b,c,d)$. Recall that the vector bundle $\xi_{h,0}$ is obtained by suing two trivial vector bundles $D_- \times \mathbb{R}^4$ and $D_+ \times \mathbb{R}^4$ over the upper and lower hemispheres D_- and D_+ by means of the clutching function $f_{h,0}\colon S^3 \to \mathrm{SO}(4)$. Each quaternion $A \in S^3$ determines two orthogonal planes P_1 and P_2 where P_1 is spanned by the vectors in the quaternions a and $bi + cj + dk$. The transformation $f_{h,0}(A)$ rotates the planes P_1 and P_2 through the angles $h\theta$ and $-h\theta$ respectively, where $a = \cos\theta$. Similarly, the vector bundle $\bar{\xi}_{h,0}$ is obtained by means of the clutching function $\bar{f}_{h,0} = r \circ f_{h,0} \circ r^{-1}$. Now that the direction with coordinate a is reversed, the transformation $\bar{f}_{h,0}$ rotates the planes P_1 and P_2 through the angles $-h\theta$. Consequently, the vector bundle $\bar{\xi}_{h,0}$ is the same as the vector bundle $\xi_{0,-h}$.

The vector bundle $\bar{\xi}_{h,0}$ is said to be obtained from $\xi_{h,0}$ by changing the fiber orientation. Note that the characteristic classes of $\bar{\xi}_{h,0}$ may or may not be different from those of $\xi_{h,0}$. For example, the Euler cochain associates with a cell D in the base of the vector bundle the degree of a map $\partial D \to S^3$. When we change the orientation of the fiber, we change the orientation of S^3 and the sign of degrees of maps $\partial D \to S^3$. Thus, the Euler obstructing cochain $e(\bar{\xi}_{h,0})$ is negative the cochain $e(\xi_{h,0})$. On the other hand, when we compute the Pontryagin classes of $\bar{\xi}_{h,0}$, we first take the complexification of $\bar{\xi}_{h,0}$. The orientation of fibers in $\bar{\xi}_{h,0}$ and $\xi_{h,0}$ is the same as we changed orientation twice: in the direction a and ia. Therefore, $p_i(\xi_{0,-h}) = p_i(\xi_{h,0})$.

Theorem 8.4. *The first Pontryagin number of $\xi_{h,j}$ is $\pm 2(h-j)$. While the Euler number of $\xi_{h,j}$ is $h+j$.*

Proof. The Euler obstructing cochain changes to its negative, when the orientation of a vector bundle is changed. Hence, we deduce that $\xi_{0,-1} = -\xi_{1,0}$. Therefore, the additivity of the Euler number with respect to taking fiberwise connected sums of vector bundles implies that the Euler number of $\xi_{h,j}$ is $h+j$. The calculation of the first Pontryagin number is similar. □

8.5 Exotic spheres of dimension 7

In view of Theorem 8.2, every manifold $\Sigma_k = M_{h,j}$ with $k = h - j$ and $1 = h + j$ is homeomorphic to a sphere as, in this case, the Euler number of the vector bundle $\xi_{h,j}$ is 1. The sphere bundle $\Sigma_k \to S^4$ bounds a disc bundle $D_k \to S^4$. If Σ_k is diffeomorphic to the standard sphere, then the boundary of the total space D_k can be capped off with a disc to produce a closed manifold N of dimension 8. There is a simple CW-structure on the manifold N. The manifold N is obtained by taking a 0-cell, attaching a 4-cell to the 0-cell, and then attaching an 8-cell to the sphere $N^{(4)}$. In particular, its cohomology groups are free generated by classes u, v, w in degrees $0, 4$ and 8 respectively, where w is an orienting cohomology class. Furthermore, we know that $v \cup v = \pm w$ as the Euler number of $\xi_{h,j} = \pm 1$.[7] We may change the orienting cohomology class of N so that $v \cup v = w$. Then $\sigma(N) = 1$.

[7] Equivalently, we may use the fact that since the intersection form of N is unimodular, we have $v \cup v = \pm w$.

By dimensional reasoning, the first Pontryagin class $p_1(N)$ is a multiple of v. If we restrict it to the sphere $S^4 = N^{(4)}$, then we obtain the class $p_1(\xi_{h,j} \oplus TS^4)$ that is $\pm 2k$ times a generator of $H^4(S^4)$.[8] Consequently, we have $p_1(N) = \pm 2kv$, and $p_1^2(N) = 4kw$.

[8] Note that the restriction of the tangent bundle of N to S^4 consists of two summands: the tangent bundle of S^4 and the normal bundle of S^4 in N. The latter is isomorphic to $\xi_{h,j}$. That is why the class $p_1(N)$ restricts to $p_1(\xi_{h,j} \oplus TS^4)$. By the Whitney formula, the restricted class equals

$$p_1(\xi_{h,j} \oplus TS^4 \oplus \varepsilon) = p_1(\xi_{h,j})$$

since

$$p_1(TS^4) = p_1(TS^4 \oplus \varepsilon) = p_1(\varepsilon^5) = 0.$$

Recall now that, by the Hirzebruch formula, we have $\sigma = (7p_2 - p_1^2)[N]/45$. In particular, the number $45\sigma + p_1^2[N] = 45 + 4k^2$ should be divisible by 7. However, it is not divisible by 7 whenever $k^2 - 1$ is not divisible by 7. On the other hand, for each odd number k there are integers h and j such that $h + j = 1$ and $k = h - j$, and therefore there is a homotopy sphere $\Sigma_k = M_{h,j}$. Thus, for example, the manifold Σ_3 is homeomorphic, but not diffeomorphic to the standard sphere S^7.

8.6 Further reading

Characteristic homological classes were initially introduced by Stiefel and Whitney. The axiomatic definition of Stiefel-Whitney as well as Chern classes was proposed by Hirzebruch.

Example 7.4 was worked out by Milnor and Kervaire

All information that we used in the chapter (and, of course, much more than that) on characteristic classes, Hirzebruch signature formula, as well as orientable cobordism groups can be found in the book **Characteristic classes** by John Milnor and James Stasheff. The elementary, obstruction theoretic definition of characteristic classes, can be found in the book **Homotopical Topology** by Anatoly Fomenko and Dmitry Fuchs. The reader may find in the same book proofs (which we omitted) of two important properties of characteristic cochains that characteristic cochains are cocylces, and that their cohomology classes do not depend on various choices in the definition of the cochains.

An excellent geometric description of the map $SU(2) \to SO(3)$ can be found in the preprint **An elementary introduction to the Hopf fibration** by David W. Lyons. The map $Spin(4) \to SO(3)$ is a universal double covering. It plays an important role especially in 4-dimensional topology. We recommend an interested reader an enjoyable book **Lecture Notes on Seiberg-Witten Invariants** by John Moore.

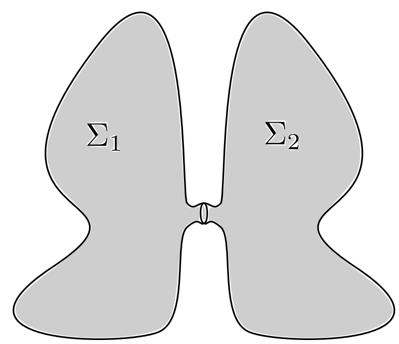
Of course, the main reference for this chapter is the original paper **On manifolds homeomorphic to the 7-sphere** by John Milnor.

9

Exotic spheres

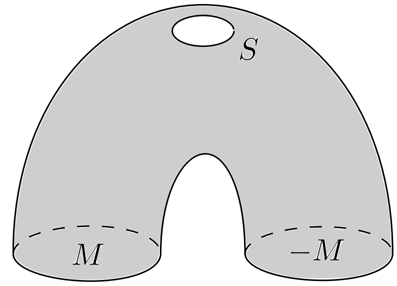
An exotic sphere of dimension n is a smooth manifold that is homeomorphic but not diffeomorphic to the standard sphere S^n. We have seen the original construction of exotic spheres by Milnor in dimension 7. Today we will discuss a general theory of exotic spheres both in dimension 7 and in higher dimensions.

9.1 The group of smooth structures on a sphere



$\Sigma_1 + \Sigma_2$

To begin with let us observe that the set of diffeomorphism classes of smooth manifolds homeomorphic to a sphere of dimension n forms an abelian monoid with connected sum operation, i.e., if Σ_1 and Σ_2 are two oriented exotic spheres, then $\Sigma_1 + \Sigma_2$ is obtained by removing a small disc from each Σ_i and identifying the two obtained boundary spheres. Of course, taking the connected sum of two topological spheres results in a topological sphere, and therefore $\Sigma_1 + \Sigma_2$ is homeomorphic to a sphere. The monoidal operation is well-defined; for example, it does not depend on the choice of representatives of Σ_1, Σ_2, on the choice of an identifying diffeomorphism of boundary spheres, and on smoothings.



It may not be immediately clear that the monoid of smooth manifolds homeomorphic to the sphere S^n is a group. However, Smale proved that for $n \neq 3, 4$ this monoid is isomorphic to the group θ_n of h-cobordisms of homotopy spheres. We will assume today that $n > 5$, unless we explicitly state otherwise.

Definition 9.1 (The group θ_n)**.** Two oriented closed manifolds M_1 and

M_2 are said to be *h-cobordant* if $M_1 \sqcup (-M_2)$ bounds an oriented compact manifold W such that each of the manifolds M_1 and M_2 is a deformation retract of W. Here $-M_2$ is the manifold M_2 with the opposite orientation. The set of h-cobordism classes of homotopy spheres forms an abelian group θ_n under the connected sum operation.[1]

[1] The zero in the group θ_n is represented by the standard sphere. The inverse of the class of a manifold M is the class of $-M$; the complement to a small disc in $M \times [0,1]$ is a cobordism from $M \sqcup (-M)$ to the standard sphere.

The Smale h-cobordism theorem asserts that *an h-cobordism W between closed simply connected manifolds M_1 and M_2 of dimension* $\dim M_1 = \dim M_2 \geq 5$ *is trivial, i.e., the cobordism W is diffeomorphic to $M_1 \times [0,1]$. In particular, the manifolds M_1 and M_2 are diffeomorphic.* Thus, two smooth n-manifolds homeomorphic to a sphere are h-cobordant if and only if they are diffeomorphic provided that $n \geq 5$. In particular, the monoid of smooth n-manifolds homeomorphic to a sphere is isomorphic to θ_n.

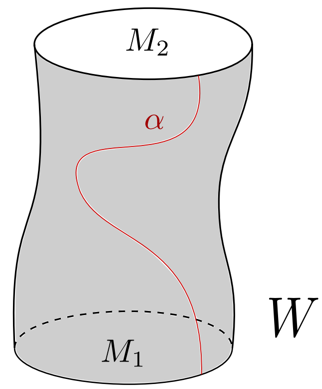


Figure 9.1: An almost framed bordism.

In this chapter we aim to study the groups θ_n. Our main tool is an observation that the groups θ_n fits a so-called *surgery exact sequence* of groups; the other entries of the exact sequence are $\mathcal{N}_*$ and P^*, which we define next.

Definition 9.2. A manifold $M \subset \mathbb{R}^{m+k}$ is *almost framed* if it is framed in the complement to one point. Suppose that for $i = 1, 2$ two manifolds M_i are framed in the complement to x_i and there is a cobordism $W \subset \mathbb{R}^{m+k} \times [0,1]$ between M_1 and M_2. We say that W is an *(almost framed) cobordism* if it is framed in the complement to a curve α joining x_1 with x_2; the framing on $W \setminus \alpha$ is required to coincide over $M_1 \setminus x_1$ and $M_2 \setminus x_2$ with the almost framings of M_1 and M_2 respectively. The set of cobordism classes of almost framed, path connected, non-empty manifolds of dimension n is denoted by $\mathcal{N}_m$.

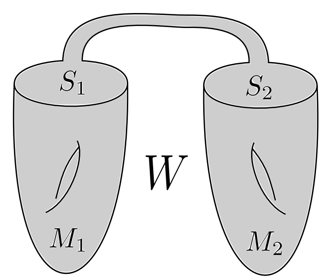


Figure 9.2: A framed cobordism in the definition of P^m for $m = 2$.

We claim that the set $\mathcal{N}_m$ forms a group with respect to taking the connected sum. Given two pointed manifolds (M_i, x_i) with $i = 1, 2$, framed in the complement to x_i, some care should be taken to form the almost framed connected sum of M_1 and M_2. To begin with, we remove an open disc D_i of dimension m from each manifold M_i so that $x_i \in \partial D_i$. After that we form the connected sum $M_1 \sharp M_2$ by attaching to $M_1 \setminus D_1$ and $M_2 \setminus D_2$ a cylinder $S^{m-1} \times [0,1]$ in such a way that the a curve $\alpha = \{x\} \times [0,1]$ joins the points x_1 and x_2. Since there is only one frame over the contractible manifold $S^{m-1} \setminus x$, there is a unique frame over $M_1 \sharp M_2$ in the complement to α that is compatible with the original frames over $M_1 \setminus D_1$ and $M_2 \setminus D_2$. Since the segment α is contractible to its midpoint x_{12}, there is a well-defined framing of $M_1 \sharp M_2$ in the complement to the point x_{12}, which we declare to be the distinguished point of the connected sum.

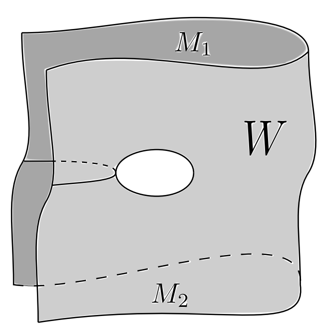


Figure 9.3: A framed cobordism in the definition of P^m for $m = 1$.

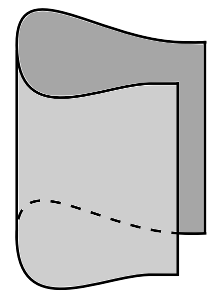
The trivial element of $\mathcal{N}_m$ is represented by the pointed standard sphere S^n with a unique frame of $S^n \setminus \{pt\}$.

Lemma 9.3. *The set $\mathcal{N}_m$ forms a group with respect to taking the connected sum. The inverse of the class of an almost framed manifold (M, x) is represented by $(-M, x)$.*

Sketch of the proof. Let D denote a small open ball in $M \subset \mathbb{R}^{m+k}$ such that $x \in \partial D$. Without loss of generality we may assume that the manifold $M \setminus D$ is placed in the half space $x_1 \leq 0$ in such a way that the boundary of $M \setminus D$ is the standard sphere S in $\{0\} \times \mathbb{R}^m \times 0^{k-1}$ and the collar neighborhood of $\partial(M \setminus D)$ is given by the trace of the translation of the standard sphere S in the direction of the coordinate vector $-e_1$. Then, the boundary of the manifold $W = (M \setminus D) \times [0,1]$ consists of three smooth pieces

$$(M \setminus D) \times \{0\}, \qquad (M \setminus D) \times \{1\}, \qquad \text{and} \qquad S^{n-1} \times [0,1],$$

and after smoothing the corners becomes diffomorphic to $M\#(-M)$. The frame over $(M \setminus D) \times \{0\}$ spreads by vertical translation to a frame over the manifold W. We remove an open disc D from the interior of W, and place the manifold $W \setminus D$ to $\mathbb{R}^{m+k} \times [0,1]$ in such a way that it is a cobordism between $M\#(-M)$ and ∂D.



$(M \setminus D) \times [0,1]$

Figure 9.4: The boundary of the manifold $W = (M \setminus D) \times [0,1]$ consists of three smooth pieces: the upper and lower horizontal copies of $M \setminus D$ and the vertical copy of $S^{n-1} \times [0,1]$.

To complete the proof we choose an arbitrary curve α in $W \setminus D$ from a point in the boundary ∂D to the distinguished point of $M\sharp(-M)$. The manifold $W \setminus D$ is a desired cobordism framed in the complement to the curve α.[2] $\square$

[2] It follows that the manifold $(-M)$ together with its frame can be obtained from the manifold M and its frame by placing M to the half space $x_1 < 0$ and then transforming M and its framing by the reflection along the hyperspace $x_1 = 0$.

The final player that we need to introduce is P^m.

Definition 9.4. The group P^m is the group of cobordism classes of compact framed manifolds M of dimension m bounded by homotopy spheres S. Two such framed manifolds M_i bounding homotopy spheres S_i for $i = 1, 2$ are called cobordant if there is a framed manifold W whose boundary consists of $M_1 \cup N \cup M_2$ where N is an h-cobordism between S_1 and S_2 and where the boundary of N is identified with the corresponding boundaries of M_1 and M_2. The operation on P^m is given by taking the exterior connected sum that joins S_1 with S_2.

Note that under our dimensional assumption $m > 5$, the h-cobordism N between the homotopy spheres S_1 and S_2 of dimension $m - 1 \geq 5$ is trivial. Thus, by the h-cobordism theorem, two framed manifolds

M_1 and M_2 represent the same element in P^m if M_1 and M_2 bound the same homotopy sphere S and there is a framed cobordism W with boundary $M_1 \cup (S \times [0,1]) \cup M_2$.

9.2 *The surgery long exact sequence for spheres*

We are now in position to write down the surgery exact sequence for spheres.

Theorem 9.5. *There is a long exact sequence of groups*

$$\cdots \longrightarrow \mathcal{N}_{m+1} \xrightarrow{p} P^{m+1} \xrightarrow{b} \theta_m \xrightarrow{\eta} \mathcal{N}_m \longrightarrow \cdots .$$

Proof. To begin with let us define the homomorphisms. The homomorphism b is defined by taking the boundary of a manifold. Next, in order to construct the homomorphism η note that the complement to a point in a topological sphere is homeomorphic to a disc, and therefore every homotopy sphere comes with a unique almost framing. Finally, the map p is defined by removing a small open disc containing the non-framed point. More precisely, if M is a manifold framed in the complement to a point x, then $p[M]$ is represented by the framed manifold $M \setminus D$ bounded the sphere ∂D, where D is a small disc centered at x.

If $\eta(\Sigma)$ is in the trivial class, then, by definition, there is a cobordism W of Σ to the standard sphere S, a curve α in W and an appropriate framing over $W \setminus \alpha$. The boundary component S can be capped off with a disc D to produce a manifold $W \cup D$ which bounding Σ. The framing over $W \setminus \alpha$ uniquely extends to a framing in the complement to α in $W \cup D$. Contracting α to the distinguished point on Σ and then removing a clollar neighborhood of Σ (together with the non-framed point point) in W, we get an element in P^{m+1}. Clearly $\eta \circ b = 0$. Thus, the sequence is exact at θ_m.

If M is a compact framed manifold which represents an element in the kernel of b, then, by the h-cobordism theorem, the boundary of M is diffeomorphic to the standard sphere. By capping the boundary of M off with a disc D we obtain a manifold $M \cup D$ representing an element in $\mathcal{N}_{m+1}$. On the other hand, a manifold $M \in P^{m+1}$ representing an element in the image of p has boundary diffeomorphic to a sphere. Therefore, the class of M belongs to the kernel of b. Thus, the sequence is exact at P^{m+1}.

Let M be a manifold representing an element in $\mathcal{N}_m$. It is framed in the complement to a point. Let D be a small open disc about that point. Suppose that $[M]$ is in the kernel of p. Then $M \setminus D$ is cobordant to the standard disc D^m by means of a cobordism W. By the h-cobordism theorem, assuming that $m > 5$, the boundary of W is diffeomorphic to $S^{m-1} \times [0,1]$. We may cup it off with $D^m \times [0,1]$ to obtain a cobordism $W \cup (D^m \times [0,1])$ of the manifold M to a manifold homeomorphic to a sphere. Furthermore, the cobordism W is framed outside the curve $\{0\} \times [0,1] \subset D^m \times [0,1]$. Thus the manifold M represents a class in the image of η. Finally, if we start with an almost framed homotopy sphere and cut out an open disc about the distinguished point with a frame, then we will be left with a standard sphere bounding a standard disc (by the h-cobordism theorem, provided that $m > 5$). Thus, the sequence is exact at $\mathcal{N}_m$. □

9.3 *Calculation of the group P^m*

In the previous chapters we considered surgery on closed manifolds. In this section we will use the same framed surgery technique to simplify framed compact manifolds with boundary homeomorphic to spheres, i.e., manifolds M representing an element in P^m. [3]

[3] Note that framed surgery on M does not change the framed cobordism class $[M] \in P^m$.

Again, we proceed by induction. If the manifold M of dimension m is $(q-1)$-connected, then for every element in $\pi_q M$ we may choose a Wall representative S. By general position argument, below the middle range $q < m/2$, the obstruction μ to the existence a framed surgery along S is trivial. Thus, we may always modify the manifold M so that it is $(\lfloor \frac{m}{2} \rfloor - 1)$-connected.

To study modifications of M near the middle range, we observe that if M bounds a topological sphere Σ, we may attach to M the disc D^m of dimension m by identifying the boundary of the disc with Σ. The resulting manifold $\tilde{M}$ is a closed topological manifold that is smooth and framed away from a disc (or, equivalently a point). For this reason, the discussion of surgeries of framed manifolds can be extended to framed manifolds M bounding a homotopy sphere. We define the signature and the Kervaire invariant of M to be those of $\tilde{M}$. [4] Let us recall the relevant theorems.

[4] To do: add corollaries about manifolds bounding homotopy spheres in the relevant chapters.

Theorem 9.6. *Let M be a compact framed manifold of dimension $m \geq 5$ bounding a homotopy sphere Σ.*

- *If m is odd, then M is cobordant to a framed contractible manifold.*
- *If $m = 2q$ with q even, then M is cobordant to a framed contractible manifold if and only if $\sigma(M) = 0$.*
- *If $m = 2q$ with q odd, then M is cobordant to a framed manifold if and only if the Kervaire invariant of M is trivial.*

Thus, we may compute the group P^m.

Theorem 9.7. *If m is odd, then $P^m = 0$. Suppose that $m = 2q$. If q is even, then $P^m = \mathbb{Z}$. If q is odd, then $P^m = \mathbb{Z}_2$.*

Proof. Suppose that m is odd. Then, by performing framed surgery in the interior or M, we may assume that the manifold M bounding a homotopy sphere Σ is contractible. Remove a standard disc D^m from the interior of M to obtain an h-cobordism $M \setminus D^m$ between Σ and S^{m-1}. By the h-cobordism theorem, the homotopy sphere Σ is actually the standard sphere S^{m-1}, while the manifold $M \setminus D^m$ is $S^{m-1} \times [0,1]$. Therefore, $M \approx D^m$, which implies $P^m = 0$.

Suppose now that $m = 2q$, and q is even. Consider the map $\sigma\colon P^m \to \mathbb{Z}$ defined by associating with M its signature. Since the signature of a manifold is an invariant of cobordisms, the map σ is well-defined, i.e., the value of the homomorphism σ does not depend on the choice of the representative of a class in P^m. Furthermore, since

$$\sigma(M_1 \# M_2) = \sigma(M_1) + \sigma(M_2),$$

the map σ is actually a homomorphism with respect to operations in P^m and $\mathbb{Z}$. We claim that the homomorphism σ is injective. Indeed, if two manifolds M_1 and M_2 bounding homotopy spheres Σ_1 and Σ_2 are of the same signature, then the boundary connected sum $M_1 \sharp (-M_2)$ is a compact framed manifold M of signature $\sigma(M) = 0$ again bounding a homotopy sphere $\Sigma = \Sigma_1 \# (-\Sigma_2)$. By Theorem 9.6, the framed manifold M is cobordant to a contractible manifold, while Σ is diffeomorphic to a standard sphere. Therefore M represents a trivial element in P^m. Thus, the homomorphism σ is injective.

At this moment, since P^m is isomorphic to a subgroup of $\mathbb{Z}$, we may conclude that either $P^m = 0$ or $P^m = \mathbb{Z}$. On the other hand, the manifold M constructed by plumbing corresponding to the matrix E_8 bounds a homotopy sphere and has signature 8. Therefore, when m is divisible by 4, the group P^m is isomorphic to $\mathbb{Z}$. [5]

[5] **Include a discussion of plumbing**

Remark 9.8. We will show later that the intersection form over $\mathbb{Z}$ of the manifold M is unimodular and even. In particular, the signature of M is divisible by 8. Therefore, the homomorphism $\sigma\colon P^m \to \mathbb{Z}$ is given by multiplication by 8. [6]

Suppose now that $m = 2q$ and q is odd. Now we consider the homomorphism $\kappa\colon P^m \to \mathbb{Z}_2$ defined by associating with a manifold M its Kervaire invariant $\kappa(M)$. Again, as in the case with the signature, the Kervaire invariant is a cobordism invariant and

$$\kappa(M_1 \# M_2) = \kappa(M_1) + \kappa(M_2).$$

Therefore, the map κ is indeed a well-defined homomorphism. By Theorem 9.6, the homomorphism κ is injective. On the other hand, consider the manifold $K(2n)$ constructed by plumbing corresponding to the unimodular matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Since the intersection form of $K(2n)$ is unimodular, it bounds a homotopy sphere. On the other hand, the Kervaire invariant of the manifold $K(2n)$ is 1. Thus, the homomorphism κ is an isomorphism. □

[6] Here is the original argument of Milnor, see **Differential manifolds which are homotopy spheres**, Lemma 3.2. We would like to show that all values

$$x \cup x[M] = Sq^q x[M] = \nu_q \cup x[M]$$

are zero mod 2. Here ν_k is the k-th Wu class defined by the requirement that

$$\nu_k \cup x[M] = Sq^{m-k}(x)[M]$$

for all x. If $x \cup x[M] \equiv 1$, then the class ν_q is non-trivial. We claim that this implies that some Stiefel-Whitney class is non-trivial. Indeed, for the total Wu class $\nu = 1 + \nu_1 + \cdots$ we have $w = Sq(\nu)$, i.e., $w_k = \sum_{i \geq 0} Sq^i(\nu_{k-1})$. Let ν_k be the non-trivial Wu class of the least degree. Then $w_k(M) = \nu_k \neq 0$ for some $k \leq q$ for the compact framed manifold with boundary homeomorphic to a sphere.

Remark 9.9. We may now reinterpret the part $\theta_m \to \mathcal{N}_m \xrightarrow{p} P^{m+1}$ of the surgery exact sequence. Namely, we may say that p is assocciating with an almost framed manifold M the algebraic invariant of the intersection form of M. It is a complete obstruction to the existence of a framed cobordism of M to a homotopy sphere.

9.4 *Calculation of the group $\mathcal{N}_m$.*

Recall that in order to calculate the group θ_m of exotic smooth structures on an m-dimensional sphere, we introduced a long exact sequence involving two other groups: the cobordism group P^m of compact framed manifolds bounding a homotopy sphere, and the cobordism group $\mathcal{N}_m$ of almost framed manifolds. We have already computed the group P^m. In this section, we will compute the group $\mathcal{N}_m$.

Theorem 9.10. *There is a long exact sequence of groups*

$$\cdots \longrightarrow \pi_m \mathrm{SO} \xrightarrow{J} \pi_m^S \xrightarrow{\eta} \mathcal{N}_m \xrightarrow{\gamma} \pi_{m-1}\mathrm{SO} \xrightarrow{J} \cdots$$

where the homomorphism η associates with a framed manifold, an almost framed manifold by forgetting the frame at one point.

Proof. To begin with, let us define the homomorphism γ. Let M be a manifold framed in the complement to a point x. Let D be a closed disc in M centered at the point x. Then over the boundary of D there are two frames: the unique frame over the disc D, and the frame that comes from the almost framing of M. We may assume that the frames over M and D are orthonormal. Therefore, the pointwise rotation of the first frame to the second over the boundary $S^{m-1} = \partial D$ is a continuous map $S^{m-1} \to \mathrm{SO}$ which represents an element $\gamma([M])$ of the homotopy group $\pi_{m-1}\mathrm{SO}$. Clearly, the so-defined map γ does not depend on the choice of the representative of the class of M and it is, in fact, a homomorphism.

The class $[M]$ is in the kernel of γ if and only if the framing over $M \setminus D$ extends to a smooth framing over M. Thus, the sequence is exact at $\mathcal{N}_m$. The composition $J \circ \gamma$ associates with an element $[M]$ the class of the boundary ∂M with the frame obtained by restricting the frame from M. However, the manifold M itself provides a framed cobordism of ∂M to an empty manifold. Therefore $J \circ \gamma = 0$. Conversely, if a framing on a sphere S^{m-1} is chosen so that the framed sphere S^{m-1} is null-cobordant, then the null-cobordism W represents an element of $\mathcal{N}_m$. Thus, the sequence is exact at $\pi_{m-1}\mathrm{SO}$.

Finally, suppose that a framed manifold M, is cobordant to the standard sphere S^{n-1} by a cobordism W framed away a curve α which connects a point in M with a point in S^{n-1}. Since the framing of $W \setminus \alpha$ extends over M, we may shrink α to a point in S^{n-1}, and then remove S^{n-1} from W together with its open collar neighborhood U. The obtained framed manifold $W \setminus U$ is a cobordism between M and a framed sphere S^{n-1}, which means that $[M]$ belongs to the image of J. On the other hand, if $[M]$ belongs to the image of J, then its complement to a point is a disc with a unique framing. Therefore, $[M]$ also belongs to the kernel of η. Thus, the sequence is exact at $\mathcal{N}_m$. □

We may thus express the group $\mathcal{N}_m$ in terms of the J-homomorphism:

$$0 \longrightarrow \operatorname{Coker} J_m \longrightarrow \mathcal{N}_m \longrightarrow \operatorname{Ker} J_{m-1} \longrightarrow 0,$$

where $\operatorname{Ker} J_{m-1}$ is isomorphic to $\mathbb{Z}$ if m is divisible by 4, and it is zero otherwise.

9.5 Calculation of homomorphisms in the surgery exact sequence

Recall that we have calculated that in the surgery long exact sequence

$$\cdots \longrightarrow \mathcal{N}_{m+1} \xrightarrow{p} P^{m+1} \xrightarrow{b} \theta_m \xrightarrow{\eta} \mathcal{N}_m \longrightarrow \cdots .$$

the group P^m is isomorphic to $\mathbb{Z}$ and $\mathbb{Z}_2$ when m respectively is of the form $4q$ and $4q+2$, and it is trivial otherwise. To begin with let us consider the homomorphism b.

We have seen that when $m = 4q$, the isomoprhism $P^m \to \mathbb{Z}$ is established by associating with a manifold M the number $\sigma(M)/8$. In particular, there is a compact manifold with boundary with signature 8. On the other hand, suppose that the manifold M represents an element in the kernel of b. In other words, suppose that the framed manifold M bounds the standard sphere. Then we may cap off the boundary to obtain an almost framed manifold $\tilde{M}$. In this case, the restriction on signature of M (or, equivalently, on the signature of an almost framed manifold $\tilde{M}$) is fairly strict.

Theorem 9.11. *The set of signatures of almost framed closed manifolds of dimension $m = 4q > 4$ forms a subgroup $t_m\mathbb{Z}$ of $\mathbb{Z}$ generated by the integer*

$$t_q = a_q 2^{2q+1}(2^{2q-1}-1)(\textit{numerator of } \frac{B_q}{4q}),$$

where $a_q = 2$ if q is odd and 1 if q is even.

Proof. Let M be a manifold of dimension $m = 4q > 4$ framed in the complement to a point x. Then the tangent bundle of $M \setminus x$ is trivial, and, in particular, all Pontryagin classes of M restricted to $M \setminus x$ are trivial. Thus, the only possibly non-trivial Pontryagin class of M is $p_q M$. In view of the Hirzebruch signature formula, the signature of M is a multiple of $p_q(M)$. In fact, according to the Hirzebruch formula, which we will discuss in greater detail in section 9.6,[7]

$$\sigma(M) = \frac{2^{2q}(2^{2q-1}-1)B_q}{(2q)!} \cdot p_q[M],$$

where $p_q[M]$ is the q-th Pontryagin number of the manifold M. The cohomology class $p_q(M)$ can be computed directly from the definition of the Pontryagin classes as obstructions to constructing multiple sections of a vector bundle.[8] It follows that $p_q M = \pm a_q(2q-1)!\gamma([M])$, where γ is the homomorphism $\mathcal{N}_m \to \pi_{m-1}\mathrm{SO} \approx \mathbb{Z}$ in Theorem 9.10. The

[7] The formula for the signature under consideration is proved in Example 9.16.

[8] For an explicit computation, see Example 7.4, and in particular, Theorem 7.11.

values of $\gamma([M])$ coincide with the values of the elements in Ker $J \subset \mathbb{Z}$. Thus, the values of $\gamma([M])$ are the multiples of the order of the image of the J-homomorphism, which is the denominator of $B_q/4q$.[9] □

[9] To summarise,

$$\sigma(M) = \frac{2^{2q}(2^{2q-1}-1)B_q}{(2q)!} \cdot p_q[M]$$
$$= \frac{2^{2q}(2^{2q-1}-1)B_q}{(2q)!} \cdot (\pm a_q(2q-1)!\gamma([M]))$$
$$= 2^{2q+1}(2^{2q-1}-1)\frac{B_q}{4q} \cdot (\pm a_q\gamma([M]))$$
$$= \pm a_q 2^{2q+1}(2^{2q-1}-1)(\text{numerator } \frac{B_q}{4q}).$$

Now we are in position to calculate the homomorphism b. If $m = 4q+2$, then the image of $P^m = \mathbb{Z}_2$ is either trivial or $\mathbb{Z}_2$, while if m is odd, then the group P^m is trivial.

Theorem 9.12. *If $m = 4q > 4$, then the image of the group $P^m = \mathbb{Z}$ in θ_{m-1} is a cyclic group of order $t_q/8$.*

Proof. In view of the exact sequence

$$\cdots \longrightarrow N_m \xrightarrow{p} P^m \xrightarrow{b} \theta_{m-1} \longrightarrow \cdots$$

the image of the homomorphism b is $P^m/\operatorname{Im} p$. The signature homomorphism identifies P_m with $8\mathbb{Z} \subset \mathbb{Z}$ and $\operatorname{Im} p$ with $t_q\mathbb{Z}$. Therefore, the order of the image of b is $t_q/8$. □

Theorem 9.12 produces a number of exotic smooth structures on spheres. For example, suppose that $m = 4q$ with $q = 2$; the lowest value of q allowed by Theorem 9.12. Then

$$t_2/8 = 1 * 2^5(2^3-1) * 1/8 = 28.$$

In particular, there are at least 28 smooth manifolds of dimension 7 homeomorphic to a sphere. In dimension 8 there are no other exotic spheres since the J-homomorphism $\pi_7(\mathrm{SO}) \to \pi_7^s$ is surjective.[10]

[10] It is known that $\pi_7^s = \mathbb{Z}_{240}$. On the other hand, the image of the J-homomorphism in degree 7 is a cyclic group of order the denominator of $B_q/4q$ for $q = 2$, i.e., the order of the image is also 240. Therefore, from the exact sequence

$$\pi_7\mathrm{SO} \xrightarrow{\simeq} \pi_7^s \to \mathcal{N}_7 \to \pi_6\mathrm{SO} = 0.$$

we deduce that $\mathcal{N}_7$ is a trivial group, and $\theta_7 = bP^8$.

9.6 *Hirzebruch genera and computation of the L-genus*

9.6.1 *Multiplicative series*

A *genus* is a ring homomorphism $\varphi\colon \Omega \otimes \mathbb{Q} \to \mathbb{Q}$ with $\varphi(1) = 1$. We aim to classify all genera of smooth manifolds.

Example 9.13. We have seen the Hirzebruch L-genus. It is the ring homomorphism that associates with each generator $\mathbb{C}P^{2n}$ in $\Omega \otimes \mathbb{Q}$ the value 1. Another important genus is the $\hat{A}$-genus. It is related to the index of the Dirac operator. The $\hat{A}$-genus is characterized by the properties that it is trivial on all quaternionic projective spaces $\mathbb{H}P^k$, while the genus of $\mathbb{C}P^2$ is $-1/8$.

To begin with, since the only invariants of rational cobordism classes are Pontryagin numbers, for any genus φ, there are homogenious polynomials $K_n = K_n(p_1, p_2, ..., p_n)$ of degree $4n$ such that for every manifold M of dimension $4n$ its genus $\varphi(M)$ is the Pontryagin number $K_n[M]$. Thus, a genus is completely determined by a formal series $K = 1 + K_1 + K_2 + \cdots$ in Pontryagin classes.

To determine which formal series K are associated with a genus, let us reformulate the multiplicative property of φ in terms of K. Let M be the product $M' \times M''$ of two manifolds with projections π' and π'' onto the two factors. Then the total rational Pontryagin class $p(M)$ can be expressed in terms of rational Pontryagin classes of M' and M'' by the formula $p = p'p''$, where p' and p'' are pullbacks of the total Pontryagin classes of M' and M'' with respect to the projections π' and π''. Thus, the multiplicativity of the genus φ can be expressed in terms of K as the requirement that[11]

$$K(p) = K(p')K(p''), \tag{9.1}$$

where we use a notation $K(p)$ for $K(p_1, p_2, ...)$. We note that the equality 9.1 should be satisfied for any formal series p, p' and p' with $p = p'p''$. For example, we may introduce formal variables $x_1, ..., x_n$ and put[12]

$$p = 1 + p_1 + p_2 + \cdots + p_n = (1 + x_1^2)(1 + x_2^2) \cdots (1 + x_n^2). \tag{9.2}$$

Then[13]

$$K(p) = K(1 + x_1^2)K(1 + x_2^2) \cdots K(1 + x_n^2).$$

In other words, the multiplicative series K is completely determined by the series $Q(x) = K(1 + x^2)$ of one variable. In fact, there is a bijective correspondence between multiplicative series K and series $Q(x) = 1 + a_2x^2 + a_4x^4 + \cdots$ in which only coefficients with even indices may be non-zero. Indeed, under the equality 9.2, the polynomials $p_1, ..., p_n$ form a basis in the space of symmetric polynomials in $x_1^2, ..., x_n^2$. In particular, for any decomposition $I = (i_1, ..., i_r)$ of the number n, there is a polynomial s_I such that

$$s_I(p) := s_I(p_1, ..., p_n) = \sum x_1^{2i_1} \cdots x_n^{2i_r}.$$

The series $Q(x)$ corresponds to the multiplicative series with terms $K_n(p) = \sum a_I s_I(p)$ where $a_I = a_{i_1} \cdots a_{i_r}$.

To summarize, the genera, i.e., the ring homomorphisms $\Omega \otimes \mathbb{Q} \to \mathbb{Q}$ are in bijective correspondence with the series $Q(x)$ in even powers of x with the free term 1.

[11] More explicitly, the condition that $\varphi(M) = \varphi(M')\varphi(M'')$ is equivalent to

$$K(p)[M] = K(p)[M'] \cdot K(p)[M''].$$

In this expression the right hand side is

$$K((\pi')^*p) \cup K((\pi'')^*p)[M' \times M''],$$

or, equivalently, $K(p')K(p'')[M]$.

[12] Thus, for example,

$$p_1 = x_1^2 + \cdots x_n^2,$$
$$p_2 = x_1^2x_2^2 + \cdots + x_{n-1}^2x_n^2.$$

[13] Let us recall our convention that for a series $p = 1 + p_1 + \cdots$, the expression $K(p)$ means $K(p_1, p_2, ...)$. In particular, the expression $K(1 + x_1^2)$ means $K(p_1, 0, 0, ...)$ for $p_1 = x_1^2$. The geometric interpretation of the decomposition 9.2 is known as the Splitting Principle.

Example 9.14. The two genera that we discussed in Example 9.13, the L-genus and the $\hat{A}$-genus, correspond to the series $Q(x) = \frac{x}{\mathrm{th}(x)}$ and $Q(x) = \frac{x/2}{\mathrm{sh}(x/2)}$ respectively. Let us compute explicitly the first two Hirzebruch polynomials L_1 and L_2 in the series $L(p) = K(p)$. As, it has been mentioned, the series $Q(x)$ corresponding to the L-genus is

$$\frac{x}{\mathrm{th} x} = 1 + \frac{1}{3}x^2 - \frac{1}{45}x^4 + \cdots + (-1)^{j-1}\frac{2^{2j}B_j}{(2j)!}x^{2j} + \cdots .$$

Therefore, the series $L(p) = Q(x_1)\cdots Q(x_n)$ starts with the terms[14]

$$\left(1 + \frac{x_1^2}{3} - \frac{x_1^4}{45}\right)\cdots\left(1 + \frac{x_n^2}{3} - \frac{x_n^4}{45}\right) = 1 + \frac{p_1}{3} + \frac{7p_2 - p_1^2}{45},$$

and, in particular, the first terms in the series L are $L_1(p) = p_1/3$ and $L_2(p) = 7p_2 - p_1^2/45$. [15]

[14] We recall that

$$p_1 = x_1^2 + \cdots x_n^2,$$
$$p_2 = x_1^2x_2^2 + \cdots + x_{n-1}^2x_n^2.$$

[15] In practice, to find the values of the genus φ of a series $Q(x)$ on the cobordism classes of manifolds $\mathbb{C}P^{2n}$, let $f(x) = 1/Q(x)$, and let $g(y)$ be the formal inverse of f, i.e., $g(f(x)) = 1$. Then

$$g'(y) = 1 + \varphi(\mathbb{C}P^2)y^2 + \varphi(\mathbb{C}P^4)y^4 + \cdots .$$

For example, for the L-genus, the function $f(x)$ is $\mathrm{th}(x)$ and $f'(x) = 1 - f(x)^2$. Hence,

$$g'(y) = 1/(1-y^2) = 1 + y^2 + y^4 + \cdots ,$$

i.e., indeed, the L-genus associates with each $\mathbb{C}P^{2n}$ the value 1. We will not use this method in what follows, and leave its validity without a proof.

9.6.2 *Genera of manifolds M^{4n} with trivial $p_1, ..., p_{n-1}$.*

In the rest of the section we will calculate a given genus φ of a manifold M^{4n} with trivial Pontryagin classes $p_1, \cdots, p_{n-1}$. We will use the symbol $'\equiv'$ to identify series in the quotient by the ideal of series generated by (x^{4n+2}), in other words the symbol $'\equiv'$ suggests that in the series we ignore terms of order $\geq 4n+1$.

Theorem 9.15. *We have $K(p) = s_n(a_2, ..., a_{2n})p_n$ when the Pontryagin classes $p_1, ..., p_{n-1}$ are trivial.*[16]

Proof. Up to higher order terms, we have

$$Q(x) \equiv 1 + a_2x^2 + \cdots + a_{2n}x^{2n} = \prod_{i=1}^{n}(1 + b_ix^2),$$

and therefore, again, up to higher order terms, $K(p) = \prod Q(x_j)$ is

$$\prod_j\prod_i(1 + b_ix_j^2) = \prod_i\prod_j(1 + b_ix_j^2) = \prod_j(1 + b_ip_1 + \cdots + b_i^n p_n).$$

When the Pontryagin classes $p_1, \cdots, p_{n-1}$ are trivial, we deduce

$$K(p) \equiv (b_1^n + \cdots + b_n^n)p_n = s_n(a_2, a_4, ..., a_{2n})p_n,$$

which is precisely the statement of the theorem. □

[16] Let us spell out the definition of $s_n(a_1, ..., a_{2n})$. Since the polynomials $p_1, ..., p_n$ form a basis in the space of symmetric polynomials in $x_1^2, ..., x_n^2$, there is a unique polynomial s_n such that

$$s_n(p) = x_1^{2n} \oplus \cdots \oplus x_n^{2n}.$$

For example, $s_1(p) = p_1$, $s_2(p) = p_1^2 - 2p_2$, and $s_3(p) = p_1^3 - 3p_1p_2 + 3p_3$. Thus,

$$s_1(a_2, ..., a_{2n}) = a_2,$$
$$s_2(a_2, ..., a_{2n}) = a_2^2 - 2a_4,$$

and

$$s_3(a_2, ..., a_{2n}) = a_2^3 - 3a_2a_4 + 3a_6,$$

where a_i is the i-th coefficient in the series $Q(x)$.

Example 9.16. Let us calculate the value $s_n(a)$ for the L-genus. We will need the so-called Cauchy formula:

$$x\frac{f'(x)}{f(x)} = \sum_{j=0}^{\infty}(-1)^j 2s_jx^{2j}.$$

where $f(x) = x/Q(x)$. [17] The expression on the left hand side as $2x/\mathrm{sh}(2x)$.[18] In its turn, the latter can be expressed in terms of Bernoulli numbers by

$$\frac{2x}{\mathrm{sh}2x} = 2\frac{x}{\mathrm{th}x} - \frac{2x}{\mathrm{th}2x} = 1 - (2^2-2)\frac{B_1}{2!}(2x)^2 + (2^4-2)\frac{B_2}{4!}(2x)^4 - \cdots .$$

In other words, $s_j = 2^{2j}(2^{2j-1}-1)B_j/(2j)!$, and $K(p) = s_n p_n$. We may reinterpret the result of this calculation by saying that for an oriented closed manifold M of dimension $4n$ with trivial Pontryagin classes $p_1, ..., p_{n-1}$, the signature is

$$\sigma(M) = K(p)[M] = 2^{2j}(2^{2j-1}-1)\frac{B_j}{(2j)!}p_n[M].$$

We used this formula for the signature in the proof of Theorem 9.11.

[17] Let $Q_m(x) = 1 + a_2x^2 + \cdots + a_{2n}x^{2n}$ be the $(2n)$-th approximation of $Q(x)$. Then

$$f_m(x) = \frac{x}{Q_m(x)} = \frac{x}{(1+b_1x^2)\cdots(1+b_nx^2)}.$$

The expression $xf'_m/f_m = x(\ln f_m)'$ is

$$x\left(\frac{1}{x} - \frac{2b_1x}{1+b_1x^2} - \cdots - \frac{2b_mx}{1+b_mx^2}\right) = 1 + \sum(-1)^j 2s_j x^{2j}.$$

This expression does not depend on m.

[18] Indeed, since $f(x) = \mathrm{th}(x)$, we have

$$x\frac{f'(x)}{f(x)} = x\frac{\mathrm{ch}^2x - \mathrm{sh}^2x}{\mathrm{ch}^2x}\Big/\frac{\mathrm{sh}x}{\mathrm{ch}\,x} = \frac{2x}{\mathrm{sh}2x}.$$

9.7 *Example: The third stable homotopy group π_3^s*

To begin with let us show that the J-homomorphism $\pi_3\mathrm{SO} \to \pi_3^s$ is surjective. To this end, let M be a framed manifold of dimension 3 in $\mathbb{R}^{3+k}$ representing a class in π_3^s. Every orientable manifold M of dimension 3 bounds a compact orientable manifold W of dimension 4. We may assume that W is simply connected by applying 1-surgery if necessary. Let F be an orientable characteristic surface embedded in W.[19] By taking the connected sum of W with $\mathbb{C}P^2\#(-\mathbb{C}P^2)$ and of F with the characteristic sphere in $\mathbb{C}P^2\#(-\mathbb{C}P^2)$ we may create a transverse sphere, and therefore we may reduce F to a sphere. By taking connected sums of (W, F) with $(\pm\mathbb{C}P^2, \mathbb{C}P^1)$ we may assume that the normal Euler number of F is 1. Thus, the boundary of a neighborhood U of F is a sphere. Consequently, the manifold $W \setminus U$ is a framed cobordims of M to a framed sphere. Thus the J-homomorphism is surjective.

[19] Its existence is proved for example in Scorpan, page170.

Since $\pi_3\mathrm{SO} = \mathbb{Z}$, the image of the J-homomorphism is a cyclic group. Let j_3 denote the order π_3^S; it is also the order of the image of the J-homomorphism. We have seen that for an orientable closed 4-manifold M with $w_2M = 0$, we have $p_1[M] = 2[f_{T^\perp M}]$, where $[f_{T^\perp M}]$ belongs to the kernel of the J-homomorphism, i.e., the number $[f_{T^\perp M}]$ is divisible by j_3. Let M be a closed even 4-manifold with $p_1[M] = 48$, e.g., we may choose M to be the manifold $K3$. Then $p_1[M]/2 = 24$ is divisible by j_3. In particular, we have that $j_3 \leq 24$.

9.8 Further reading

10

The h-cobordism theorem

In this chapter we will discuss various techniques of simplifying cobordisms of (non-framed) manifolds of dimension m in the Euclidean space $\mathbb{R}^{m+k}$ with $k \gg m$.

We have seen that a cobordism W in $\mathbb{R}^{m+k} \in [0,1]$ between manifolds W_0 and W_1 can be represented as a composition of spherical cobordisms. In particular, the cobordism manifold W is diffeomorphic to the union

$$W_0 \times [0, \varepsilon] \cup H_1 \cup \cdots \cup H_n$$

with smoothened corners, where H_i is the i-th handle, see Figure 10.1. Three comments are in order.

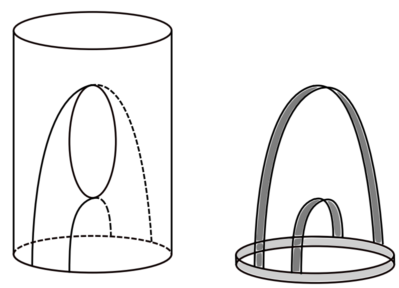


Figure 10.1: A cobordism W, and a presentation of the manifold W as a union of $W_0 \times [0, \varepsilon]$ and handles.

- A union presentation of a cobordism W determines the cobordism W up to a diffeomorphism. For this reason we will often describe the cobordism W by its union presentation.

- To simplify the discussion, we will use manifolds with corners in our constructions whenever it is convenient. We already know that corners can always be smoothened.

- The handles in a presentation of a cobordism are attached consecutively in the order of their appearance in the presentation.[1]

In general, a presentation of a cobordism W is not unique. For example, the cobordism W in Figure 10.1 with two handles H and H' has at least two obvious different presentations in which the handles H and H' are attached in different orders. According to the Handle Rearrangement Lemma 10.1, handles in a presentation of a cobordism can always be rearranged in such a way that the handles are attached

[1] For example, the handle H_1 is attached before any other handle to the upper boundary $W_0 \times \{\varepsilon\}$ of the collar neighborhood of W_0, while the handle H_2 is attached after H_1 along the upper boundary of the cobordism $W_0 \times [0, \varepsilon] \cup H_1$, not necessarily along $W_0 \times \{\varepsilon\}$. The attaching sphere of the k-th handle H_k may touch any previously attached handle.

consecutively in ascending order of their indices. By the Handle Creation Lemma 10.2, whenever it is desired, two consecutive new handles H and H' of indices i and $i+1$ can be added to a presentation of a cobordism. Informally, the handle H creates a hole, while H' caps the hole off. Conversely, the Handle Cancellation Lemma 10.3 asserts that whenever in a presentation there are two consecutive handles H and H' of indices i and $i+1$ such that the attaching sphere of H' intersects the belt sphere of H transversally at a unique point, the handles can be cancelled. The Handle Slide Lemma 10.4 offers a modification of handles H and H' of the same index $i < m-1$. It asserts, for example, that if the handles H and H' are attached to a simply connected manifold along attaching spheres in homotopy classes $[S_a]$ and $[S_b]$, then the handle H can be replaced by a handle attached along an attaching sphere in the homotopy class $[S_a] + [S_b]$. Finally, the Handle Trade Lemma 10.5 suggests a procedure of trading a handle of index i for a handle of index $i+2$ provided that the index i of the handle is below the middle range, $i \leq m/2 - 1$. A handle trading is possible only if a certain relative homotopy group is trivial.

If the described techniques of handle manipulation seems poorly motivated to a reader, the reader may want to begin with the proof of the h-cobordism thereom and refer to sections with a detailed description of techniques whenever is necessary.

10.1 Handle manipulation techniques

10.1.1 Handle rearrangement

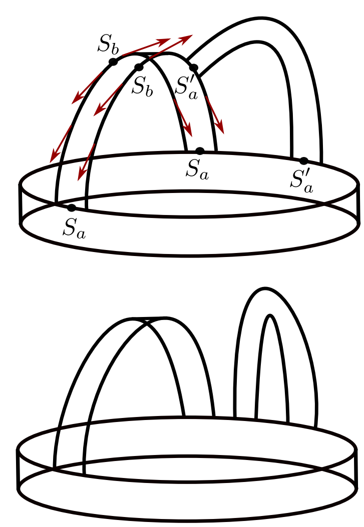


Figure 10.2: Since S'_a is disjoint from S_b, there is a radial isotopy which brings S'_a to the complement to h_j.

In some occasions two consecutively attached handles can be rearranged. For this modification of the cobordism W, we may assume that W has only two handles, H and H', of indices i and i' respectively. The handle $H = D^i \times D^j$ is attached along the thickening $S^{i-1} \times D^j$ of the attaching sphere S_a. The result of the corresponding spherical surgery on W_0 is a manifold $W_{1/2}$. It contains the thickening $D^i \times S^{j-1}$ of the belt sphere S_b. Note that the belt sphere is of dimension $j - 1 = m - i$. The handle $H' = D^{i'} \times D^{j'}$ is attached to the intermediate manifold $W_{1/2}$ along an attaching sphere S'_a of dimension $i' - 1$.

Lemma 10.1. *Suppose that $i' \leq i$, then the cobordism W is diffeomorphic to a cobordism $W_0 \times [0, \varepsilon] \cup H' \cup H$.*

Proof. Without loss of generality we may assume that the attaching sphere S'_a of the handle H' is transverse to the belt sphere S_b of the handle H in $W_{1/2}$. Since the dimension of S'_a is $i'-1$, while the dimension of S_b is $m-i$, we conclude that S'_a is disjoint from S_b if $i'-1+m-i<m$. The latter holds whenever $i' \leq i$. When the attaching sphere S'_a is disjoint from the belt sphere S_b, we may assume that the thickening of S'_a is disjoint from the thickening of S_b in $W_{1/2}$, see Figure 10.3.[2] and therefore we may first attach to $W_0 \times [0,\varepsilon]$ the handle H', and then the handle H. □

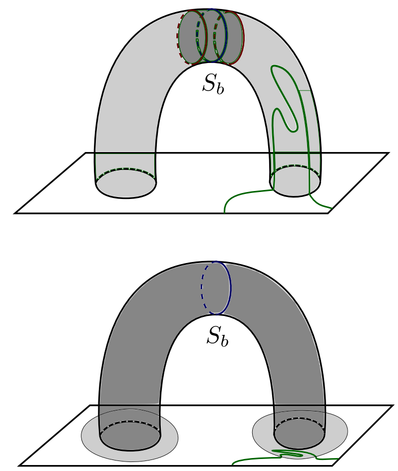


Figure 10.3: There is a radial isotopy in $\mathbb{R}^i$, along the vector field $v(x)=x$. It preserves the origin 0, and in finite time stretches a small neighborhood of 0 over the unit disc D^i. More generally, suppose a spherical surgery is performed on a manifold W_0 producing a manifold $W_{1/2}$, i.e., the thickening h_i of an attaching sphere is replaced with the thickening of $h_j = D^i \times S^{j-1}$ of a belt sphere S_b. There is a radial vector field over h_j assigning to each point (x,s) the radial vector $v(x)=(x,0)$. The radial vector field v can be extended to a vector field over $W_{1/2}$ with support in a neighborhood of h_j. In finite time the radial isotopy along v stretches a small (dark grey) neighborhood U of S_b over all h_j. In particular, it caries the complement $h_j \setminus U$ outside of h_j.

10.1.2 Handle creation

Let W be a trivial cobordism $W_0 \times [0,1]$. We will show that it has a structure of a cobordism with two handles H_i of index i and H_{i+1} of index $i+1$. In other words, we may create two handles of consecutive indicies on a cobordism without changing the diffeomorphism type of the cobordism.

[2] We can use the radial isotopy to replace the thickening $h_{i'}$ of the attaching sphere S'_a with a thickening of a new attaching sphere S''_a disjoint from h_j. The spherical cobordism along S'_a is diffeomorphic to the spherical cobordism along S''_a.

Recall that the i-handle $H_i = D^i \times D^j$ is bounded by h_i and $h_j = D^i \times S^{j-1}$, see Figure 10.4. Furthermore, the part h_j itself splits as the union of $h_j^+ = D^i \times S_+^{j-1}$ and $h_j^- = D^i \times S_-^{j-1}$, where S_+^{j-1} and S_-^{j-1} are the upper and lower hemispheres. Similarly, the boundary of the $i+1$-handle $H^{i+1} = D^{i+1} \times D^{j-1}$ is a union of $h_{i+1} = S^i \times D^{j-1}$ and $h_{j-1} = D^{i+1} \times S^{j-2}$. The part h_{i+1} itself can be represented as a union of $h_{i+1}^+ = S_+^i \times D^{j-1}$ and $h_{i+1}^- = S_-^i \times D^{j-1}$. Since S_+^i is the upper hemisphere of S^i diffeomorphic to D^i, and S_+^{j-1} is the upper hemisphere diffeomorphic to D^{j-1}, we may identify h_{i+1}^+ with h_j^+ to form a disc $H_i \cup H_{i+1}$, see Figure 10.5.

Attaching the disc $H_i \cup H_{i+1}$ along the disc $h_i \cup h_{i+1}^-$ to $W_0 \times [0,1]$ results in a cobordism $W \times [0,1] \cup H_i \cup H_{i+1}$. On the other hand, attaching the disc $H_i \cup H_{i+1}$ along the disc does not change the diffeomorphism type of $W_0 \times [0,1]$. Thus, we proved the following lemma.

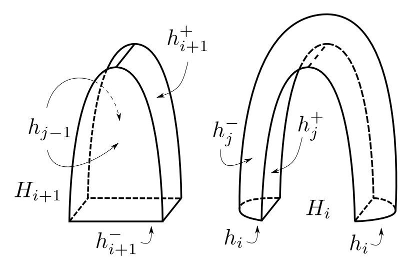


Figure 10.4: The handles H_i and H_{i+1}.

Lemma 10.2. *Let W be a trivial cobordism $W_0 \times [0,1]$. Let S_-^i denote an embedded disc in $W_0 \times \{1\}$. Then there is a handle H_i with attaching sphere ∂S_-^i, and a handle H_{i+1} attached to $W_0 \times [0,1] \cup H_i$ such that $W_0 \times [0,1] \cup H_i \cup H_{i+1}$ is a trivial cobordism.*

10.1.3 Handle cancellation

Handle cancellation is a procedure inverse to the handle creation. We begin with a cobordism $W = W_0 \times [0, \varepsilon] \cup H_i \cup H_{i+1}$ where $H_i = D^i \times D^j$ is a handle of index i and H_j is a handle of index j, and determine when the two handles can be cancelled. We note that the cobordism W is a composition of a spherical cobordism from W_0 to $W_{1/2}$ of index i, and a spherical cobordism from $W_{1/2}$ to W_1 of index $i+1$.

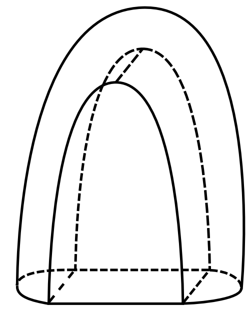


Figure 10.5: The disc $H_i \cup H_{i+1}$.

Lemma 10.3. *Suppose that the attaching sphere S'_a of H_{i+1} intersects the belt sphere S_b of the handle H_i transversally in $W_{1/2}$ at a unique point. Then the cobordism W is trivial.*

Proof. Since S'_a intersects S_b transversally, we may assume that near each intersection point the sphere S'_a is parallel to the core disc of H_i, i.e., near the intersection point p in $S'_a \cap S_b$ the sphere agrees with $D^i \times \{p\} \subset H_i$. Let v denote the radial vector field on $h_j = D^i \times S^{j-1}$.[3] It is a smooth vector field which we may extend over $W_{1/2}$ with support in a neighborhood of h_j. An isotopy along the vector field v stretches a small neighborhood of the belt sphere $0 \times S^{j-1}$ over the whole solid torus h_j. In particular, it brings S'_a to an attaching sphere which intersects h_j along discs $D^i \times \{p\}$ parallel to the core. Therefore we may assume that the handle H_{i+1} is attached to $W_{1/2}$ along $h_{i+1} = h^+_{i+1} \cup h^-_{i+1}$ where h^+_{i+1} coincides with h^+_j, while h^-_{i+1} is a thickening of a disc in the closure of $W_{1/2} \setminus h_j$. Therefore, the attaching of the handles H_i and H_{i+1} is the same as in the handle creation procedure. Thus, the cobordism W is trivial. □

[3] Recall that the boundary of H_i consists of the union $h_i \cup h_j$. The spherical surgery on W_0 corresponding to H_i consists of replacing $h_i = S^{i-1} \times D^j$ with $h_j = D^i \times S^{j-1}$.

10.1.4 Handle slides

Suppose now that the cobordism W of a manifold W_0 consists of attaching two handles H and H' of index $i < m-1$, where m is the dimension of W_0. In this case it is possible to change the homotopy classes of attaching spheres without changing the diffeomorphism type of the cobordism W.

Suppose that H is attached along an attaching sphere S_a, while H' is attached along S'_a. By the Handle Rearrangement Lemma we may assume that both handles H and H' are attached at the same time to $W_0 \times \varepsilon$. The sphere S'_a has a neighborhood h_i which can be identified with $S'_a \times D^j$. Pick a point q on the boundary of D^j. Then $S'_a \times \{q\}$

is a parallel displacement of S'_a which we will denote by S''_a. Let γ be an embedded path in $W_0 \times \{\varepsilon\}$ parametrized by $[0,1]$ from a point $\gamma(0)$ on the sphere S_a to a point $\gamma(1)$ on a parallel displacement S''_a of the sphere S'_a. Finally, let S'''_a denote the connected sum of S_a with S''_a along the path γ. More precisely, let $[0,1] \times D^{i-1}$ denote an i-dimensional tube about the curve $\gamma = [0,1] \times \{0\}$ such that the disc $\{0\} \times D^{i-1}$ is a neighborhood of $\gamma(0)$ in S_a, and the disc $\{1\} \times D^{i-1}$ is a neighborhood of $\gamma(1) = \{s\} \times \{q\}$ in S''_a, and suppose that there is no other intersection points between the tube $[0,1] \times D^{i-1}$ and the two attaching spheres. The connected sum S'''_a is formed by removing $\{0\} \times S^{i-1}$ from S_a, removing $\{1\} \times S^{i-1}$ from S''_a, and attaching the remaining part $[0,1] \times \partial D^{i-1}$ of the boundary of the tube.

Lemma 10.4. *Let W be a cobordism $W_0 \times [0,\varepsilon] \cup H \cup H'$ of a path connected manifold W_0. Then W is diffeomorphic to a cobordism $W_0 \times [0,\varepsilon] \cup H' \cup H'''$, where H''' is a handle of index i attached along S'''_a.*

Proof. Let $W_{1/2}$ be the manifold obtained from W_0 by spherical surgery corresponding to H'. In other words, the manifold $W_{1/2}$ is obtained from W_0 by replacing the solid torus h'_i with the solid torus $h'_j = D^i \times S^{j-1}$. We note that the union of the tube $[0,1] \times D^{i-1}$ along γ together with the disc $D^i \times \{q\}$ is a disc B in $W_{1/2}$. We may push the arc $S_a \cap \partial B$ through the disc B to the arc $\partial B \setminus S_a$. This isotopy brings S_a to the sphere S'''_a. Therefore, attaching the handle H along S_a is equivalent to attaching a handle H''' along S'''_a. □

Suppose now that W_0 is simply connected. Then elements in $\pi_{i-1}W_0$ are free homotopy classes of maps $S^{i-1} \to W_0$, and $[S'''_a] = [S_a] \pm [S''_a]$ in $\pi_{i-1}W_0$, where the sign depends on whether the connected sum of S_a and S''_a along γ is orientation preserving or not. If $i < m-1$, then we may perform the connected sum along γ so that the connected sum sphere S'''_a is in the class $[S_a] + [S''_a]$, and we may also perform the connected sum so that the class $[S'''_a]$ is $[S_a] - [S''_a]$.

10.1.5 Handle trading

We continue to study a cobordism W of a manifold W_0 of dimension m. It turns out that if the relative homotopy group $\pi_i(W, W_0)$ is trivial, and i is below the middle range (more precisely, $i \leq m/2 - 1$), then every handle H_i of index i can be traded for a handle of index $i+2$.

Lemma 10.5. *Let* $W = W_0 \times [0,\varepsilon] \cup \alpha H'_i \cup H_i \cup \beta H_{i+1}$ *be a cobordism of a manifold* W_0 *of dimension* m *which consists of attaching* $\alpha + 1$ *handles of index* $i > 0$*, and* β *handles of index* $i+1$*. Suppose that* $m \geq 2i+2$*, and* $\pi_i(W, W_0) = 0$*. Then* W *is diffeomorphic to a cobordism of the form* $W_0 \times [0,\varepsilon] \cup \alpha H'_i \cup \beta H_{i+1} \cup H_{i+2}$*.*

Proof. The core $D^i \times \{pt\}$ of the handle H_i represents a trivial element in the trivial group $\pi_i(W, W_0)$. Hence there is a disc B^{i+1} in W bounded by $\partial B^{i+1} = S^i_+ \cup S^i_-$ where S^i_+ is the core $D^i \times \{pt\}$ of the handle H_i, while S^i_- is a disc in $W_0 \times \{\varepsilon\}$. By slightly perturbing B^{i+1} we may assume that it is an embedded disc in the manifold W of dimension $m+1 \geq 2i+3$, and it is disjoint from the cores D^i and D^{i+1} of all other handles of the cobordism W. Then B^{i+1} can be pushed to W_1.[4] Since we may assume that B^{i+1} is an embedded disc in W_1, we may use it to create cancelling handles H'_{i+1} and H_{i+2} on top of W_1. The attaching sphere of H'_{i+1} is the sphere B^{i+1} which intersects the belt sphere of H_i at a unique point. Therefore the handles H'_{i+1} and H_i can be cancelled. Thus, the handle H_i is replaced with the handle H_{i+2}. □

[4] Note that the parts $B^{i+1} \cap H'_i$ and $B^{i+1} \cap H_{i+1}$ can be pushed radially to h'_j and h_{j-1}, while the rest of B^{i+1}, which is in $W_0 \times [0,\varepsilon]$ can be pushed upward to $W_0 \times \{\varepsilon\}$.

10.2 The h-cobordism theorem

We say that a cobordism W between manifolds W_0 and W_1 is an *h-cobordism* if the inclusions $W_0 \subset W$ and $W_1 \subset W$ are homotopy equivalences.

Lemma 10.6. *Let* W *be an h-cobordism of a path connected manifold* W_0*. Then* W *has a structure of a cobordism with no* 0*-handles as well as no* $m+1$ *handles.*

Proof. By the Handle Rearrangement Lemma we may assume that the handles in the cobordism W are attached in the ascending order of their indices. Since attaching handles of index ≥ 2 does not change the number of path components of the cobordism, we deduce that the initial part $W_0 \times [0,\varepsilon] \cup \alpha_0 H_0 \cup \alpha_1 H_1$ of the cobordism W which contains all handles of W of indices 0 and 1 is path connected. In fact, we may choose α_0 handles of index 1 and attach them before any other handle of index 1 so that the initial part of the cobordism W that consists of attaching all α_0 handles of index 0, and α_0 chosen handles of index 1 is path connected. Consider one of the 0-handles H_0 and one of the chosen 1-handles H_1 abutting H_0. Then the attaching sphere of

H_1 intersects the belt sphere of H_0 at precisely one point, and therefore the two handles H_0 and H_1 can be cancelled. Thus, all 0-handles of a path connected cobordism W can be paired with some of the 1-handles and cancelled.

We may now reverse the cobordism W and cancel all 0-handles of the reversed cobordism from W_1 to W_0. Cancelling 0-handles of the reversed cobordism corresponds to cancelling $(m+1)$-handles of the original cobordism W. □

Lemma 10.7. *Let W be an h-cobordism of a connected manifold W_0 of dimension $m \geq 5$. Then W has a structure of an h-cobordism handles of indices q and $q+1$ if $m = 2q$ and $q, q+1$ and $q+2$ if $m = 2q+1$.*

Proof. By Lemma 10.6 all 0 and $m+1$ handles in W can be cancelled. Next, since the relative fundamental group $\pi_1(W, W_0)$ is trivial, we may trade 1-handles for 3-handles. Similarly, since $\pi_1(W, W_1)$ is trivial, we may trade 1-handles of the reversed cobordism for 3-handles (such a trade corresponds to trading m-handles of the original cobordism W for $m-2$-handles).

In general, we may trade handles of an h-cobordism by induction in the ascending order of indices $i = 1, 2, ...$ of handles until all handles of indices $i \leq m/2 - 1$ are traded, and then in the descending order of indices of handles $i = m, m-1, ...$ until all handles of indices $m + 1 - i \leq m/2 - 1$ are traded (the latter corresponds to $i \geq m/2 + 2$). □

Let W be a manifold with boundary which is obtained from a manifold with boundary W' by attaching n handles H_i^α of the same index i along the boundary of W'. Here the upper index α distinguishing the i-handles runs over the set $1, ..., n$. Each handle H_i^α is attached to W' along a solid torus h_i^α and contains a core D_α^i. We observe that there is a continuous radial deformation of the pair (H_i^α, h_i^α) to the pair $(D_\alpha^i, \partial D_\alpha^i)$. It defines a homotopy equivalence between W and the topological space W_{CW} obtained from W' by attaching the cores D_α^i of the handles H_i^α.

Lemma 10.8. *Let W be a manifold with boundary obtained from a simply connected manifold with boundary W' by attaching n handles of the same index i. Then the group $\pi_k(W, W')$ is trivial for $k < i$, and it is free abelian of rank n generated by the homotopy classes $[D_\alpha^i]$ of the n cores of the handles.*

Proof. We have seen that W is homotopy equivalent to the union W_{CW}

of W' and the cores $\cup D^i_\alpha$. When $k < i$, the image of every generic map f representing an element in $\pi_k(W_{CW}, W')$ misses a point in each D^i_α, and therefore the representing map f can radially be deformed off the discs D^i_α to a map with image in W'. Thus, the groups $\pi_k(W, W')$ with $k < i$ are trivial.

Since W' is simply connected and the pair (W, W') is $(i-1)$-connected, we conclude that $\pi_i(W, W')$ is isomorphic to $\pi_i(W_{CW}/W') = \pi_i(\vee S^i)$. The latter group is isomorphic to $\mathbb{Z}^n$. □

In fact, the proof of Lemma 10.8 gives another interpretation of the group $\pi_k(W, W')$ which we will record as a corollary.

Corollary 10.9. *Under the hypotheses of Lemma 10.8, the group $\pi_i(W, W')$ is isomorphic to the group $\pi_i(W/W')$, while the space W/W' is homotopy equivalent to the wedge of spheres $D^i_\alpha/\partial D^i_\alpha$.*

Theorem 10.10. *Every h-cobordism W of a simply connected manifold W_0 of dimension $2q \geq 5$ is trivial.*

Proof. We have seen that W reduces to an h-cobordism with handles of indices q and $q+1$. Let $W_{1/2}$ denote the manifold obtained from W_0 by spherical surgeries corresponding to the q-handles of W. Let $W_{0,1/2}$ denote the initial part of the cobordism W of W_0 to $W_{1/2}$. It follow that the boundary homomorphism

$$\partial\colon \pi_{q+1}(W, W_{0,1/2}) \to \pi_q(W_{0,1/2}, W_0)$$

in the long exact sequence of the triple $(W, W_{0,1/2}, W_0)$ is an isomorphism.[5] By Lemma 10.8, the relative homotopy group $\pi_q(W_{0,1/2}, W_0)$ of the initial part of the cobordism W is generated by the homotopy classes $[D^q_\alpha]$ of the core discs of the q-handles, while the relative homotopy group $\pi_{q+1}(W, W_{0,1/2})$ of the terminal part of the cobordism W is generated by the homotopy classes of the core discs D^{q+1}_β of the $(q+1)$-handles.

[5] Note that since W is an h-cobordism all groups $\pi_*(W, W_0)$ are trivial.

As an element of the group generated by $[D^q_\alpha]$, each class $\partial[D^{q+1}_\beta]$ can uniquely be written as a linear combination

$$\partial[D^{q+1}_\beta] = \sum k_{\alpha,\beta}[D^q_\alpha]$$

with some coefficients $k_{\alpha,\beta}$. To determine the geometric meaning of coefficients $k_{\alpha,\beta}$ it is convenient to postcompose the boundary homomorphism ∂ with the isomorphism of Corollary 10.9 onto the homotopy group of the wedge of spheres $D^q_\alpha/\partial D^q_\alpha$. We can identify now each coefficient $k_{\alpha,\beta}$ with the degree of the map

$$S^q = \partial D_\beta^{q+1} \to W_{0,1/2} \to \vee(D_\alpha^q/\partial D_\alpha^q) \to D_\alpha^q/\partial D_\alpha^q \qquad (10.1)$$

which maps S^q to the attaching sphere in $W_{1/2} \subset W_{0,1/2}$ and then collapses first $W_{0,1/2}$ to the wedge of spheres, and then all spheres in the wedge except for $D_\alpha^q/\partial D_\alpha^q$ to the distinguished point. In other words, the coefficient $k_{\alpha,\beta}$ is the algebraic number of times the attaching sphere ∂D_β^{q+1} of the handle H_β^{q+1} wraps around the core disc D_α^q of the handle H_β^{q+1}. We claim that this is the same as the algebraic intersection number of the attaching sphere of the handle H_β^{q+1} and the belt sphere of H_α^q. Indeed, the degree of the map (10.1) is the number of signed preimages of the center $\{0\} \in D_\alpha^q$. Each of the signed preimages corresponds to the signed intersection of the attaching sphere and the belt sphere, which proves the claim.

Since the homomorphism ∂ is an isomorphism the matrix $[k_{\alpha,\beta}]$ can be diagonalized by means of elementary row and column operations. We note that reordering rows and columns correspond to reordering q and $q+1$-handles. A multiplication of a row or a column by -1 corresponds to changing the orientation of a q or $q+1$-handle. Finally, adding or subtracting one row to another, or one column to another, corresponds to sliding handles. Thus, without loss of generality, we may assume that the matrix $[k_{\alpha,\beta}]$ is the identity matrix. In particular, each handle H_α of index q corresponds to a handle H_β of index $q+1$ such that the algebraic number of intersection points of the belt sphere of H_α and the attaching sphere of H_β is 1. Since $W_{1/2}$ is of dimension $2q \geq 6$, we may use the Whitney trick to arrange that the belt sphere of H_α and the attaching sphere of H_β has precisely one intersection point. Then, by the Handle Cancellation Lemma, the handles H_α and H_β can be cancelled.

Since the number of handles in W can always be decreased, we may iterate the process and obtain a structure of a cobordism with no handles. In other words, W is a trivial cobordism. □

Theorem 10.11. *Every h-cobordism W of simply connected manifold W_0 of dimension $2q+1 \geq 5$ is trivial.*

Let us suppose that the cobordism W is a composition of a cobordism $W_{[0,1]}$ from W_0 to W_1 with only handles of index q, a cobordism $W_{[1,2]}$ from W_1 to W_2 with handles of index $q+1$, and a cobordism $W_{[2,3]}$ from W_2 to W_3 with handles of index $q+2$.

Lemma 10.12. *We have* $\pi_*(W_{[0,2]}, W_0) = 0$ *for* $* \leq q$.

Proof. In the long exact sequence

$$\to \pi_{*+1}(W, W_0) \to \pi_{*+1}(W, W_{[0,2]}) \to \pi_*(W_{[0,2]}, W_0) \to \pi_*(W, W_0) \to$$

of the triple $(W, W_{[0,2]}, W_0)$, the groups $\pi_*(W, W_0)$ are trivial since W is an h-cobordism. On the other hand, the groups $\pi_{*+1}(W, W_{[0,2]})$ are trivial for $* \leq q$ by Lemma 10.8. Therefore, $\pi_*(W_{[0,2]}, W_0)$ are trivial for $* \leq q$. □

Lemma 10.13. *The homomorphism* $\partial\colon \pi_{q+1}(W_{[0,2]}, W_{[0,1]}) \to \pi_q(W_{[0,1]}, W_0)$ *is surjective.*

Proof. Note that the homomorphism ∂ in the long exact sequence of a triple is followed by the homomorphism to the zero relative homotopy group $\pi_q(W_{[0,2]}, W_0)$. Thus, ∂ is surjective. □

Proof of Theorem 10.11. We may identify the relative homotopy group $\pi_{q+1}(W_{[0,2]}, W_{[0,1]})$ with the group $\pi_{q+1}(\vee D^{q+1}_\beta / \partial D^{q+1}_\beta)$ generated by classes represented by the core discs D^{q+1}_β of the handles of W of index $q+1$. Similarly, we may identify the group $\pi_q(W_{[0,1]}, W_0)$ with the q-th homotopy group of the wedge of spheres $\vee D^q_\alpha / \partial D^q_\alpha$, generated by classes $[D^q_\alpha]$. Then

$$\partial[D^{q+1}_\beta] = k_{\alpha,\beta}[D^q_\alpha],$$

for some numbers $k_{\alpha,\beta}$. Furthermore, $k_{\alpha,\beta}$ is the algebraic intersection number of the attaching sphere of H^{q+1}_β and the belt sphere of H^q_α in W_1. We can use the elementary row and column operations to simplify the matrix $[k_{\alpha,\beta}]$ so that $k_{1,1} = 1$. The elementary row and column operations can be realized by handle moves. Therefore, we may find a handle decomposition of W such that the belt sphere of a q-handle algebraically intersects the attaching sphere of a $(q+1)$-handle at one point.

Suppose that $2q + 1 \geq 7$. Then using the Whitney trick we may modify the attaching sphere of the $(q+1)$-handle so that it intersects the belt sphere of the q-handle at precisely one point. Then the two handles can be cancelled. We may repeat the argument to cancel all q-handles. Next, we may flip the cobordism W up side down to obtain an h-cobordism with only q and $q+1$-handles. Finally, we repeat the argument to eliminate all q-handles. This eliminates the $q+1$ handles as well, since in this case the homomorphism ∂ is an isomorphism.

Suppose now that $2q + 1 = 5$. In this case we need to justify our application of the Whitney trick. We note that the complement to the belt sphere of H^q_α in W_1 is diffeomorphic to the complement of the attaching sphere of H^q_α in W_0. Since the dimension of the attaching sphere of H^q_α is 1, the complement to the belt sphere of H^q_α in W_1 is simply connected. For this reason, we may choose the Whitney disc in the Whitney trick disjoint from the belt sphere of H^q_α, and the attaching sphere of H^{q+1}_α. Thus, the Whitney trick is possible even in the case $2q + 1 = 5$. □

11

Surgery on maps of simply connected manifolds

One of the main problems of Surgery Theory is to classify smooth closed oriented manifolds of a given dimension.

Question 11.1. Let X be a finite simply connected CW-complex. Is X homotopy equivalent to a smooth closed oriented manifold of a given dimension n?

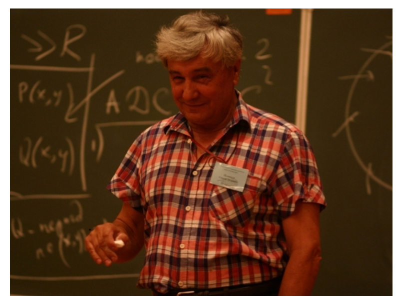


Figure 11.1: Sergey Novikov, 1938–

The obvious obstruction is the Poincare duality. If X is a homotopy equivalent to a closed oriented manifold of dimension n, then its cohomology and homology groups satisfy the Poincare duality: there is an integral homology class $[X] \in H_n(X)$, called the *fundamental class* of X, such that the cap product $[X]\cap$ is an isomorphism of the cohomology group $H^i(X)$ with the homology group $H_{n-i}(X)$. A CW-complex with Poincare duality will be called a *geometric Poincare complex* of dimension n.

11.1 *Normal maps*

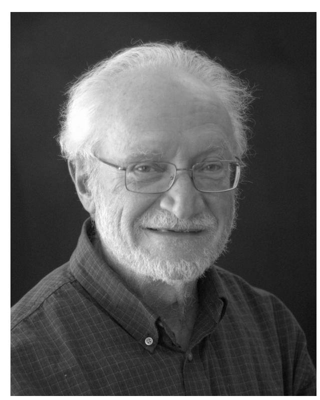


Figure 11.2: William Browder, 1934–

In this chapter all spaces will be assumed to be simply connected. For such a space X, every element in the homotopy group is represented by a free (i.e., not pointed) map of a sphere to X.

To begin with let us observe that if there exists a homotopy equivalence $f\colon X \simeq M$ of a finite CW-complex to a closed oriented manifold, then there also exists a vector bundle $\nu_X = f^* T^{\perp} M$ over X that plays the role of the normal bundle. To slightly simplify the notation, let $\nu = \nu_M$

denote the perpendicular bundle of the manifold M in $\mathbb{R}^{m+k}$, and let ν_X be a vector bundle over X of dimension k.

Definition 11.2. A *normal map* $(f, b)\colon (M, \nu) \to (X, \nu_X)$ consists of a map f, and a fiberwise isomorphism $b\colon \nu \to \nu_X$ covering f.

Our strategy will be to start with an arbitrary continuous normal map (f, b) from a smooth closed oriented manifold of dimension m and then try to modify it by means of *normal surgery* so that it is a homotopy equivalence.

A *normal cobordism* between maps (f_0, b_0) of (M_0, ν_0) and (f_1, b_1) of (M_1, ν_0) to (X, ν_X) consists of a cobordism W between M_0 and M_1 with normal bundle ν as well as a normal map $(f, b)\colon (W, \nu) \to (X, \nu_X)$ that restricts to the normal maps (f_0, b_0) and (f_1, b_1) over the two boundary components of W. Given a normal cobordism, we say that (f_1, b_1) is obtained from (f_0, b_0) by a *normal surgery*. When W is a spherical cobordism between M_0 and W_1, we say that (f, b) is a *spherical normal cobordism*.

When necessary we may embed the finite CW-complex X into a high dimensional Euclidean space, and then replace X with its ε-neighborhood in the Euclidean space, which is homotopy equivalent to X. In other words, whenever it is convenient we may assume that the CW-complex X is a high dimensional manifold. In particular, we may assume that $f\colon M \to X$ is an embedding. We will denote the group $\pi_{k+1}(X, M)$ by $\pi_k(f)$. In other words, an element of $\pi_k(f)$ is represented by a map $i\colon S^k \to M$ as well as a map $D^{k+1} \to X$ extending the map $f \circ i$. The groups $\pi_k(f)$ fit a long exact sequence.

$$\cdots \to \pi_*(f) \to \pi_*(M) \xrightarrow{f_*} \pi_*(X) \to \pi_{*-1}(f) \cdots$$

We aim to modify the normal map f by normal surgery to a homotopy equivalence, i.e., we aim to kill homotopy groups of f.

Suppose not all homotopy groups of f are trivial. Suppose there is an element $\lambda \neq 0$ in $\pi_q(f)$ where $q \leq m/2$. It is represented by an embedding $i\colon S^q \to M$ as well as an extension $j\colon D^{q+1} \to X$ of the map $f \circ i$. We may assume that j is an inclusion. Denote the image of i by S' and the image of j by D_X. Let us investigate under what conditions there exists a normal surgery along λ and what effect it has on $\pi_*(f)$. We will only sketch the argument as it very similar to one in Chapter 5.

As any vector bundle over a disc, the restriction of ν_X to the disc D_X is trivial, i.e., there exists a frame $e_1, .., e_k$ of the vector bundle ν_X over the disc D_X. Since the vector bundle ν over S' is fiberwise isomorphic to ν_X over ∂D_X, the frame $e_1, ..., e_k$ defines a frame $\tau_1, ..., \tau_k$ of the perpendicular bundle of M over S'. We may extend the vector fields $\tau_1, ..., \tau_k$ to a frame of the perpendicular bundle of M near S', and then extend it further to a neighborhood of S' in $\mathbb{R}^{m+k}$. We will regard the directions of the vector fields $\tau_1, ..., \tau_k$ as vertical up.

Let D' be a disc in $\mathbb{R}^{m+k} \times [0,1]$ bounded by $S' \in \mathbb{R}^{m+k} \times \{0\}$ such that the collar neighborhood of S in D is of the form $S \times [0, \varepsilon)$. Choose an orthonormal frame $v' = \{v'_i\}$ of the perpendicular bundle of D'. By the Multicompression theorem, there is a small ambient isotopy F_t of S to a sphere $F_1 S$ in $\mathbb{R}^{m+k}$ that brings vector fields v'_i over S' to τ_i over $F_1 S'$ for $i = 1, ..., k$. The ambient isotopy extends to an isotopy of $\mathbb{R}^{m+k} \times [0,1]$ and brings the disc D' with vector fields $\{v'_i\}$ to a disc D with normal vector fields $\{v_i\}$.

We then compress $F_1 S'$ to its projection S in M along the vertical up planes $\langle \tau_1, ..., \tau_k \rangle$, and slightly perturb S so that S is immersed. We say that S is the *Wall representative* of the class λ. We note that the frame $\tau_1, ..., \tau_k$ of M over S extends to the normal vector fields $v_1, ..., v_k$ over D. Furthermore, when S is embedded, it has a frame $v_1, ..., v_{m-q+k}$ which extends over D.

If the Wall representative S is embedded, then a normal surgery along λ is possible. Indeed, the disc D together with $v_1, ..., v_{m-q+k}$ is a base for framed surgery. Thus there exists a spherical cobordism $W \approx M \cup D$ of the manifold M. Furthermore, the vector bundle ν extends to a vector bundle ν_W over W by means of vector spaces $\langle v_1, ..., v_k \rangle$. Finally, there is a map $(W, \nu_W) \to (X, \nu_X)$ which restricts to the map (f, b) over (M, ν), takes D isomorphically to its copy D_X in X and sends ν_W over D to ν_X over D_X by $v_i \mapsto e_i$.

In general the Wall representative S' may not be embedded. Let $\mu(\lambda)$ denote the algebraic number of self-intersection points of S'. If $q < m/2$, then $\mu(\lambda)$ is trivial. Otherwise, it is an integer when q is even, and an element in $\mathbb{Z}_2$ when q is odd.[1] As in the case of framed cobordisms, we can show that the invariant $\mu(\lambda)$ is well-defined. On the other hand, if $m \geq 5$, then a normal surgery along λ is possible if and only if $\mu(\lambda) = 0$, as in the case $\mu(\lambda) = 0$ we may assume by the Whitney trick that the Wall representative S of λ is embedded.

When a normal surgery along λ is performed the class λ is killed, while the homotopy groups $\pi_i(f)$ are preserved for $i < q$.[2]

[1] In literature, the invariant $\mu(\lambda)$ is often defined differently, as a homotopy obstruction $\mathcal{O}(\lambda)$.

Theorem 11.3. *The obstruction to the existence of a spherical surgery along S is an element $\mathcal{O}(\lambda) \in \pi_q V_k(\mathbb{R}^{m+k-q})$.*

Proof. As in the case of surgery on manifolds, we observe that the normal bundle of D in $\mathbb{R}^{m+k} \times [0,1]$ is trivial and admits an essentially unique trivialization $v_1, .., v_{m+k-q}$. Then, for each point $x \in S$, the vectors $\tau_1, ..., \tau_k$ define an element in $V_k(\mathbb{R}^{m+k-q})$. Thus we defined an element $\mathcal{O}(\lambda)$ in $\pi_q V_k(\mathbb{R}^{m+k-q})$. It can be shown that $\mathcal{O}(\lambda)$ is a complete obstruction to the existence of a spherical surgery along S. ☐

[2] Indeed, recall the following lemma where $n = q$.

Lemma 11.4. *Let W be a spherical cobordism between W_0 and W_1 of index $n + 1 \leq m/2$, where m is the dimension of the manifolds W_0 and W_1. Suppose that the attaching sphere of the corresponding surgery represents an element $x \in \pi_n W_0$. Then $\pi_i W_0 = \pi_i W_1$ for $i < n$, while $\pi_n W_1$ is a factor group of $\pi_n W_0$ by a subgroup containing x.*

Theorem 11.5. *A normal map* $(f,b)\colon(M;\nu)\to(X;\nu_X)$ *of a manifold of dimension* $m=2q$ *or* $m=2q+1$ *with* $m\geq 5$ *is normally cobordant to a q-connected normal map.*

In order to make a normal map $(q+1)$-connected we need to study intersection forms on M and X. Unfortunately, there is no nice geometric interpretation for intersection forms on homotopy groups of a CW-complex X. For this reason, we need to pass from homotopy groups to (co)homology. In the next section we formulate and prove some of the properties of (co)homology groups which need to study surgery on maps.

11.2 Products in homology and cohomology

11.2.1 Tensor products

Let A and B be abelian group. Then the *tensor product* $A\otimes B$ is the abelian group generated by elements $a\otimes b$ where $a\in A$ and $b\in B$ subject to the relations

$$(a+a')\otimes b = a\otimes b + a'\otimes b,$$

$$a\otimes(b+b') = a\otimes b + a\otimes b'.$$

We note that $a\otimes b$ is a (complicated) name for a single element in $A\otimes B$. Not every element in $A\otimes B$ is of the form $a\otimes b$. In general, an element in $A\otimes B$ is of the form $a_1\otimes b_1+\cdots+a_n\otimes b_n$.

Frequently, the tensor product of abelian groups $A\otimes B$ is defined differently, by its universal product. Namely, we say that the tensor product $A\otimes B$ of abelian groups A and B is an abelian group together with a bilinear map $A\times B\to A\otimes B$ which satisfies the universal property that for any abelian group C, any bilinear map $A\times B\to C$ can be uniquely written as a composition of the canonical map $A\times B\to A\otimes B$ and a homomorphism $A\otimes B\to C$.

Example 11.6. Let X and Y be two CW-complexes. Then the chain complex $C_*(X\times Y)$ is canonically isomorphic to the tensor product $C_*X\otimes C_*Y$ of chain complexes, i.e., $C_n(X\times Y)=\sum C_iX\otimes C_{n-*}Y$.[3] It follows that the differential on $C_*(X\times Y)$ is defined by $d(c\otimes c')=dc\otimes c'+(-1)^{deg(c)}c\otimes dc'$. Similarly, we can identify $C^*(X\times Y)$ with $C^*X\otimes C^*Y$ and $\delta(c\otimes c')=\delta c\otimes c'+(-1)^{deg(c)}c\otimes\delta c'$.

[3] It is tempting (but wrong) to assume that $C_n(X\times Y)=\sum C_iX\times C_{n-i}Y$. For example, suppose that X consists of just one cell D^i and Y consists of one cell D^{n-i}. Then $C_iX=C_{n-i}Y=\mathbb{Z}$. We note that $X\times Y$ consists of just one cell $D^n=D^i\times D^{n-i}$ and therefore $C_n(X\times Y)$ is isomorphic to $\mathbb{Z}=\mathbb{Z}\otimes\mathbb{Z}$, not $\mathbb{Z}\times\mathbb{Z}$.

11.2.2 Naturality of the cup and cap products

Recall that the diagonal map $X \to X \times X$ of a CW-complex X can always be approximated by a cellular map $\Delta \colon X \to X \times X$ which sends each cell of degree i to cells of degrees $\leq i$. The cup product $[x] \cup [y]$ of cohomology classes in X is defined to be the class $[\Delta^*(x \otimes y)]$. It follows that the cup product is natural in the sense that for any map $f \colon X \to Y$ of CW-complexes, and any cohomology classes a, b in H^*Y, we have [4]

$$f^*(a \cup b) = f^*a \cup f^*b.$$

The slant product $H_n(X \times X) \otimes H^k X \to H_{n-k}X$ is defined in terms of cochains by $a \otimes a'/b \mapsto b(a')a$. Finally, the cap product $H_n X \otimes H^k X \to H_{n-k}X$ is defined again in terms of cochains by $a \otimes b \mapsto \Delta_* a/b$. The cap product also satisfies the naturality property that for any map $f \colon X \to Y$ of CW-complexes, and any classes $x \in H_n X$ and $y \in H^k Y$, we have[5]

$$f_*(x \cap f^*y) = f_*x \cap y.$$

The cap and cup product are related by means of the *cap-cup* formula

$$u(x \cap v) = (u \cup v)(x)$$

for any $u \in C^k X, v \in C^{n-k}X$ and $x \in C_n X$.

Exercise 11.7. Prove the naturality properties of the cup and cap product, as well as the cap-cup formula.[6]

The products on homology and cohomology groups can also be defined for pairs (Y, X) of CW-complexes. We recall that the chain complex $C_*(Y, X)$ is defined to be the factor chain complex C_*Y/C_*X. Hence, we may define the diagonal map

$$\bar{\Delta} \colon C_*(Y, X) \to C_*Y \otimes C_*(Y, X).$$

To this end, we observe that every chain $c \in C_*(Y, X)$ is represented by a chain $b \in C_*Y$, and therefore we may define $\bar{\Delta}(c)$ by the projection of $\Delta_* b$. The map $\bar{\Delta}$ is well defined. Indeed, if b and b' are two different representatives of c, then $b' = b + a$, where $a \in C_*X$. On the other hand, the projection of $\Delta_* a$ is zero. The diagonal map $\bar{\Delta}$ allows us to define products of pairs of spaces. The cup product

$$\cup \colon H^*Y \otimes H^*(Y, X) \longrightarrow H^*(Y, X)$$

is defined by $a \otimes b \mapsto \bar{\Delta}^*(a \otimes b)$. The two cap products

$$\cap \colon H_n(Y, X) \otimes H^k X \longrightarrow H_{n-k}(Y, X),$$

[4] Equivalently, there is a commutative diagram

$$\begin{array}{ccc} H^*X & \xleftarrow{f^*} & H^*Y \\ {\scriptstyle \cup f^*b}\downarrow & & \downarrow{\scriptstyle \cup b} \\ H^*X & \xleftarrow{f^*} & H^*Y. \end{array}$$

[5] Equivalently, there is a commutative diagram

$$\begin{array}{ccc} H^*X & \xleftarrow{f^*} & H^*Y \\ {\scriptstyle x\cap}\downarrow & & \downarrow{\scriptstyle f_*x\cap} \\ H_{n-*}X & \xrightarrow{f_*} & H_{n-*}Y. \end{array}$$

[6] For example, to prove the naturality property for the cap product, let us denote representatives of homology class x and the cohomology class y by the same letters. Then, by definitions of products,

$$\begin{aligned} f_*(x \cap f^*y) &= f_*[\Delta_* x/f^*y] \\ &= f_*[\textstyle\sum x_i \otimes x_i'/f^*y] \\ &= f_*[\textstyle\sum \langle f^*y, x_i' \rangle x_i] \\ &= \textstyle\sum \langle f^*y, x_i' \rangle f_* x_i \\ &= \textstyle\sum \langle y, f_* x_i' \rangle f_* x_i \\ &= \textstyle\sum f_* x_i \otimes f_* x_i'/y \\ &= \textstyle\sum \Delta_*(f_* x)/y \\ &= f_* x \cap y. \end{aligned}$$

The cap-cap product is proved similarly:

$$\begin{aligned} u(x \cap v) &= u(\Delta_* x/v) \\ &= u(\textstyle\sum x_i \otimes x_i'/v) \\ &= u(\textstyle\sum v(x_i')x_i) \\ &= \textstyle\sum v(x_i')u(x_i) \\ &= \textstyle\sum \langle u \otimes v, x_i \otimes x_i' \rangle \\ &= \langle u \otimes v, \Delta_* x \rangle \\ &= \langle \Delta^*(u \otimes v), x \rangle \\ &= (u \cup v)(x). \end{aligned}$$

$$\cap: H_n(Y,X) \otimes H^k(Y,X) \longrightarrow H_{n-k}Y$$

are defined by $a \otimes b \mapsto \bar{\Delta}(a)/b$ and $a \otimes b \mapsto \Delta'(a)/b$, where Δ' is defined similarly to $\bar{\Delta}$ except that the target space of Δ' is $C_*(Y,X) \otimes C_*Y$.

11.2.3 *Poincaré Duality*

A *Poincaré complex* of formal dimension m is a CW-comples X together with a so-called funamental class $[X]$ in H_mX such that the cap product map

$$[X]\cap: H^kX \longrightarrow H_{m-k}X$$

is an isomorphism for all k. For example, every closed oriented connected manifold X of dimension m is a a Poincaré complex of formal dimension m. To define its fundamental class, choose a CW-structure on X such that the orientation of any cell e^m_α of dimension m agrees with the orientation of X. Then the fundamental class $[X]$ is represented by the chain $\sum[e^m_\alpha]$ where the sum ranges over all cells of dimension m. Similarly, a pair (Y,X) of CW-complexes with a homology class $[Y] \in H_m(Y,X)$ is a Poincaré pair if the two cap product homomorphisms

$$[Y]\cap: H^k(Y,X) \longrightarrow H_{m-k}Y,$$

$$[Y]\cap: H^kY \longrightarrow H_{m-k}(Y,X)$$

are isomorphisms for all k.

Given a cohomology class U of a Poincaré complex X, we will denote the *Poincaré dual* class $[X] \cap U$ by u. There is a so-called Poincaré pairing:

$$H_kX \otimes H_{m-k}X \longrightarrow \mathbb{Z}$$

which is a homomorphism that takes $u \otimes v$ to the integer $(U \cup V)[X]$ which we denote by $u \cdot v$. The number $u \cdot v$ can also be computed as $U(v)$.[7]

[7] Indeed, we have $u \cdot v = (U \cup V)[X] = U([X] \cap V) = U(v)$.

Maps $f: X \to Y$ of Poincare duality spaces of formal dimensions m and n respectively define the so-called *Umkehr* homomorphisms in homology and cohomology groups in the "wrong" direction. Namely, there is a well-defined homomorphism

$$f_!: H_*Y \longrightarrow H_*X$$

defined by taking a class $v \in H_*Y$, converting it to a cohomology class V, pulling it back to the class $U = f^*V$ in H^*X, and then converting it back to a homology class u in H_*X. Similarly, the Umkehr map

$$f^!: H^*X \longrightarrow H^*Y$$

is defined by coverting a cohomology class $U \in H^*X$ to a homology class u, sending it to class $v = f_*u$ in H_*Y, and then converting it to a class V in H^*Y.

Definition 11.8. A map $f: X \to Y$ of Poincaré complexes is of *degree one* if $f_*[X] = [Y]$.

Lemma 11.9. *If $f: X \to Y$ is a map of degree one, then $f_* \circ f_! = \mathrm{id}$ and $f^! \circ f^* = \mathrm{id}$.*

Proof. Recall that by the naturality of the cap product, we have $f_*([X] \cap f^*V) = [Y] \cap V$. Since the homomorphism $([Y]\cap)$ can be inverted, we have

$$([Y]\cap)^{-1} f_*([X] \cap f^*V) = V,$$

where on the left hand side we have $f^! \circ f^*V$. On the other hand, since $f_!([Y] \cap V) = [X] \cap f^*Y$, the naturality property can also be written as

$$f_*f_!([Y] \cap V) = [Y] \cap V,$$

which completes the proof of the Lemma. □

Similarly, for maps $f: (Y, X) \to (Y', X')$ of pairs of Poincaré complexes, there are Umkehr homomorphisms[8]

$$f_!: H_*(Y', X') \to H_*(Y, X), \qquad f^!: H^*(Y, X) \to H^*(Y', X').$$

When $f_*[Y] = [Y']$, i.e., when f is of degree one, we have $f_* \circ f_! = \mathrm{id}$ and $f^! \circ f^* = \mathrm{id}$.

[8] The homomorphism $f_!$ is defined by converting a homology class v to a cohomology class V such that $v = [Y'] \cap V$, pulling it back to $U = f^*V$ and then converting it to a homology class $u = [Y] \cap U$. The homomorphism $f^!$ is defined similarly by converting U to a homology class u, pushing it forward to v, and the converting it to a cohomology class V.

11.3 Degree one maps

We recall that a map $f: M \to X$ of Poincaré complexes is of degree one if $f_*[M] = [X]$. We have seen that if f is a map of degree one, then f_* has a left inverse $f_!$. In particular, the map f_* is split surjective.[9] Similarly, the map f^* is split injective. In particular, there are canonical isomorphisms

$$H_*M \approx H_*X \oplus K_*M \qquad H^*M \approx H^*X \oplus K^*M$$

[9] To see that the map f_* is surjective, note that every element $x \in H_*X$ is the image under f_* of the element $f_!(x)$. Indeed, $f_*f_!(x) = \mathrm{id}(x) = x$.

where H_*X and H^*X are identified with the images of $f_!$ and f^*, while K_*M and K^*M are the subgroups $\ker f_*$ and $\ker f^!$.

The groups K_*M and K^*M will be the primary objects of our study. These groups measure the difference between the (co)homology groups of M and X. Note that in the study of cobordisms of framed manifolds, the main objective is to simplify a given framed manifold M by killing all homotopy or, equivalently, (co)homology groups of M by surgery. In the study of cobordisms of normal maps, in order to approximate X by M, the main objective is not to kill all (co)homology classes of M, but to kill all classes in the canonical subgroups K_*M and K^*M of the (co)homology groups of M.

We will show that the groups K_*M and K^*M share many properties of homology and cohomology groups of manifolds. For example, recall that for a closed connected oriented manifold M of dimension $m = 2q$ there is a non-degenerate intersection form, called the Poincaré paring,

$$H_q(M; k) \otimes H_q(M; k) \longrightarrow k$$

where k is either $\mathbb{Q}$ or $\mathbb{Z}_2$. It turns out that when $f\colon M \to X$ is a degree one map to a Poincaré complex, the subspaces $K_q(M; k)$ and $H_q(X; k)$ are perpendicular. Indeed, if u and $f_!v$ are two homology classes in the two subspaces respectively, then $u \cdot f_!v = 0$.[10] In particular, the intersection form on $H_q(M; k)$ restricts to a non-degenerate intersection form on $K_q(M; k)$. If we now restrict the cap homomorphism $[M]\cap$ to K^*M, then we obtain a map $U \mapsto [M] \cap U$ to $K_{m-*}M$. Indeed, suppose that U is a class in $K^*M = \ker f^!$. Then the class $u = [M] \cap U$ is in the kernel of f_* which implies that $u \in K_{m-*}M$. Similarly, the cap product $[M]\cap$ sends the subgroup H^*X of H^*M to the subgroup $H_{m-*}X$ of $H_{m-*}M$.[11] By dimensional counting now, the map $[M]\cap\colon K^*M \to K_{m-*}M$ is an isomorphism, which we call *the Poincaré* isomorphism.[12] The Universal coefficient theorem for the groups K^*M and K_*M also holds true:

$$K^kM \simeq \operatorname{Tor} K_{k-1}M \oplus \operatorname{Free} K_kM.$$

Theorem 11.10. *Let M be a simply connected closed manifold of dimension $m = 2q$ or $m = 2q + 1$. If $f\colon M \to X$ is a map of degree 1 into a simply connected Poincaré complex such that $K_iM = 0$ for $i \le q$, then f is a homotopy equivalence.*

Proof. Suppose that $K_*M = 0$ in degrees $\le q$. By the Poincaré duality, the groups K^*M are trivial for $* \ge m - q$. Therefore, by the Universal Coefficient Theorem, the groups K_*M are trivial for $* \ge m - q$. It

[10] We have

$$u \cdot ([M] \cap f^*V) = \langle f^*V, u\rangle = \langle V, f_*u\rangle = 0,$$

since $u \in \ker f_*$.

[11] If $U = f^*V$, then $u = [M] \cap U$ is $f_!([X] \cap V)$.

[12] In fact, we have shown that there is a commutative diagram of Poincaré isomorphisms:

$$\begin{array}{ccccc} K^*M & \longleftarrow & H^*M & \longleftarrow & H^*X \\ \downarrow & & \downarrow & & \downarrow \\ K_{m-*}M & \longrightarrow & H_{m-*}M & \longrightarrow & H_{m-*}X. \end{array}$$

follows that f is a homology isomorphism. Since M and X are simply connected, the map f is a homotopy equivalence. □

We note that a normal cobordism of a map $M \to X$ is a normal map of pairs $f\colon (W, M) \to (Y, X)$ where $Y = X \times [0,1]$. For this reason, we will need to extend our discussion to normal maps of pairs, which is straightforward. For example, there are canonical isomorphisms

$$H_*(W, M) \approx H_*(Y, X) \oplus K_*(W, M),\ H^*(W, M) \approx H^*(Y, X) \oplus K^*(W, M),$$

where $H_*(Y, X)$ and $H^*(Y, X)$ are identified with the images of $f_!$ and f^*, while $K_*(W, M)$ and $K^*(W, M)$ are the subgroups $\ker f_*$ and $\ker f^!$. As above, we have Poincaré duality isomorphisms

$$K^*(W, M) \approx K_{m-*}W, \qquad K^*W \approx K_*(W, M).$$

Next, as in the case of cobordisms of framed manifolds, we will need to define invariants of normal maps f: the signature $\sigma(f)$ and the Arf invariant $\mathrm{Arf}(f)$.

When $m = 2q$ with q even, we choose the coefficient field k to be $\mathbb{Q}$. In this case, since the intersection form on $H_q(M; k)$ is the direct sum of the intersection forms on $H_q(X; k)$ and $K_q(M; k)$, the signature $\sigma(M)$ is the sum of $\sigma(X)$ as well as the signature $\sigma(f)$ of the intersection form on $K_q(M; k)$. Since $\sigma(M)$ and $\sigma(X)$ are invariant with respect to cobordisms, we deduce that $\sigma(f)$ is an invariant of cobordisms as well.

Suppose now that $m = 2q + 1$, and the map f is q-connected. By the Relative Hurewicz Theorem,[13] the group $K_q(M)$ is isomorphic to $\pi_q(f)$, and therefore the function μ which we previously defined on $\pi_q(f)$ can be regarded as a function on $K_q(M; \mathbb{Z}_2)$. As in the case of the framed cobordisms, μ is a quadratic form associated with the intersection form on $K_q(M; \mathbb{Z}_2)$. We define the Arf-invariant $\mathrm{Arf}(f)$ to be the Arf-invariant of the quadratic form μ. Again, as in the case of framed cobordisms, we can show that $\mathrm{Arf}(f)$ is well-defined, and, in fact, it can be defined for arbitrary (not necessarily q-connected) normal maps $M \to X$ of a simply connected closed manifold M to a simply connected Poincaré complex X.

[13] **Relative Hurewicz Theorem**: Let (X, A) be a pair of simply connected CW-complexes. Suppose that the groups $\pi_*(X, A)$ are trivial for $* < n$. Then the groups $H_*(X, A)$ are isomorphic to $\pi_*(X, A)$ for $* \leq n$.

11.4 The Browder-Novikov theorem

Theorem 11.11. *(Browder-Novikov) Let $f\colon M \to X$ be a normal map of degree 1 of a manifold of dimension $2q > 4$ to a simply connected Poincaré* *space. If q is even, then f is normally cobordant to a homotopy equivalence if and only if the signature of f is zero. If q is odd, then f is normally cobordant to a homotopy equivalence if and only if the Kervaire invariant is zero.*

Proof. We have seen that when q is even, signature is an invariant of normal cobordism, and when q is odd, the Kevaire invariant does not change under normal cobordism. Thus, the existence of a normal cobordism of f to a homotopy equivalence implies that $\sigma(f) = 0$ when q is even and $\mathrm{Arf}\,(f) = 0$ when q is odd.

Suppose now that q is even and $\sigma(f) = 0$. By Theorem 11.5, we may assume that $K_iM = 0$ for $i < q$. By the Poincaré duality together with the Universal Coefficient theorem, we deduce that K_qM is a free abelian group.[14] As in the case of framed cobordisms, we deduce that there is a class $\lambda \in K_qM$ such that $\mu(\lambda) = 0$. Without loss of generality, we may assume that λ is a free generator of K_qM. Then it is also a free generator of H_qM, and, as in the case of framed cobordisms, a normal surgery along λ kills the factor in K_qM generated by λ. After finitely many normal surgeries along generators of K_qM we get a trivial group K_qM. By Theorem 11.10, then, the map f is a homotopy equivalence.

[14] Indeed, the group $K_qM \approx K^qM$ is isomorphic to $\mathrm{Free}\,K_qM \oplus \mathrm{Tor}\,K_{q-1}M$, while the group $K_{q-1}M$ is trivial by assumption.

Suppose now that q is odd and $\mathrm{Arf}\,(f) = 0$. We may assume that $K_iM = 0$ for $i < q$, and, in particular, K_qM is free. By the Hurewicz isomorphism, we may identify K_qM with π_qM. Therefore, applying the argument as in the case of framed cobordisms, we find finitely many surgeries which eliminate K_qM. Then f is a homotopy equivalence. □

Theorem 11.12. *(Browder-Novikov) Every normal map $f\colon M \to X$ of degree 1 of a simply connected manifold of dimension $2q+1 \geq 5$ to a simply connected Poincaré complex is normally cobordant to a homotopy equivalence.*

Proof. The argument is completely the same as in the case of the framed surgery. By Theorem 11.5, we may assume that $K_iM = 0$ for $i < q$, and K_qM is finite. When q is even, a surgery along a torsion class $\lambda \in K_qM$, results in creating a free class λ' which may then be cancelled, see Lemma 6.17.[15] Thus when q is even, pairs of surgeries reduce M to a manifold homotopy equivalent to X. When q is odd, we may use series of surgeries to kill $K_qM \otimes \mathbb{Z}_p$ without increasing the order of K_qM. This implies that until K_qM is trivial, its order of K_qM can be decreased.Therefore, again, the manifold M can be modified by surgery to a manifold homotopy equivalent to X. □

[15] Recall that $H_qM \approx H_qX \oplus K_qM$

12

The surgery long exact sequence

In this section we aim to introduce and prove the long exact sequence for simply connected finite CW-complexes X:

$$\cdots \longrightarrow N_\partial(X \times I) \longrightarrow L_{n+1} \longrightarrow \mathcal{S}(X) \longrightarrow \mathcal{N}(X) \longrightarrow L_n,$$

where L_n is $0, \mathbb{Z}_2, 0, \mathbb{Z}$ depending on the residue class of n mod 4.

In short, for the existence of homotopy equivalence $M \to X$ we need to assume that X is a Poincare complex, and there exists a vector bundle ξ over X that plays the role of the normal bundle. We will see that the spherization class of ξ is essentially unique. The set $\mathcal{S}(X)$ is the set of manifolds homotopy equivalent to M. If $M \in \mathcal{S}(X)$ exists, then the homotopy equivalence $M \to X$ is a normal map. The normal cobordism group of normal maps $M \to X$ is denoted by $\mathcal{N}(X)$. We have already seen that the obstruction to surging a map $M \to X$ to a homotopy equivalence is precisely an element in L_n.

Let F and X are finite CW-complexes. A homotopy F-fibration $Y \to X$ is a map whose homotopy fiber is homotopy equivalent to F. Homotopy fibrations $f\colon Y \to X$ and $f'\colon Y' \to X$ are said to be equivalent if there exists a homotopy equivalence $h\colon Y \to Y'$ such that $f = f'h$. Let $H(F)$ denote the monoid of homotopy self-equivalences of F equipped with the compact-open topology. There is a simplicial construction of the classifying space $BH(F)$. The space $BH(F)$ enjoys the standard classifying property.

Theorem 12.1 (Stashef)**.** *For every finite CW-complex X there is a bijection between $[X, BH(F)]$ and the set of equivalence classes of homotopy F-fibrations to X.*

We will be interested in homotopy spherical fibrations. To begin with observe that a space S is a homotopy sphere iff it is a simply connected homology sphere.

Theorem 12.2. *Let X be a simply connected finite Poincare duality space of dimension m embedded into $\mathbb{R}^{m+k}$. Let $V \to X$ is the homotopy projection of a tubular neighborhood to X, ie., homotopy inverse to the incusion $X \to V$. Then the restriction map $\partial V \to X$ is a homotopy S^{k-1}-fibration.*

Remark 12.3. A tubular neighborhood can be defined by means of a triangulation. Since X is finite we may assume that it is triangulated and its triangulation agrees with some triangulation on $\mathbb{R}^{m+k}$. A tubular neighborhood can be defined to be the star of X in the second barycentric triangulation of $\mathbb{R}^{m+k}$. The space X is a deformation retract of its tubular neighborhood and $V \setminus X$ is homotopy equivalent to ∂V.

Proof. Let F denote the homotopy fiber of the inclusion $\partial V \to V \simeq X$. By the Poincare duality $H^q(V/\partial V) \simeq H_{m+k-q}V$; in particular, there is a class U such that $\cap U$ takes the fundamental class of the former space to the fundamental class of the latter space. It follows that $\cup U : H^q V \to H^{k+q}(V/\partial V)$ is an isomorphism. Since X is finite, a general position argument shows that $\pi_i(V \setminus X) \to \pi_i V$ is an isomorphism for $i < k-1$ and onto for $i = k-1$. We get

$$\pi_i(\partial V) \to \pi_i(V \setminus X) \to \pi_i(V)$$

is an isomorphism for $i < k-q$ and onto for $i = k-1$. By the Relative Hurevicz theorem, $\pi_{i-1}(F) = \pi_i(V/\partial V) = 0$ for $i < k-1$. Also $\pi_{k-1}(F) = H_k(V/\partial V) = \mathbb{Z}$ since in this dimension there is no torsion ($H^{k+1}(V/\partial V) = 0$) and $H^k(V/\partial V) = \mathbb{Z}$. Thus it remains to show that $H^i F = 0$ for $i \geq k$. Consider the spectral sequence for a homotopy fibration $\partial V \to V$.

$k-1$	$\mathbb{Z}$	$*$	$*$	$*$
$*$	0	0	0	0
$*$	0	0	0	0
2	0	0	0	0
1	0	0	0	0
0	$\mathbb{Z}$	$*$	$*$	$*$
	0	1	2	3

Clearly, in E_2 we have that the 0 line is isomorphic to the $(k-1)$-st line and the isomorphism is given by taking the cup product with U. Chasing the spectral sequence one shows that F is a homology sphere. Therefore the map $\partial U \to U$ is a homotopy S^{q-1}-fibration. □

In other words, for every finite CW-complex X we obtain a map $\nu\colon X \to BG_k \to BG$ classifying the spherical fibration $\partial V \to X$, where G_k is the space of homotopy self-equivalences of the sphere S^{k-1} and BG the colimit of spaces BG_k. It turns out that the map ν does not depend on choices we made in the construction and is called the normal Spivak fibration. We observe now that if X is homotopy equivalent to a manifold, then its classifying map ν lifts to a a map to BO. There is a fibration

$$G/O \longrightarrow BO \longrightarrow BG.$$

One can show that the obstruction to the existence of a lift is an element in $[X, B(G/O)]$ (see Madsen-Milgram, p.40) and a fixed lift identifies the set of lifts with $[X, G/O]$.

Now, fix a lift $X \to BO$. Then we have considered the group of normal cobordisms $M \to X$. Such a group is denoted by $\mathcal{N}(X)$. Let $\mathcal{S}(X)$ denote the set of h-cobordant equivalences $M \to X$. In other words, an element in $\mathcal{S}(X)$ is represented by a homotopy equivalence $f\colon M \to X$. Two homotopy equivalences f, f' represente the same element in $\mathcal{S}(X)$ if there exists a homotopy equivalence $(F, f, f')\colon (W, M, M') \to (X \times [0,1], X \times \{0\}, X \times \{1\}$ which is an h-cobordism. Suppose $M \to X$ is a homotopy equivalence representing a class $\alpha \in \mathcal{S}(X)$; choose a homotopy inverse $g\colon X \to M$. Then $g^*\nu$ is a lift of the normal Spivak fibration where ν is the normal bundle of M. Consequently α defines an element in $\mathcal{N}(X)$. In other words we have a map $\mathcal{S}(X) \to \mathcal{N}(X)$. An element in $\mathcal{N}(X)$ lifts to $\mathcal{S}(X)$ if its invariant in L_n is trivial. In other words we have an exact sequence

$$\mathcal{S}(X) \to \mathcal{N}(X) \longrightarrow L_n.$$

It turns out that this exact sequence can be extended from the left hand side; it is called the exact sequence of surgery theory.

13

(Co)homology with twisted coefficients

In this chapter we consider a more general question. Namely, let $f: M \to X$ be a map of a closed manifold $M \subset \mathbb{R}^{m+k}$ of dimension m into a CW-complex X. We would like to determine under what conditions there is a surgery on f which turns f into a homotopy equivalence.

If X is homotopy equivalent to a manifold $M' \subset \mathbb{R}^{m+k}$, then the homotopy equivalence takes the perpendicular vector bundle $T^{\perp}M'$ to a vector bundle ν over X, which leads us to studying *normal maps* $f: M \to X$, i.e., maps f equipped with fiberwise isomorphisms $T^{\perp}M \to \nu$ covering f. Also, if X is homotopy equivalent to a manifold, then certain homology and cohomology groups of X should be Poincaré dual to each other. The homology and cohomology groups here are those with coefficients in the group ring $\mathbb{Z}[\pi_1 X]$.

In this section we will introduce homology and cohomology of a CW complex X with coefficients in $\mathbb{Z}[\pi_1 X]$. In section 13.1 we introduce the group ring $\mathbb{Z}[\pi_1 X]$, and use it to extend the Whiney trick construction to the case of spheres embedded into a non-simply connected manifold. Next, in section 13.2 we review the definition of right and left $\mathbb{Z}[\pi]$-modules. In particular, we note that $\pi_n X$ is a left $\mathbb{Z}[\pi_1 X]$-module. In §13.3 we define (co)homology with local coefficients in a $\mathbb{Z}[\pi]$-module, and in §13.4 review various products. In particular, we review the definition of the cap product and formulate the Poincaré duality theorem for closed (possibly non-orientable) manifolds.

A CW-complex X is said to be a Poincaré duality complex if its homology $H_*\tilde{X}$ and cohomology $H^*\tilde{X}$ with coefficients in $\mathbb{Z}[\pi]$ are related by means of the Poincaré duality, see §13.5. In particular, the CW-complex possesses a distinguished homology class $[X]$ called the fundamental class. A map $f\colon M \to X$ is a degree one map if it takes the fundamental class of M to the fundamental class of X. It follows that for a degree one map $H_*\tilde{M}$ is isomorphic to $H_*\tilde{X} \oplus K_*M$ for a $\mathbb{Z}[\pi_1 X]$-module K_*M. We will see in the next chapter that when $m = 2q$ or $m = 2q+1$, there is no obstruction in modifying f by normal surgery into a q-connected map for which $K_i M = 0$ when $i < q$. On the other hand, if f is $(q+1)$-connected, then it is a homotopy equivalence. Thus it remains to study the question under what conditions the $\mathbb{Z}[\pi_1 X]$-module $K = K_q M$ can be eliminated by normal surgery of f. In §13.7 we show that when $m = 2q$, the module $K_q M$ is a finitely generated stably free $\mathbb{Z}[\pi_1 X]$-module.

13.1 *The Whitney trick in non-simply connected manifolds*

In the case of non-simply connected manifolds M, the role of the coefficient ring is played by the group ring $\Lambda = \mathbb{Z}[\pi_1 M]$; it is the ring of finite formal linear combinations $n_1 g_1 + \cdots + n_k g_k$ where $n_i \in \mathbb{Z}$ and $g_i \in \pi_1 M$ with ring operations

$$\sum n_g g + \sum m_g g = \sum (n_g + m_g) g, \qquad \sum n_g g \cdot \sum m_h h = \sum n_g m_h gh.$$

Exercise 13.1. Show that $\mathbb{Z}[\pi_1 S^1]$ consists of Laurant polynomials

$$a_{-k}x^{-k} + \cdots + a_{-2}x^{-2} + a_{-1}x^{-1} + a_0 + a_1 x + a_2 x^2 + \cdots + a_s x^s,$$

while $\mathbb{Z}[\pi_1 \mathbb{R}P^2]$ is the polynomial ring $\mathbb{Z}[x]$ quotiened by $x^2 - 1$.

We note that without the ring multiplication, Λ is a free abelian group with generators corresponding to elements in $\pi_1 M$.

The group ring Λ appears already in the Whitney trick for non-simply connected manifolds. Indeed, let M be a connected manifold of dimension $m = 2q > 4$ oriented at a reference point s,[1] and S_1, S_2 be two oriented spheres of dimension q immersed into M with reference points s_1 and s_2 respectively. Suppose that the spheres S_1 and S_2 are *threaded*, i.e., each sphere S_i is equipped with a path w_i in M from the reference point s of M to the reference point s_i on S_i. For each intersection point x_j of S_1 and S_2, choose a path u_{ij} on S_i from s_i to x_j. Then for each intersection point $x = x_j$, the space $T_x M$ inherits two orientations. One orientation of $T_x M$ comes from the orientation of $T_x S_1$ followed by the orientation of $T_x S_2$.

[1] In other words, we fix an equivalence class of bases for $T_s M$, where two bases are equivalent if the determinant of the linear transformation which takes the first basis into the second one is positive.

For the second orientation, choose a basis $\{e_i(t)\}$ of the vector space T_tM for every point t on the concatenation[2] curve $u_{1j} * w_1$ so that the components of the vectors $e_i(t)$ change continuously with t and so that $\{e_i(s)\}$ is a positive basis of T_sM. The second orientation of T_xM is defined to be that of $\{e_i(x)\}$.[3] If the two orientations on T_xM agree, then the intersection point x is *positive* and we write $\varepsilon(x) = 1$, otherwise x is *negative* and we put $\varepsilon(x) = -1$. The obstruction to canceling all intersection points of the spheres S_1 and S_2 by regular homotopy of S_1 and S_2 is the element

$$\lambda(S_1, S_2) = \sum \varepsilon(x_j)g(x_j)$$

in $\Lambda = \mathbb{Z}[\pi_1 M]$, where $g(x_j)$ is the concatenation $w_2 * u_{2j} * u_{1j}^{-1} * w_1^{-1}$ of the chosen paths from s to x_j through s_2 and then from x_j to s through s_1.

[2] We follow the convention which is used in the definition of the group operation in $\pi_1 M$. Recall that the group operation in $\pi_1 M$ is defined by concatenation. Namely in terms of representatives the product of two loops γ_1 and γ_2 is an appropriately reparametrized loop $\gamma_1 * \gamma_2$ which is the path that first follows γ_1 and then follows γ_2. Thus, the curve $w_1 * u_{1j}$ denotes the path which first follows w_1 and then u_{1j}.

[3] If the manifold M is oriented, then we may use the orientation on M to define the second orientation on T_xM.

Exercise 13.2. Show that $\lambda(S_1, S_2)$ is a complete obstruction to canceling all intersection points by Whitney moves. In particular, if $\lambda(S_1, S_2) = 0$ and S_1 and S_2 are embedded, then there is an isotopy of S_1 to a submanifold in M disjoint from S_2. If $\lambda(S_1, S_2) = 0$, and S_1 and S_2 are immersed, then there is a regular homotopy of S_1 to an immersed surface in M disjoint from S_2.[4]

[4] **Hint for Exercise 13.2** If $\lambda(S_1, S_2) = 0$, then each $\epsilon(x_i)g(x_i)$ appears in pair with $\epsilon(x_j)g(x_j)$ where $g(x_i) = g(x_j)$ and $\epsilon(x_i) = -\epsilon(x_j)$. The points x_i and x_j can be canceled by the Whitney trick along a disc D providing a null-homotopy of $u_{1i} * u_{2i}^{-1} * u_{2j} * u_{1j}^{-1}$; note that a neigbhorhood of the disc D in M is oriented and simply connected, and therefore we may refer to the Whitney trick that we discussed in the simply connected case.

Exercise 13.3. Show that if S_1' and S_2' are two immersed spheres homotopic to S_1 and S_2, then $\lambda(S_1, S_2) = \lambda(S_1', S_2')$.[5]

[5] **Hint for Exercise 13.3**: Choose a homotopy $\mathbf{S}$ of S_1 to S_1' and a homotopy $\mathbf{S}'$ of S_2 to S_2' in $M \times [0, 1]$. We may assume that $\mathbf{S}$ and $\mathbf{S}'$ intersect transversally along a submanfold Σ of $M \times [0, 1]$ of dimension 1. The endpoints of the submanifold Σ establish an equivalence between $\lambda(S_1, S_2)$ and $\lambda(S_1', S_2')$.

A similar argument works for self-intersections. Indeed, let S be a sphere of dimension q immersed into a manifold M of dimension $m = 2q$. Again, choose a basepointe p in M and s in S, an orientation of T_pM, a path w from p to s, and for each self-intersection point x_j two paths u_{1j} and u_{2j} from s to x_j in S so that u_{1j} and u_{2j} approch x_j along different intersecting sheets. Finally, define $\mu(S)$ to be the linear combination $\sum \varepsilon(x_j)g(x_j)$ in Λ, where $g(x_j)$ is the concatenation $w * u_{2j} * u_{1j}^{-1} * w^{-1}$ and $\varepsilon(x_j)$ is the orientation of the self-intersection point x_j. If $\mu(S) = 0$, then all self-intersection points can be eliminated by regular homotopy of S. If $\mu(S) \neq 0$, then it might be possible to change the order of u_{1j} and u_{2j} so that the invariant is zero.

Exercise 13.4. Show that interchanging u_{1j} with u_{2j} replaces g with g^{-1}, and ϵ with $(-1)^q w(x_j)\epsilon$, where $w(x_j) = w(g(x_j))$ is 1 if the orientation of M along $g = g(x_j)$ is preserved and $w(x_j) = -1$ otherwise. In other words, ϵg is replaced with $(-1)^q \epsilon \bar{g}$, where $\bar{g}$ stands for $w(g)g^{-1}$.[6]

[6] **Hint for Exercise 13.4** Note that if $e_1, ..., e_q$ is the orientation at x_j of the sheet of S containing u_{1j} and $f_1, ..., f_q$ is the orientation at x_j of the sheet of S containing u_{2j}, then interchanging u_{1j} with u_{2j} leads to a replacement of bases:

$$(e_1, ..., e_q, f_1, ..., f_q) \mapsto (f_1, ..., f_q, e_1, ..., e_q).$$

This is an orientation preserving change of basis if q is even, and orientation reversing otherwise.

To make μ independent from the choices of orderings of u_{1j} and u_{2j}, we need to put $g = \bar{g}$ if q is even, and $g = -\bar{g}$ if q is odd. In other words, consider the quotient group $\Lambda_\varepsilon = \Lambda / \langle g - (-1)^q \bar{g} | g \in \Lambda \rangle$. Then μ is a well-defined element in Λ_ε.

Exercise 13.5. The immersed sphere S is regularly homotopic to an embedding if and only if $\mu(S) = 0$ in Λ_ε.

13.2 *Right and left Modules*

A *left action* of a group π on a set C is a function $\pi \times C \to C$ which associates with a pair (g, c) an element denoted by $g(c)$. We also say that the element g takes c to $g(c)$. A left action is required to satisfy two axioms. First, the identity element 1 in π acts trivially, i.e., $1(c) = c$ for every element in C, and, second, the element gh takes any point c to the same point as the element g takes $h(c)$, i.e., $g(h(c)) = (gh)(c)$. Similarly, a *right action* of a group π on a set C is a function $C \times \pi \to C$ denoted by $(c, g) \mapsto (c)g$ that satisfies the properties that $(c)1 = c$ and $((c)g)h = (c)(gh)$, for all elements $c \in C$ and $g, h \in \pi$. Note that under the right action the element gh takes c to the same point as the element h takes $(c)g$, i.e., the order of actions of g and h under the right action is different from that under the left action.

When C is an abelian group, we also require that the left action $c \mapsto g(c)$ (respectively, the right action $c \mapsto (c)g$) of each element g on C is a homomorphism. In particular, a left action of a group π on an abelian group C is a homomorphism $\pi \to \mathrm{Aut}(C)$.

For example, for a pointed topological space X, there is a left action of the fundamental group $\pi = \pi_1 X$ on the higher homotopy groups $\pi_n X$ for $n > 1$. Indeed, the homotopy group $\pi_n X$ can be interpreted as the group of homotopy classes of threaded spheres (f, γ), i.e., pairs of continuous maps $f : S^n \to X$ together with a path γ in X from the base point of X to the image of the south pole of the sphere S^n.[7] When there are two threaded maps (f, γ) and (f', γ'), their sum is the connected sum of the maps f and f' along the path $\gamma * \gamma'$. Note that the cylinder of the connected sum passes through a small neighborhood U of the distinguished point x. We choose the distinguished point on the connected sum to be near x and choose an arbitrary thread which entirely lies in U.

[7] To related this definition of $\pi_n X$ to the standard one, let s and x denote the distinguished points of S^n and X respectively. We note that there is a unique up to homotopy way to modify f in a neighborhood U of s in such a way that $f(U)$ stretches along the thread γ, and s maps to x.

The interpretation of homotopy groups $\pi_n X$ with $n > 1$ in terms of threaded spheres may be quite helpful. For example, if $\pi_1 X$ is trivial, it follows immediately that all threads can be disregarded since threads are defined up to homotopy and in a simply connected space all choices of threads are equivalent. It is also easy to define a left action of $\pi = \pi_1 X$ on $\pi_n X$ in terms of threaded spheres. Indeed, put

$$g(f, \gamma) = (f, g * \gamma).$$

In other words, an element g acts on a threaded sphere (f, γ) by changing its thread to the path which first traverses g and then follows γ. We will need actually to convert the left action of π on $\pi_n X$ into a right action:

$$(f, \gamma)g = (f, g^{-1} * \gamma).$$

A left $\mathbb{Z}[\pi]$-module is an abelian group C together with a homomorphism $\pi \to \mathrm{Aut}(C)$. The *action* of the group ring $\mathbb{Z}[\pi]$ on the abelian group C is defined by $(\sum n_g g)c \mapsto \sum n_g g(c)$. A right $\mathbb{Z}[\pi]$-module C is defined similarly. It is an abelian group with a right action of π.

For example, when π is the fundamental group of a manifold M, the abelian group Λ_ε[8] is a right module over $\Lambda = \mathbb{Z}[\pi]$. Under this action an element $h \in \pi$ sends a point $\gamma \in \Lambda_\varepsilon$ to the point $w(h)h^{-1} * \gamma * h$, where $w(h) = +1$ when h is an orientation preserving loop and $w(h) = -1$ otherwise.[9]

[8] Recall that

$$\Lambda_\varepsilon = \Lambda / \langle g - (-1)^q \bar{g} | g \in \Lambda \rangle.$$

[9] To show that the action is well-defined, note that

$$(\gamma - (-1)^q \bar{\gamma})g = w(g)[\delta - (-1)^q \bar{\delta}]$$

where $\delta = g^{-1} * \gamma * g$. In other words the action $(\gamma)h = \bar{h} * \gamma * h$ on Λ defines an action on the factor group Λ_ε.

13.3 Homology and cohomology with twisted coefficients

Let X be a pointed path-connected CW-complex with fundamental group $\pi_1 X = \pi$. The distinguished point of X will be denoted by $*$. There is a cellular chain complex $C_* X$, as well as a threaded cellular chain complex $C_*(\tilde{X})$. In the threaded cellular chain complex the group $C_n(\tilde{X})$ is a free abelian group generated by threaded n-cells in X, where a *threaded cell* is a cell together with a choice of a homotopy class of a path from the distinguished point $*$ of X to any point of the cell. We note that the terminal point of the path is not essential since any two points in a cell can be joined by an essentially unique path in the (contractible) cell. In particular, the boundary of a threaded n-cell is a linear combination of threaded $(n-1)$-cells.

The threaded cellular chain complex $C_*(\tilde{X})$ can be identified with the cellular chain complex of the universal covering $\tilde{X}$ of X; thus, the notation. Indeed, recall that the universal covering space $\tilde{X}$ of X consists of pairs (x, γ) where x is a point in X and γ is a homotopy class of paths from the distinguished point $*$ to x. In particular, every threaded cell in X corresponds to a unique cell in $\tilde{X}$. Similarly, the boundary homomoprhism in the threaded cellular chain complex of X corresponds to the boundary homomorphism in the cellular chain complex of $\tilde{X}$.

Out next observation is that the chain complex $C_*(\tilde{X})$ is a right $\mathbb{Z}[\pi]$-module. To describe the corresponding action of π on $C_n(\tilde{X})$, let D^n be a cell in X threaded by means of a path γ, and g an element of π. Then $(D^n, \gamma)g = (D^n, g^{-1} \circ \gamma)$.[10] Since $C_*(\tilde{X})$ is a right $\mathbb{Z}[\pi]$-module, for any left $\mathbb{Z}[\pi]$-module A, there is a well-defined chain complex $C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi]} A$ of abelian groups. Its homology group is called the *homology group of X with local coefficients in A*, and it is denoted by $H_*(X; A)$.

[10] Recall that $g^{-1} \circ \gamma$ is the path which first traverses g^{-1} and then follows γ.

Example 13.6. Suppose that the module A is the abelian group $\mathbb{Z}$ with a trivial action by π. Then the chain complex $C_*(X; A)$ is isomorphic to the chain complex C_*X. On the other hand, if the module A is a free $\mathbb{Z}[\pi]$ on one generator, then

$$C_*(X; A) = C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[\pi] = C_*(\tilde{X}),$$

and therefore the chain complex $C_*(X; A)$ is isomorphic to the chain complex $C_*(\tilde{X})$.

Next, we will define the cohomology group of X with local coefficients in a left $\mathbb{Z}[\pi]$-module A. To begin with, we observe that the abelian groups $C_*(\tilde{X})$ have canonical structures of left $\mathbb{Z}[\pi]$-modules, namely, with the action $g(D^n, \gamma) = (D^n, g * \gamma)$. Since each $C_i(\tilde{X})$ is a left $\mathbb{Z}[\pi]$-module, there are well defined abelian groups $C^*(X; A)$ of homomorphisms $\mathrm{Hom}_{\mathbb{Z}[\pi]}(C_*(\tilde{X}), A)$ of left $\mathbb{Z}[\pi]$-modules. An element in $C^*(X, A)$ is a function that assigns with each threaded cell (D^n, γ) an element a in the abelian group A in a way compatible with the left action by π. Namely, the function assigns to $(D^n, g * \gamma)$ the number $g(a)$. The coboundary homomorphism $\delta_n \colon C^n(X; A) \to C^{n+1}(X, A)$ is defined by the standard formula $\langle \delta_n f, x \rangle = \langle f, d_{n+1} x \rangle$. It follows that the abelian groups $C^*(X; A)$ together with the coboundary homomorphisms form a cochain complex. The corresponding cohomology groups are called *the cohomology groups of X with local coefficients in A.*

Example 13.7. If A is the abelian group $\mathbb{Z}$ with a trivial action by π, then $C^*(X, A)$ is isomorphic to the cochain complex $C^*(X)$. If $A = \mathbb{Z}[\pi]$, and X is a finite CW-complex, then $H^k(X; \mathbb{Z}[\pi])$ is isomorphic to compactly supported cohomology group $H^k_c(X)$. To simplify notation, we will write $H^k(\tilde{X})$ for $H^k(X; \mathbb{Z}[\pi])$.

13.4 Products in (co)homology with twisted coefficients

We have seen that the free abelian group $C_*(\tilde{X})$ of threaded cellular chains is a left $\mathbb{Z}[\pi]$-module, where $\pi = \pi_1 X$ and an element $g \in \pi$ acts on $C_*(\tilde{X})$ by taking a cell D^n threaded by γ to the same cell D^n threaded by $g * \gamma$. The abelian group $C^*(\tilde{X}) \otimes C^*(\tilde{X})$ is also a left $\mathbb{Z}[\pi]$-module with the action defined by $g(x \otimes y) = (gx \otimes gy)$ for any generator $x \otimes y$ of the tensor product of groups, and any element g in the fundamental group π.

Suppose now that the CW complex X is a manifold M with fundamental group $\pi_1 M = \pi$. Let $\mathbb{Z}_\omega$ denote the right $\mathbb{Z}[\pi]$-module which consists of the abelian group $\mathbb{Z}$ as well as the left action by π given by $n \cdot g = n$ if g is orientation preserving and $n \cdot g = -n$ otherwise. The $\mathbb{Z}[\pi]$-module $\mathbb{Z}_\omega$ is called the *orientation sheaf* over the manifold M. For a general CW-complex X with a homomorphism $\omega \colon \pi \to \mathrm{Aut}(\mathbb{Z})$, the right $\mathbb{Z}[\pi]$-module $\mathbb{Z}_\omega$ is defined to be the abelian group $\mathbb{Z}$ with the action of π on $\mathbb{Z}$ given by ω, i.e., with the action $n \cdot g = w(g)n$.

An element in the abelian group $\mathbb{Z}_\omega \otimes_{\mathbb{Z}[\pi]} C_n \tilde{X}$ is a linear combination with integral coefficients of elements of the form $1 \otimes x$, where x is a cell D^n threaded by a homotopy class γ of paths subject to the identifications $1 \otimes gx = w(g) 1 \otimes x$. We can similarly interpret generators of the abelian group $\mathbb{Z}_\omega \otimes_{\mathbb{Z}[\pi]} C_* \tilde{X} \otimes C_* \tilde{X}$ as triples $1 \otimes x \otimes y$ with identifications $1 \otimes gx \otimes gy = \pm w(g) 1 \otimes x \otimes y$.

The diagonal map $X \to X \times X$ is not cellular in general, but it can be modified by a homotopy to a cellular map Δ. In induces a homomorphism of abelian groups $C_* \tilde{X} \to C_* \tilde{X} \otimes C_* \tilde{X}$, and takes any generator x to a $\mathbb{Z}$-linear combination $\sum x' \otimes x''$. In fact, it is a map of left $\mathbb{Z}[\pi]$-modules as it takes gx to $\sum gx' \otimes gx''$, and therefore it defines a homomorphism of abelian groups

$$\Delta_* \colon \mathbb{Z}_\omega \otimes_{\mathbb{Z}[\pi]} C_* \tilde{X} \longrightarrow \mathbb{Z}_\omega \otimes_{\mathbb{Z}[\pi]} (C_* \tilde{X} \otimes C_* \tilde{X}).$$

Recall that $C_* \tilde{X}$ also has a structure of a right-$\mathbb{Z}[\pi]$ module. There is an isomorphism of abelian groups

$$\mathbb{Z}_\omega \otimes_{\mathbb{Z}[\pi]} (C_* \tilde{X} \otimes C_* \tilde{X}) \longrightarrow C_* \tilde{X} \otimes_{\mathbb{Z}[\pi]} C_* \tilde{X}.$$

which takes $1 \otimes x \otimes y$ to $x \otimes y$.[11] Thus, the homomorphism Δ_* takes an element $1 \otimes x$ to $\sum x_i' \otimes x_i''$ where x_i' is an element in the left $\mathbb{Z}[\pi]$-module $C_* \tilde{X}$, while x_i'' is an element in the right $\mathbb{Z}[\pi]$-module $C_* \tilde{X}$.

[11] To show that the homomorphism is well-defined, let g be an element in π preserving the orientation, i.e., $w(g) = +1$. Then

$$1 \otimes (gx \otimes gy) \mapsto gx \otimes gy$$
$$= xg^{-1} \otimes gy = x \otimes y.$$

Similarly, if g is an element in π reversing the orientation, then

$$-1 \otimes (gx \otimes gy) \mapsto -gx \otimes gy$$
$$= -xg^{-1} \otimes gy = x \otimes y.$$

We are now in position to define the cap product. Namely, let x be an element in $C_m(X;\mathbb{Z}_\omega) = C_m(\tilde{X}) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}_\omega$. Then there is a a map

$$x\cap\colon C^i\tilde{X} \mapsto C_{m-i}\tilde{X}$$

defined by $y \mapsto \sum \overline{y(x_i')} x_i''$.[12] Here we use the involution map on $\mathbb{Z}[\pi]$ defined by $\sum n_g g \mapsto \sum n_g w(g) g^{-1}$, where $w(g) = 1$ when g is orientation preserving and $w(g) = -1$ otherwise. The map $x\cap$ in (co)chains depends on the choice of the cellular diagonal map Δ. However, when x is a homology class, in (co)homology the cap product is a well-defined homomorphism

$$x\cap\colon H^i\tilde{X} \longrightarrow H_{m-i}\tilde{X}.$$

[12] If $\sum w(g) x_i' g^{-1} \otimes g(x) x_i''$ is another representative of $\Delta_*(x)$, then the resulting chain is

$$\begin{aligned}\sum w(g)\overline{y(x_i'g^{-1})}gx_i'' &= \sum w(g)\overline{y(gx_i')}gx_i'' \\ &= \sum w(g)\overline{gy(x_i')}gx_i'' \\ &= \sum w(g)w(g)g^{-1}\overline{y(x_i)'}gx_i'' \\ &= \sum \overline{y(x_i')}x_i''.\end{aligned}$$

It turns out that if M is a closed connected manifold of dimension m, then $H_m(M;\mathbb{Z}_\omega)$ is isomorphic to $\mathbb{Z}$. Let $[X]$ be a generator of this group. We will use the following theorem without proof.[13]

[13] Another version of the Poincaré duality theorem asserts that the homomorphisms

$$[X]\cap\colon H^i(M;\mathbb{Z}_\omega) \to H_{m-i}M,$$

$$[X]\cap\colon H^iM \to H_{m-i}(M;\mathbb{Z}_\omega)$$

are isomorphisms.

Theorem 13.8 (The Poincare duality). *There is a $\mathbb{Z}[\pi]$-isomorphism*

$$[X]\cap\colon H^i\tilde{M} \to H_{m-i}\tilde{M}.$$

We note that for every finitely generated free $\mathbb{Z}[\pi]$-module $A = \mathbb{Z}[\pi] \oplus \cdots \oplus \mathbb{Z}[\pi]$ we have $H_i(M;A) = H_i\tilde{M} \oplus \cdots \oplus H_i\tilde{M}$. A similar equality is true for cohomology groups. Thus, there is a Poincare duality isomorphism $H^i(M;A) \approx H_{m-i}(M;A)$ for every finitely generated free $\mathbb{Z}[\pi]$-module A.

Corollary 13.9. *Suppose that for every projective $\mathbb{Z}[\pi]$-module P the homology groups $H_i(M;P)$ are trivial for $i < q$. Then the groups $H^{m-i}(M;P)$ are trivial for $i < q$.*

Proof. Indeed, since P is projective, it is a summand of a free $\mathbb{Z}[\pi]$-module $P \oplus Q$, while $H^{m-i}(M;P)$ is a summand of the trivial group

$$H^{m-i}(M;P) \oplus H^{m-i}(M;Q) = H^{m-i}(M;P\oplus Q) = H_i(M;P\oplus Q) = 0,$$

since the group $H_i(M;P\oplus Q)$ is the direct sum of the trivial groups $H_i(M;P)$ and $H_i(M;Q)$. □

13.5 Poincaré complexes

Let X be a finite CW-complex, equipped with a homomorphism $\omega\colon \pi \to \mathrm{Aut}(\mathbb{Z})$ on its fundamental group π. The homomorphism ω defines an action of the group ring $\mathbb{Z}[\pi]$ on $\mathbb{Z}$ by $g(n) = n\omega(g)$, and therefore turns the abelian group $\mathbb{Z}$ into a $\mathbb{Z}[\pi]$-module, which we denote by $\mathbb{Z}_\omega$. In practice, we will consider a CW-complex X with a vector bundle ν_X. The homomorphism ω is the orientation sheaf associated with ν_X. Namely, $\omega(g)$ is the identity homomorphism when g is a homotopy class of orientation preserving loops, while $\omega(g)$ is given by the multiplication by -1 when g is an orientation reversing class.

We say that X is a (finite) *Poincaré complex* of dimension m if there is a class $[X] \in H_m(X; \mathbb{Z}_\omega)$ such that the homomorphisms $[X]\cap\colon H^i\tilde{X} \to H_{m-i}\tilde{X}$ are isomorphisms for all i. We note that the homomorphism $\omega = \omega_X$ as well as the homology class $[X]$ are parts of the Poincaré complex structure on X.

Let $f\colon M \to X$ be a map of Poincaré complexes. It defines a homomorphism $f^*w_X\colon \pi_1 M \to \mathrm{Aut}(\mathbb{Z})$ by taking the homotopy class γ of a loop to $\omega_X(f_*\gamma)$. If $f^*w_X = w_Y$, then the map f defines homomorphisms of (co)homology with coefficients in $\mathbb{Z}_{\omega_X}$ and $\mathbb{Z}_{\omega_Y}$. We say that the map f is of *degree one*, if $f_*[M] = [X]$.

Given a normal map f of degree 1 of a connected closed manifold M of dimension $m \geq 5$ into a Poincare complex X, we aim to determine under what conditions the map f is cobordant to a homotopy equivalence. Suppose that the map f induces an isomorphism of fundamental groups of M and X.[14] Furthermore, since f is of degree 1, by definition, it pulls the twisted coefficients $\mathbb{Z}_{\omega_X}$ back to $\mathbb{Z}_{\omega_M}$. We will write $\mathbb{Z}_\omega$ for both. Then the map f not only defines homomorphisms

$$f_*\colon H_*\tilde{M} \to H_*\tilde{X}, \qquad f^*\colon H^*\tilde{X} \to H^*\tilde{M},$$

but also homomorphism

$$f^!\colon H^*\tilde{M} \to H^*\tilde{X}, \qquad f_!\colon H_*\tilde{X} \to H_*\tilde{M},$$

where for example $f^!$ is defined by first converting a cohomology class U to a homology class $u = [M] \cap U$, then pushing it to a class $v = f_*u$, and finally converting it back to a cohomology class V such that $[X] \cap V = v$. It follows that $f_* \circ f_! = \mathrm{id}$ and $f^! \circ f^* = \mathrm{id}$, when f is of degree 1, see Lemma 11.9. In particular,

$$H_*\tilde{M} = H_*\tilde{X} \oplus K_*M, \qquad H^*\tilde{M} = H^*\tilde{X} \oplus K^*M,$$

where K_*M is $\ker f_*$, and $K^*M = \ker f^!$. Furthermore, in view of Poincaré dualities on M and X, we deduce the existence of Poincaré duality isomorphisms: $[M]\cap\colon K^iM \to K_{m-i}M$. Similarly, for a normal map of pairs $(W, M) \to (Y, X)$, of degree 1 there are well-defined groups $K_*(W, M)$ and $K^*(W, M)$.

[14] In fact, by Theorem 14.1 below, we may assume that the map $f\colon M \to X$ is q-connected, where $m = 2q$ or $m = 2q + 1$. In particular, we may assume that f induces an isomorphism of fundamental groups of M and X.

13.6 Homological algebra

Let (C, d) be a chain complex of finitely generated projective Λ-modules such that the modules C_i are trivial for $i \neq m, m+1, ..., n$, i.e., it is of the form

$$0 \to C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_{m+2}} C_{m+1} \xrightarrow{d_{m+1}} C_m \to 0.$$

We say that the chain complex (C, d) is *split exact* if each module C_i is isomorphic to a direct sum $B_i \oplus B_{i-1}$ of modules such that d_i is trivial on B_i and takes B_{i-1} by identity to the factor B_{i-1} in C_{i-1}. In this case B_{i-1} is the module of boundaries $d_i(B_{i-1})$ in C_{i-1}.

Lemma 13.10. *Suppose that the homology groups of (C, d) are trivial. Then (C, d) is split exact.*

Proof. Since $B_m = C_m$ is projective, there is a homomorphism $\Gamma_m : B_m \to C_{m+1}$ left inverse to d_{m+1}, i.e., a homomorphism such that $d_{m+1} \circ \Gamma_m = \mathrm{id}$. Therefore, the projective module C_{m+1} is the direct sum of projective modules $B_{m+1} \oplus B_m$ where the summands B_{m+1} and B_m are identified with the images of d_{m+2} and Γ_m respectively. Furthermore, there is an exact sequence of projective modules

$$0 \to C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_{m+3}} C_{m+2} \xrightarrow{d_{m+2}} B_{m+1} \to 0.$$

Therefore the argument can be repeated, and Lemma 13.10 follows by induction. □

Remark 13.11. Note that the conclusion of Lemma 13.10 still holds even if the module C_n is not assumed to be projective. In fact, in this case Lemma 13.10 implies that $C_n = B_{n-1}$ is projective.

Lemma 13.12. *Suppose that for all choices of coefficients, the cohomology groups of (C, d) are trivial. Then (C, d) is split exact.*

Proof. Choosing the coefficient module to be C_n, we deduce that the sequence

$$0 \longleftarrow \mathrm{Hom}_\Lambda(C_n, C_n) \xleftarrow{d_n^*} \mathrm{Hom}_\Lambda(C_{n-1}, C_n) \longleftarrow \cdots$$

is exact. In particular, there is a homomorphism $f : C_{n-1} \to C_n$ such that $d_n^*(f) = \mathrm{id}$, i.e., $f \circ d_n = \mathrm{id}$. Therefore, $C_{n-1} \approx B_{n-1} \oplus B_{n-2}$, and there is an exact sequence $0 \to B_{n-2} \to C_{n-2} \to C_{n-3} \to \cdots$. Thus, the argument can be iterated, and Lemma 13.12 follows by induction. □

Theorem 13.13. *Suppose that $H_iC = 0$ for $i < q$, and for all choices of projective coefficients $H^jC = 0$ for $j > q$. Then H_qC is a finitely generated stably free Λ-module, and $H^qC = (H_qC)^*$.*

Proof. By Remark 13.11, the module $Z_{q-1} = B_{q-1}$ is projective. Therefore the exact sequence $0 \to \operatorname{Ker} d_q \to C_q \to \operatorname{Im} d_q \to 0$ where $\operatorname{Ker} d_q = Z_q$ and $\operatorname{Im} d_q = B_{q-1}$ is split exact. Therefore, $C_q = Z_q \oplus Z_{q-1}$. By the argument in the proof of Lemma 13.12, we deduce that there is a sequence

$$0 \to B_q \xrightarrow{d_q} C_q \to C_{q-1} \to \cdots$$

whose cohomology groups are trivial for all choices of coefficients. Consequently, by the same argument, the group C_q is a direct sum of B_q and some module $B_q^{\perp}$. It follows that $C_q = B_q \oplus H_q \oplus Z_{q-1}$.[15] If we remove H_q from C_q in the chain complex C, then we obtain an exact sequence

$$\cdots \to C_{q+2} \to C_{q+1} \to B_q \oplus Z_{q-1} \to C_{q-1} \to C_{q-2} \to \cdots .$$

By Lemma 13.10, this sequence is split exact. In particular, we have

$$B_q \oplus Z_{q-1} \oplus \bigoplus_{ev \neq 0} C_{q+ev} = \bigoplus C_{n+odd}$$

where *ev* and *odd* range over all even and odd integers respectively. By adding H_q to both sides, we obtain

$$\bigoplus_{ev \neq 0} C_{q+ev} = H_q \oplus \bigoplus C_{n+odd},$$

which implies that H_q is a finitely generated stably free Λ-module.

Finally, let us show that $H^qC = (H_qC)^*$. We have seen that C is split exact at all terms except at C_q:

$$\to B_{q+1} \oplus B_q \to B_q \oplus H_q \oplus B_{q-1} \to B_{q-1} \oplus B_{q-2} \to \cdots$$

The same is true for the complex $\operatorname{Hom}(C, \mathbb{Z}[\pi])$; hence the claim. □

[15] Indeed, let $f : C_q \to B_q$ be a left inverse of d_q, i.e., a homomorphism such that $f \circ d_q = id$. Since the image of d_q is in Z_q, we have $f|_{Z_q} \circ d_q = id$, where $f|_{Z_q} : Z_q \to B_q$ is the restriction of f. In other words, B_q is a direct summand of Z_q, which implies that $Z_q = B_q \oplus H_q$. To summarize, $C_q = Z_q \oplus Z_{q-1} = B_q \oplus H_q \oplus Z_{q-1}$.

13.7 *The kernels K_qM and K^qM*

We will see that every normal map $f : M \to X$ of degree 1 can be turned by surgery into a q-connected map. We will next formulate this fact in terms of kernels K_iM. We say that a module is *stably free* if its direct sum with some free module is free.[16]

[16] **The following is Lemma 2.3 in Wall.**

Theorem 13.14. *Let $f: M \to X$ be a q-connected degree one map from a closed connected manifold of dimension m to a finite Poincaré complex for $q \geq 2$. If $m = 2q$, then the kernels $K_i M$ are trivial for $i \neq q$, and $K_q M = \pi_{q+1}(f)$ is a finitely generated stably free $\mathbb{Z}[\pi]$-module. If $m = 2q+1$, then the kernels $K_i M$ are trivial for $i \neq q, q+1$, and $K_q M \approx \pi_{q+1}(f)$.*

Proof. We may assume that the map f is an embedding of a CW-complex into a CW-complex. Let $\tilde{X} \to X$ be a universal covering. Since $q \geq 2$, it restricts to a universal covering $\tilde{M} \to M$ over M. The inclusion $\tilde{M} \to \tilde{X}$ is still a q-connected map.[17] Consequently, the kernel $K_i M = H_{i+1}(\tilde{X}, \tilde{M})$ is trivial for $i < q$, and

$$K_q M = H_{q+1}(\tilde{X}, \tilde{M}) \approx \pi_{q+1}(\tilde{X}, \tilde{M}) \approx \pi_{q+1}(X, M) = \pi_{q+1}(f).$$

by the Relative Hurewicz Theorem. Let now C denote the chain complex of $\mathbb{Z}[\pi]$-modules of the pair (X, M). Its homology $H_i(C) = K_i M$ are trivial for $i < q$. Therefore the chain complex C is of the form[18]

$$\to C_{q+1} \to Z_q \oplus B_{q-1} \to B_{q-1} \oplus B_{q-2} \to B_{q-2} \oplus B_{q-3} \to \cdots$$

Consequently, the homology groups $H_i(C; P)$ are trivial for $i < q$ and any coefficient $\mathbb{Z}[\pi]$-module P. By the Poincaré duality, $H^{m-i}(C; P)$ is trivial for any projective $\mathbb{Z}[\pi]$-module P.[19] If $m = 2q$, then $m - i$ ranges over all $j > q$ as i ranges over integers $< q$. Therefore, by Theorem 13.13, the module $K_q M$ is a finitely generated stably free $\mathbb{Z}[\pi]$-module. If $m = 2q$, we deduce that $K_i M$ may be non-trivial only for $i = q$ and $i = q + 1$. □

Thus, every normal map $f: M \to K$ of degree 1 is cobordant to a map with trivial kernels $K_i M = 0$ for $i \leq q - 1$.[20] On the other hand, the map f is a homotopy equivalence if and only if it is $(q+1)$-connected, or, equivalently $K_i = 0$ for $i \leq q$. Thus, we need to study when the $\mathbb{Z}[\pi]$-module $K = K_q M$ can be eliminated by surgery.

[17] Indeed, we have $\pi_i \tilde{M} \approx \pi_i M$ and $\pi_i \tilde{X} \approx \pi_i X$ for $i > 1$.

[18] The splitting is constructed in the proof of Lemma 13.10 except for the splitting of C_q. To split C_q we note that in

$$0 \to \operatorname{Ker} d_q \to C_q \to \operatorname{Im} d_q \to 0$$

the module $\operatorname{Ker} d_q$ is the module Z_q of cycles, while $\operatorname{Im} d_q$ is the module B_{q-1} of boundaries. This sequence splits since B_{q-1} is projective; it is a summand of a free module $C_q = B_{q-1} \oplus B_{q-2}$.

[19] Indeed, if $P \oplus Q$ is free, then $H^{m-i}(C; P)$ is a direct summand of the zero module

$$H^{m-i}(C; P) \oplus H^{m-i}(C; Q) = H_i(C; P \oplus Q).$$

[20] Without loss of generality we may assume that f is an inclusion. Then a map f is q-connected if all groups $\pi_i(X, M)$ are trivial for $i \leq q$. On the other hand, for $i > 1$, we have $K_i = \pi_i(X, M)$.

14

Surgery on non-simply connected manifolds

14.1 Surgery below the middle range

Let $M \subset \mathbb{R}^{m+k}$ be a manifold of dimension m, and X a finite CW-complex equipped with a vector bundle ν. We say that $f: M \to X$ is a *normal map* if it is covered by a fiberwise isomorphism $f: T^{\perp}M \to \nu$. In other words, for each point $x \in M$, there is a chosen isomorphism of the perpendicular space $T_x^{\perp}M$ and the space $\nu_{f(x)}$, and, furthermore, the chosen isomorphisms change continuously as x changes in M.

Without loss of generality we may assume that M has a chosen CW-structure, and the map f is an embedding. In fact, we may replace the CW-complex X with the mapping cylinder of f, and therefore we may assume that the CW-complex X is obtained from $M \times [0, \varepsilon]$ by attaching finitely many cells.

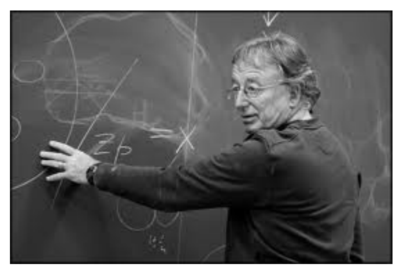


Figure 14.1: Dennis Sullivan (1941–)

A *normal cobordism* of normal maps $f_i: M_i \to X$, with $i = 0, 1$, is a cobordism $W \subset \mathbb{R}^{m+k} \times [0, 1]$ of the manifolds M_0 and M_1 together with a normal map $W \to X$ extending the normal map f.

Theorem 14.1. *Every normal map $f: M \to X$ of a closed connected manifold of dimension $m \geq 2q$ to a connected CW-complex is cobordant to a q-connected normal map.*

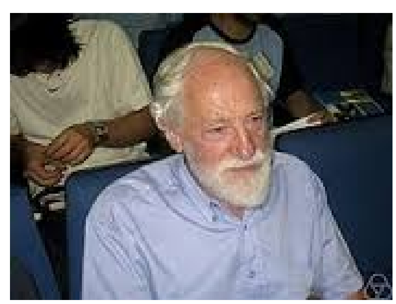


Figure 14.2: Charles Terence Clegg Wall (1936–)

Proof. We may assume that f is an embedding. Then the CW-complex X is obtained from the manifold M by attaching finitely many cells. Without loss we may assume that the cells of dimension $> q$ are attached after all cells of dimension $\leq q$ are attached. Let k be the number of cells in $X \setminus M$ of dimension $\leq q$.

Let X_s denote the CW-complex obtained from $M \times [0, \varepsilon]$ by attaching the first s cells so that X_0 coincides with $W_0 = M \times [0, \varepsilon]$. The first cell D^i is attached along an attaching sphere of dimension $\leq q - 1$ to the manifold $M \times \{\varepsilon\}$ of dimension $\geq 2q$,[1] and therefore, the attached cell can be thickened to a handle $D^i \times D^j$ where $j = m - i + 1$. The thickening of the cell turns X_1 into a cobordism W_1 from M to M_1, while the compression $D^i \times D^j \to D^i$ defines a deformation retraction $W_1 \to X_1$. The second cell $X_2 \setminus X_1$ is attached along an attaching sphere in $X_1 \subset W_1$. We may modify it by homotopy to obtain an attaching sphere in $M_1 \subset \partial W_1$, and then thicken the second cell to a handle, and turn X_2 into a cobordism W_2 between M and M_2.

[1] We have seen before that the normal bundle of such a sphere in M is necessarily trivial. Indeed, the normal bundle of an embedded sphere $S \subset M$ of dimension $i-1$ in $\mathbb{R}^{m+k}$ consists of the sum of the perpendicular bundle $T^{\perp}M$ over S and the perpendicular bundle ν_S of S in M. Since the vector bundle $T^{\perp}M \approx \nu$ over S extends over D^i, it is trivial, i.e., it can be identified with ε^k. Thus we have

$$\varepsilon^{m+k-i+1} = \nu_S \oplus \varepsilon^k$$

which implies that ν_S is trivial when the dimension of ν_S is strictly greater than the dimension of S.

Continuing by induction, we construct a sequence of manifolds W_i with $i \leq k$ together with deformation retractions $W_i \approx X_i$ such that each W_i for $i \geq 1$ is obtained from W_{i-1} by attaching a single handle. In particular, the composition of the homotopy equivalence $W_i \to X_i$ with the inclusion $X_i \subset X$ is a normal cobordism of f to a map $f_i \colon M_i \to X$.

Since the space X is obtained from X_k by attaching cells of dimension $\geq q$, the map $W_k \approx X_k \to X$ is q-connected.[2] On the other hand, the cobordism W_k is obtained from M_k by attaching handles of high indices $m - i + 1$ where $i \leq q$. Therefore the relative homotopy groups $\pi_i(W_k, M_k)$ are trivial for $j \leq m - q$, which means that the inclusion $M_k \to W_k$ is also a q-connected map as $m - q \geq q$. As a composition of q-connected maps, the map $M_k \to W_k \to X$ is also q-connected. □

[2] There is a long exact sequence of homotopy groups

$$\pi_q X_k \to \pi_q X \to 0 \to \pi_{q-1} X_k$$

since $\pi_i(X, X_k) = 0$ for $i \leq q$.

14.2 Surgery on maps of manifolds of even dimension

14.2.1 Wall representatives

Let $f \colon M \to X$ be a normal map of a manifold $M \subset \mathbb{R}^{m+k}$ of dimension $m \geq 5$ into a Poincaré complex X of dimension m, with a fiberwise isomorphism $T^{\perp}M \to \nu$ covering the map f. Suppose that f is of degree 1. Then f is normally cobordant to a q-connected map where $m = 2q$ or $m = 2q + 1$. By an appropriate version of the Poincare duality, the map f is a homotopy equivalence if and only if it is $q + 1$ connected, i.e., $K = \pi_{q+1}(X, M)$ is zero. An element in K is represented by a threaded immersion $i \colon S^q \to M$ together with an extension of $f \circ i$ to a map $D^{q+1} \to X$.[3] In fact, by the Whitney trick we may assume that the

[3] We recall that a threaded map $S^q \to M$ is a map together with a curve w from the basepoint of M to the image of the basepoint of S^q.

threaded immersion i is a threaded embedding. We note then that near the embedded sphere $S' = i(S)$, the manifold M is framed. Indeed, since the disc D^{q+1} is contractible, the pull back of ν over D^{q+1} has a unique up to homotopy frame. In view of the fiberwise isomorphism $T^{\perp}M \approx \nu$ over S', the frame over $\partial D^{q+1} = S'$ defines a distinguished frame $\tau_1, ..., \tau_k$ of M over S' which can be extended to a neighborhood of S'.

We will next define a Wall representative of the homotopy class i. In fact, the construction of a Wall representative is the same as in the case of a framed manifold M except that in general the manifold M is framed only near the sphere S'. Such a frame is sufficient as the construction of the Wall representative takes place in a neighborhood of S'.

We will briefly review the construction.

The sphere S' bounds a unique up to isotopy disc D' in $\mathbb{R}^{m+k} \times [0,1]$ where the space $\mathbb{R}^{m+k}$ containing the manifold $M \supset S'$ is identified with the horizontal slice $\mathbb{R}^{m+k} \times \{0\}$. Since the disc D' is contractible, there is a unique up to isotopy orthonormal frame $v = \{v_i\}$ over D'. By the Multicompression Theorem there is an ambient isotopy F which slightly perturbs S' in $\mathbb{R}^{m+k}$ and brings the vector fields $v_1, ..., v_k$ to the vector fields $\tau_1, ..., \tau_k$ over S'. The ambient isotopy F extends to an ambient isotopy of $\mathbb{R}^{m+k} \times [0,1]$ and modifies D' together with its frame appropriately. The sphere S' may not be in M anymore, but it is still in a tubular neighborhood of M. Therefore it can be projected by a regular homotopy to a sphere $S \subset M$. The regular homotopy of S' extends to a regular homotopy of D' and its frame, and results in a framed disc D in $\mathbb{R}^{m+k} \times [0,1]$ bounding S.

We note that S is a threaded sphere, representing the homotopy class $[i]$. It is equipped with a frame $v_1 = \tau_1, ..., v_k = \tau_k, v_{k+1}, ..., v_{m-q+k}$ in $\mathbb{R}^{m+k}$ which extends to a frame $\{v_i\}$ over D in $\mathbb{R}^{m+k} \times [0,1]$. We say that S is a *Wall representative* of the homotopy class $[i]$. Furthermore, we say that a representative $(g, \tilde{g})$ of $x \in K$ is a Wall representative if $g(S^k)$ is a Wall representative. Similarly, we say that $(g, \tilde{g})$ is an embedded Wall representative if it is a Wall representative and g is an embedding.

Theorem 14.2. *Suppose that $m \geq 2q$ and $m \geq 5$. Let x be an element in K. If x admits an embedded Wall representative, then any embedded representative S of x is a Wall representative.*

Proof. The proof is similar to the proof of Theorem 5.6. Namely, choose an embedded Wall representative S_x of x in M. Since S_x and S are homotopic, by the Whitney strong immersion theorem there is an immersion $g\colon S^q \times [0,1] \to \mathbf{M}$ to the cylinder $\mathbf{M} = M \times [0,1]$ such that over $S^q \times \{t\}$ the immersion g agrees with the inclusion of S_x and S when $t \in [0,\varepsilon]$ and $t \in [1-\varepsilon, 1]$ respectively. The image of g is an immersed cylinder $\mathbf{S}$. In $\mathbb{R}^{m+k} \times [0,1] \times \mathbb{R}$ there is an immersed disc $\mathbf{D} = D^{q+1} \times [0,1]$ such that $D^{q+1} \times \{1\}$ is the core D_x of a framed surgery along the Wall representative S_x, the disc $D^{q+1} \times \{1\}$ is an embedded disc in $\mathbb{R}^{m+k} \times \{1\} \times \mathbb{R}$, and $\partial D^{q+1} \times [0,1] = \mathbf{S}$, see Figure 14.3.

As in the construction of the Wall representative, there is a perpendicular frame $\tau_1, ..., \tau_k$ of M over S_x. Since $S^q \times \{0\}$ is a deformation retract of $S^q \times [0,1]$, the frame $\tau_1, ..., \tau_k$ can be extended over the immersion g. In other words, there is a function which associates with each $u \in S^q \times [0,1]$ a frame of $T^{\perp}M$ at the point $g(u)$.[4]

[4] We note that since g is an immersion, there may exist points u and v in $S^q \times [0,1]$ such that $g(u) = g(v)$. In this case, there are two possibly different frames at $g(u) = g(v)$ corresponding to u and v. For this reason, the frame $\tau_1, ..., \tau_k$ is a frame not over $\mathbf{S}$, but a frame along an immersion.

The core D_λ of the surgery is equipped with perpendicular vector fields $v_1, ..., v_{m-q+k}$ over D such that the vector fields v_i coincide with τ_i for $i \le k$ over S_x. As above, we may extend the vector fields $v_i | S_x$ along the immersion g in such a way that $v_i = \tau_i$ for $i \le k$. Next, since $\mathbf{S} \cup D_\lambda$ is a deformation retract of $\mathbf{D}$, we may extend the vector fields v_i over $\mathbf{D}$. It remains to observe that the disc $D^{q+1} \times \{1\}$ together with vector fields v_i over it is a base of framed surgery along S. □

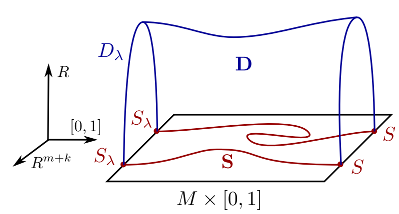


Figure 14.3: The immersed disc $\mathbf{S}$ in $\mathbf{M}$ as well as the disc $\mathbf{D}$.

14.2.2 *The intersection pairing λ and self-intersection form μ*

Let $f\colon M \to X$ be a q-connected normal map of degree one of a manifold of dimension $m = 2q$. We may assume that that f is an inclusion, in which case the group $K = K_q M$ can be identified with the group $\pi_{q+1}(X, M)$. In particular its elements are represented by pairs $x = (g, \tilde{g})$ of immersions $g\colon S^q \to M$ and extensions $\tilde{g}\colon D^{q+1} \to X$ of g. We define the intersection pairing

$$\lambda\colon K \times K \longrightarrow \Lambda$$

by $\lambda(x,y) = \lambda(g(S^q), h(S^q))$ where $x = (g, \tilde{g})$ and $y = (h, \tilde{h})$ any two elements in K. We note that by Exercise 13.3, the intersection pairing λ is well-defined.

Exercise 14.3. Show that λ is non-degenerate, i.e., its adjoint map $K \to K^*$ is an isomorphism.

Next, let us define the invariant μ. Recall that on the group ring $\Lambda = \mathbb{Z}[\pi]$, where $\pi = \pi_1 M$, there is an involution $g \mapsto \bar{g}$ which takes g to $w(g)g^{-1}$, where $w(g) = 1$ if g is orientation preserving and $w(g) = -1$ otherwise. Let Λ_ε denote the quotient group of Λ by $\langle g - (-1)^q \bar{g} | g \in \Lambda \rangle$. For an immersion $S \looparrowright M$ of a sphere we have defined the algebraic number $\mu(S)$ of intersection points with coefficients in Λ_ε. The invariant $\mu \colon K \to \Lambda_\varepsilon$ associates with an element $[i]$ the algebraic number $\mu(S)$ of intersection points with coefficients in Λ_ε of the Wall representative S of i.

As in the case of framed manifolds, we deduce that the invariant μ is well-defined, and $\mu(x) = 0$ for $x \in K$ if and only if there is a surgery along a representative of x. Furthermore, if $\mu(x) = 0$, then any embedded representative of x is a Wall representative provided that $m \geq 2q$ and $m \geq 5$ by Theorem 14.2.

To begin with let us recall that an element in $K = \pi_{q+1}(X, M)$ is represented by a threaded immersion $i \colon S^q \to M$ together with an extension $\tilde{i} \colon D^{q+1} \to X$. Explicitly, the addition of elements $(i, \tilde{i})$ and $(i', \tilde{i}')$ can be described as follows. The concatenation $w' * w^{-1}$ obtained from threads w and w' of the threaded immersions i and i' respectively is a path from the distinguished point $i(S^q)$ of the first immersed sphere to the distinguished point $i'(S^q)$ of the second immersed sphere. Take a connected sum of the two immersed spheres along the path $w' * w^{-1}$, and choose a short thread for the newly obtained immersed sphere; note that a connected sum is obtained by attaching to the two immersed spheres a cylinder passing near the distinguished point of M, and therefore there is a short path from the distinguished point of M to a point on the cylinder. The new threaded sphere is bounded in X by the boundary connected sum of the discs $\tilde{i}(D^{q+1})$ and $\tilde{i}'(D^{q+1})$ where the connected sum is taken along $w' * w^{-1}$.

There is also an action of Λ on K; if S is an immersed sphere equipped with a curve w from the basepoint of M to the basepoint of S and g a curve representing an element in $\pi_1 M$, then Sg is the same immersed sphere but equipped with the curve $g^{-1} * w$. By Lemma **??** the module K is a finitely generated stably free Λ-module. There is also an involution $\Lambda \to \Lambda$ defined by $g \mapsto \omega(g)g^{-1}$ where $\omega(g) = 1$ if g is orientation preserving, and $\omega(g) = -1$ otherwise.

Theorem 14.4. *The intersection pairing λ and the self-intersection form μ satisfy*

$$\lambda(S_1, S_2) = \varepsilon \overline{\lambda(S_2, S_1)},$$

$$\lambda(S, S_1 g_1 + S_2 g_2) = \lambda(S, S_1) g_1 + \lambda(S, S_2) g_2.$$

$$\mu(S_1 + S_2) = \mu(S_1) + \mu(S_2) + \lambda(S_1, S_2),$$

$$\mu(Sg) = \bar{g}\mu(S)g,$$

$$\lambda(S, S) = \mu(S) + \varepsilon\overline{\mu(S)}.$$

where $\varepsilon = (-1)^q$.

Exercise 14.5. Prove Theorem 14.4.

In other words $\mathbf{K} = (K, \lambda, \mu)$ is a non-degenerate ε-quadratic form over Λ, $\varepsilon = (-1)^k$,[5] see Exercise 14.3.

Definition 14.6. An ε-quadratic form over Λ is a pairing $\lambda : K \times K \to \Lambda$ and a map $\mu : K \to \Lambda_\varepsilon$ on a projective Λ-module K that satisfy the equalities of Theorem 14.4. We say that an ε-quadratic form is non-degenerate if λ is non-degenerate, i.e., the associated map $K \to \mathrm{Hom}_\Lambda(K, \Lambda)$ defined by $x \mapsto \lambda(x, -)$ is an isomorphism.

There is an important example of an ε-quadratic form over Λ. Let L be a finitely generated projective Λ-module, and $K = L \oplus L^*$ where $L^* = Hom_\Lambda(L, \Lambda)$. Put $\mu(x, f) = f(x)$, and $\lambda(x, f; y, g) = f(y) + \varepsilon\overline{g(x)}$. We say that $\mathbf{H}_\varepsilon = (K, \lambda, \mu)$ is a hyperbolic ε-quadratic form.[6]

There is a semigroup structure on the set of ε-quadratic forms; namely,

$$\mathbf{K_1} \oplus \mathbf{K_2} = (K_1 + K_2, \lambda_1 \oplus \lambda_2, \mu_1 \oplus \mu_2).$$

Example 14.7. Let $f : M \to X$ be a normal q-connected map of degree one of a closed manifold of dimension $m = 2q$. A surgery along a trivial element in $\pi_{q+1}(f)$ replaces the manifold M with a manifold $M\#(S^q \times S^q)$. Consequently, the group $K_q M$ is replaced with the group $K_q M \oplus (\mathbb{Z}\pi \oplus \mathbb{Z}\pi)$, while the ε-quadratic form $\mathbf{K}$ is replaced with its direct sum $\mathbf{K} \oplus \mathbf{H}_\varepsilon$.

7

Definition 14.9. The abelian semigroup $L_{2q}(\Lambda)$ is the quotient semigroup of non-degenerate ε-quadratic forms on finitely generated free Λ-modules under the equivalence relation generated by $\mathbf{K} \simeq \mathbf{K} \oplus \mathbf{H}_\varepsilon$.

Exercise 14.10. Show that $L_{2q}(\Lambda)$ is a group.[8]

[5] In literature, given a ring A with involution, a pairing λ on an A-module A is said to be *sesquilinear* if the second property of Theorem 14.4 holds, ε-symmetric, if the first property holds, and μ is said to be an ε-quadratic form, if the last three properties hold. Finally, $\mathbf{K}$ is nonsingular, if the map $K \to K^*$ adjoint to λ is an isomorphism of A-modules.

[6] For example, let L be the free module Λ with generator e. Then L^* is a free module with generator e^* such that $e^*(e) = 1$, where 1 is the generator of $\mathbb{Z}[\pi]$. The matrix of the form λ is given by

$$\begin{bmatrix} 0 & 1 \\ \varepsilon & 0 \end{bmatrix}.$$

The form μ is defined so that $\mu(e) = \mu(e^*) = 0$. This implies $\mu(e + e^*) = 1$.

[7]

Exercise 14.8. Let $f : M \to X$ be a q-connected normal map of a manifold of dimension $m = 2q$. Suppose that f' is a normal map obtained from f by a normal spherical surgery along a sphere $S^{q-1} \subset M$. Show that $\mathbf{K}(f') = \mathbf{K}(f) \oplus \mathbf{H}_q$, where $\mathbf{H}_q = (\Lambda \oplus \Lambda, \lambda, \mu)$ is the ε-quadratic form on the free Λ-module with generators x and y such that $\lambda(x, y) = 1$, $\mu(x) = \mu(y) = 0$.

[8] **Hint for Exercise 14.10.** Show that $-\mathbf{K} = (K, -\lambda, -\mu)$. More precisely, show that the ε-quadratic form

$$(K, \lambda, \mu) \oplus (K, -\lambda, -\mu)$$

is isomorphic to the hyperbolic form $\mathbf{H}_\varepsilon(K)$. Let us construct the map

$$K \oplus K \mapsto K \oplus K.$$

which takes (x, y) to $(x - \varepsilon\overline{\lambda(y,)}, x + \lambda(, y))$.

14.2.3 *Effect of surgery on homotopy groups*

Suppose that K is a free Λ-module with a basis $\{a_1, ..., a_k, b, c\}$ such that with respect to the intersection form λ, the class b is orthogonal to all a_i and b, the class c is orthogonal to all a_i and c, and $\lambda(b, c) = 1$. Since $2q \geq 6$, we may choose representatives of the classes a_i, b and c, so that the geometric numbers of intersections of the representatives agrees with the algebraic numbers.[9] Suppose that $\mu(b) = 0$. Then there is a normal surgery along b which results in a normal q-connected map of a manifold M' with kernel K' and maps λ' and μ'.

[9] In other words, we may choose representing spheres $S_{a_1}, ..., S_{a_k}, S_b, S_c$ together with extending maps $D \to X$ such that S_b and S_c are embedded spheres disjoint from S_{a_i}, there is a unique intersection point in $S_b \cap S_c$, and the path from the distinguished point in M through the distinguished point in S_b, the intersection point, the distinguished point in S_c back to the distinguished point in M is null-homotopic.

Lemma 14.11. *The group K is a free Λ-submodule of K' generated by $\{a_1, ..., a_k\}$. The maps λ' and μ' are restrictions of λ and μ.*

Proof. The manifold M' is obtained from M by removing a torus h_q, and then attaching a torus h'_q to $M_0 = M \setminus h_q$. As in Lemma 6.7, consider the long exact sequence of $\mathbb{Z}[\pi]$-modules

$$H_{q+1}(\tilde{M}, \tilde{M}_0) \xrightarrow{\partial} K_q M_0 \xrightarrow{i_*} K_q M \xrightarrow{j_*} H_q(\tilde{M}, \tilde{M}_0)$$

The source of the homomorphism ∂ is a trivial module, while the homomorphism j_* is a surjective homomorphism onto a module isomorphic to $\mathbb{Z}[\pi]$ which takes the class c to a generator. Thus $K_q M_0$ is a free $\mathbb{Z}_[\pi]$-module generated by $\{a_1, ..., a_k, b\}$. As in Lemma 6.8, consider the long exact sequence

$$H_{q+1}(\tilde{M}', \tilde{M}_0) \xrightarrow{\partial} K_q M_0 \to K_q M' \to 0.$$

The homomorphism ∂ takes the generator of $H_{q+1}(\tilde{M}', \tilde{M}_0) \approx \mathbb{Z}[\pi]$ to b, and therefore, the module $K' = K_q M'$ is a free $\mathbb{Z}[\pi]$-module generated by $\{a_1, ..., a_k\}$. □

14.2.4 *The main theorem*

Every normal map f on a manifold of dimension $m = 2q$ is normally cobordant to a q-connected map f'. Define $\mathbf{K}(f) = \mathbf{K}(f')$.

Theorem 14.12. *The invariant $\mathbf{K}(f)$ is well-defined, and it is an invariant of a normal cobordism. In particular, if f is normally cobordant to a homotopy equivalence, then $\mathbf{K}(f) = 0$.*

Proof. It suffices to show that if $W \to X \times [0, 1]$ is a normal cobordism between q-connected maps $f : M \to X$ and $f' : M \to X$, then

$\mathbf{K}(f) = \mathbf{K}(f')$. By performing surgery below the middle range, we may assume that W is obtained from $M \times [0, \varepsilon]$ by attaching handles of indeces q and $q+1$. The cobordism W is a composition of a cobordism W' on M which consists of trivially attached handles of index q,[10] and the inverse W' to a cobordism on M' which also consists of trivially attached handles of index q. By Example **??**, the element $\mathbf{K}(f)$ does not change under the corresponding surgery. □

[10] Suppose that a handle of index q is attached along an attaching sphere S of dimension $q-1$. Since the cobordism is normal, the image of the sphere S in X is null-homotopic. On the other hand, since f is q-connected, it follows that S is null-homotopic in M. Thus all handles of index q are attached along spheres that are null-homotopic. **A bit care is necessary. The sphere has to be null isotopic.**

Theorem 14.13. *Let $f\colon M \to X$ be a normal map of degree one of a manifold of dimension $m = 2q$. Then $\mathbf{K}(f) \in L_{2q}(\Lambda)$ is a well-defined complete obstruction to the existence of a normal cobordism of f to a homotopy equivalence.*

Sketch. We may assume that f is q-connected. If $\mathbf{K}(f)$ is zero in $L_{2q}(\Lambda)$, then after performing finitely many spherical surgeries along spheres of dimension $q-1$, we may assume that $\mathbf{K}(f) = \mathbf{H}_q^k$ for some finite k. A spherical surgery along a basis vector of K eliminates a factor of $\mathbf{H}_q^k$; indeed, the inverse spherical surgery is perfomed along a sphere of dimension $q-1$ in q-connected manifold and therefore results in addition of a factor of $\mathbf{H}_q^k$. Thus, in finitely many steps, we obtain $K = 0$. □

14.3 Surgery on maps of manifolds of odd dimension

14.3.1 Heegaard splittings

Let $f\colon M \to X$ be a normal q-connected map of degree one from a closed manifold of dimension $m = 2q+1$ such that the groups K_iM are non-trivial only in degrees q and $q+1$. Let U be an embedded ball with k handles of degree q in M, and M_0 the closure of the complement to U in M. Suppose that X is obtained from a Poincaré pair (X, S^{2q}) by attaching a cell $B = B^{2q+1}$ along an embedded "boundary" sphere S^{2q}. We say that the decompositions $M = M_0 \cup U$ and $X = X \cup B$ form a *degree one splitting* if $f(M_0) \subset X_0$, $f(U) \subset B$ and the maps

$$f|\colon \partial U \longrightarrow S^{2q},$$

$$f|\colon (U, \partial U) \longrightarrow (B^{2q+1}, S^{2q}),$$

$$f|\colon (M, M_0) \longrightarrow (X, X_0)$$

are degree one maps. We note that the k spheres S_α of dimension q in U corresponding to the k handles of U are send to the contractible set

B in X, and therefore the inclusions $S_\alpha \subset M$ together with extensions $D^{q+1} \to B^{2q+1}$ of $f|S_\alpha$ represent elements x_α in $K_q M$. We say that the degree one splitting is associated with elements $\{x_\alpha\}$. A degree one splitting is a *Heegaard splitting* if $\{x_\alpha\}$ is a set of generators of $K_q(M)$.

In the rest of the section we will construct Heegaard splittings.

Choose a set of generators of the $\mathbb{Z}[\pi]$-module $K_q M$, and represent each generator by an embedded sphere S_α in M, and a map of a disc $D^{q+1} \to X$ extending $f|S_\alpha$. Choose a thickening $S_\alpha \times D^{q+1}$ of each sphere S_α, and connect the chosen tori by thin tubes to obtain a copy of a ball $U = \#_b S_\alpha \times D^{q+1}$ with, say, k handles in M. The closure of the complement will be denoted by M_0. We may modify the map f by homotopy so that $f|U$ is a constant map onto the center x_0 of the cell B^{2q+1}.

If $f|M_0$ is transverse to x_0, then the inverse image of x_0 has only finitely many points x in M_0. For each x we may modify f so that f takes a disc neighborhood of x to x_0. Then we may perform a trivial surgery of index $(q+1)$-along an attaching sphere in the disc neighborhood of x. This results in a map f' of $M' = M \# S^q \times S^{q+1}$, with meridian S_x of dimension q. The sphere $S_x \in M'$ together with the constant extension $D^{q+1} \to X$ of the map $f|S_x$ represents a new generator. By performing such a surgery for each inverse image x of x_0 in M_0, we may assume that f takes M_0 to $X \setminus \{x_0\}$.

Since any two maps to B are homotopic, we may modify the map f by homotopy in a neighborhood of U so that $f|\partial U$ is a degree one map to ∂B, and f still takes U to B. We claim that $f|U$ is also a map $(U, \partial U) \to (B^{2q+1}, S^{2q})$ of degree 1. Indeed, there is a diagram of short exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_{2q+1}(\tilde{U}, \partial\tilde{U}) & \longrightarrow & H_{2q}\partial\tilde{U} & \longrightarrow & H_{2q}\tilde{U} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H_{2q+1}(\tilde{B}, \partial\tilde{B}) & \longrightarrow & H_{2q}\partial\tilde{B} & \longrightarrow & H_{2q}\tilde{B} & \longrightarrow & 0.
\end{array}$$

Since the vertical homomorphism in the middle of the diagram takes the fundamental class $[\partial U]$ to $[\partial B]$, and $[U]$ and $[B]$ are unique classes for which $\partial[U] = [\partial U]$ and $\partial[B] = [\partial B]$, we conclude that $[U]$ maps to $[B]$, and therefore $(U, \partial U) \to (B, \partial B)$ is of degree one.

Since $U \cap M_0 = \partial U$, by the relative Mayer-vietoris sequence,

$$H_i(\tilde{M}, \tilde{\partial} U) \approx H_i(\tilde{M}_0, \partial\tilde{M}_0) \oplus H_i(\tilde{U}, \partial\tilde{U}).$$

Since $[U]$ maps to $[B]$, we deduce that $[M_0]$ maps to $[X_0]$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_{2q+1}\tilde{M} & \longrightarrow & H_{2q+1}(\tilde{M},\partial\tilde{U}) & \longrightarrow & H_{2q}\partial\tilde{U} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H_{2q+1}\tilde{X} & \longrightarrow & H_{2q+1}(\tilde{X},\tilde{S}^{2q}) & \longrightarrow & H_{2q}\partial\tilde{S}^{2q} & \longrightarrow & 0. \end{array}$$

A priori, the center x_0 in B may be in the image of the map $(M_0, \partial M_0) \to (X, S^{2q})$. However, since the component $[U]$ of the image of $[M_0]$ is zero, we deduce that the inverse image of x_0 with respect to $f|M_0$ consists of even number of points $z_1, ..., z_{2s}$ of alternating signs. All points $z_1, ..., z_{2s}$ belong to a simply connected neighborhood of U, and therefore we may use the Whitney trick to further modify the map f so that $f(M_0)$ does not contain x_0. Next we may use the radial homotopy of f over $f^{-1}(B) \cap M_0$ to obtain a map which we still denote by f such that $f(M_0)$ contains no interior points of B.

Then the map f together with decompositions $M = M_0 \cup U$ and $X = X_0 \cup B$ is a degree one splitting.

Lemma 14.14. *There is a short exact sequence*[11]

$$0 \to K_{q+1}(M_0, \partial M_0) \to K_q(\partial M_0) \to K_q M_0 \to 0,$$

of stably free modules. If $K_{q+1}(M_0, \partial M_0)$ is free, then $K^{q+1}(M_0, \partial M_0)$ is the dual module for $K_{q+1}(M_0, \partial M_0)$. By performing trivial surgeries in the interior of M_0, we may assume that all modules in this short exact sequence are free.

[11] The map $M_0 \to X_0$ is not a degree one map, but we still have $H_k\tilde{M}_0 = H_k\tilde{X}_0 \oplus K_k M$ where $K_k M_0$ is the kernel of the homomoprhism $H_k M_0 \to H_k X_0$, and we still have

$$K_k M_0 \approx K^{m-k}(M_0, \partial M_0).$$

Proof. We may assume that the map $f\colon (M_0, \partial M_0) \to (X_0, S^{2q})$ is an inclusion of a pair of CW-complexes. Then the map of the corresponding cellular chain complexes is injective, and there is a chain complex D_* such that there is a short exact sequence

$$0 \to C_*(M_0, \partial M_0) \to C_*(X_0, S^{2q}) \to D_{*-1} \to 0.$$

and $H_i(D_*) = K_i(M_0, \partial M_0)$. On the other hand, $K_i(M_0, \partial M_0) = 0$ for $i \neq q+1$.[12] Since $H_i(D_*) = 0$ for $i \neq q+1$, we deduce that $D_i = B_i \oplus B_{i-1}$ for $i < q+1$, and $D_{q+1} = Z_{q+1} \oplus B_q$. In other words, the complex D_* has the form

[12] Indeed, we have $K_i(M_0, \partial M_0) = K_i(M, U)$. Note that $K_q U \to K_q M$ is surjective, and therefore, $K_q(M, U) = 0$.

$$\to D_{q+2} \to Z_{q+1} \oplus B_q \to B_q \oplus B_{q-1} \to \cdots.$$

This implies that $H^i(D_*) = 0$ for $i < q+1$. Thus, we have

$$K_{q+1} M_0 \approx K^q(M_0, \partial M_0) = H^q(D_*) = 0,$$

where the first isomorphism is the Poincaré duality isomorphism. This shows that the sequence in the statement of Lemma 14.14 is exact.

Next, we observe that if $H_{q+1}(D_*) = 0$, then the short exact sequence $B_{q+1} \to Z_{q+1} \to H_{q+1}(D_*)$ splits. Hence, $H^{q+1}D = \mathrm{Hom}_\Lambda(H_{q+1}D, \Lambda)$.

Next, let us show that $K_q M_0$ is stably free. To this end consider the relative chain complex E_* such that $C_* M_0 \to C_* X_0 \to E_*$ is a short exact sequence of chain complexes. Then $H_i(E) = K_i M_0 = 0$ for $i < q$.

On the other hand, since the complex D_* splits in degrees $i < q+1$, we deduce that its homology $K_i(M_0, \partial M_0; P)$ are trivial for $i < q+1$ for any $\mathbb{Z}[\pi]$-module P. Therefore, by the argument in the proof of Corollary 13.9, the Poincaré dual groups $K^i(M_0; P)$ are also trivial for $i > q$.[13] Thus, by Theorem 13.13, the module $K_q M_0$ is stably free. By performing trivial surgeries in the interior of M_0, we may make $K_q M_0$ free. Then the short exact sequence in the statement of Lemma 14.14 splits. Hence $K_{q+1}(M_0, \partial M_0)$ is also stably free.

[13] If $P \oplus Q$ is free, then $K^i(M_0, P)$ is a direct summand of the trivial module $K^i(M_0; P \oplus Q)$.

Let us look more closely at a trivial surgery in the interior of M_0. Such a surgery replaces the manifold M with $M' = M \# S^q \times S^{q+1}$. We choose the sphere $S^q \times \{pt\}$ as a representative of an additional generator of $K_q M$. Then U is replaced with $U' = U \#_b S^q \times D^{q+1}$. Consequently, the group $K_q(\partial M_0)$ is replaced with $K_q(\partial M_0) \oplus \mathbb{Z}[\pi] \oplus \mathbb{Z}[\pi]$, while each group $K_{q+1}(M_0, \partial M_0)$ and $K_q M_0$ is stabilized by a copy of $\mathbb{Z}[\pi]$. Thus, indeed, by performing trivial surgeries in the interior of M_0 we may make $K_{q+1}(M_0, \partial M_0)$ free. $\square$

14.3.2 *ε-quadratic formations*

To motivate the definition of ε-quadratic formations, let $M = M_0 \cup U$ and $X = X_0 \cup B$ be a Heegaard splitting for a normal degree one map f. Recall that the boundary $\partial U = \partial M_0$ is a sphere with k handles $\# S_\alpha \times S^q$. Let K denote its homology in degree q, but threaded in M. Then K is a free $\mathbb{Z}[\pi]$-module with $2k$-generators $\{m_1, ..., m_k, \ell_1, ..., \ell_k\}$ where m_i and ℓ_i are respectively the meridian and longitude in the i-th summand. The module K comes with forms λ and μ such that (K, λ, μ) is a hyperbolic ε-quadratic form over $\mathbb{Z}[\pi]$. The meridians m_i generate a $\mathbb{Z}[\pi]$-submodule of K denoted by F. Next, consider the $\mathbb{Z}[\pi]$-module G given by the image of the injective boundary homomorphism $K_{q+1}(M_0, \partial U) \to K_q \partial U$.[14] The module F is a free $\mathbb{Z}[\pi]$-submodules of rank k. Without loss of generality we may also assume

[14] The homomorphism is injective by Exercise 14.14. Note also that F is the image of the injective boundary homomorphism $K_{q+1}(U, \partial U) \to K_q U$.

that G is a free $\mathbb{Z}[\pi]$-submodule.[15] The set $\mathbf{K}(f) = (K, \lambda, \mu, F, G)$ is an example of an *ε-quadratic formation*.

[15] See Lemma 14.14.

Definition 14.15. Let (K, λ, μ) be an ε-quadratic form on a free $\mathbb{Z}[\pi]$-module K. Given a submodule L of K, the *orthogonal module* $L^\perp$ to L consists of all elements $y \in K$ orthogonal to all elements $x \in L$, i.e., $\lambda(x, y) = 0$. We say that a submodule L in K is *lagrangian* if it is a free direct summand, and $L = L^\perp$, and $\mu|L = 0$. An *ε-quadratic formation* (K, λ, μ, F, G) over $\mathbb{Z}[\pi]$ consists of a non-singular quadratic form (K, λ, μ) as well as an ordered pair of lagrangians F and G.

Exercise 14.16. Show that the set $\mathbf{K}(f)$ is indeed an ε-quadratic formation.

Proof. The submodule F is clearly a lagrangian. Let us show that $G = K_{q+1}(M_0, \partial M_0)$ is also a lagrangian in $K = K_q(\partial M_0)$. An element $x \in G$, corresponds to an element ∂x in K; the latter is represented by an immersed framed sphere in $M_0 = S^q \times S^q$. In view of the short exact sequence of the pair $(M_0, \partial M_0)$, the sphere ∂x represents a trivial element in $K_q M_0 = \pi_{q+1}(f|M_0)$. Therefore, it bounds an immersed disc D_x of dimension $q+1$ in M_0. Given two elements x, x' in G, we may assume that the discs D_x and $D_{x'}$ associated with x and x' are intersected transversally along arcs and circles. As in Lemma 6.1, we deduce that $\lambda(\partial x, \partial x') = 0$. To prove that $\mu(x) = 0$ for every $x \in G$, we consider the arcs of the self-intersection points of D_x, instead of the arcs in $D_x \cap D_{x'}$.

It remains to show that $G = G^\perp$. Suppose that there is an element $x \in G$ such that $\lambda(\partial x, \partial x') = 0$ for all $x' \in G$. By the Poincaré duality isomorphism the module $K_q M_0$ is isomorphic to $K^{q+1}(M_0, \partial M_0)$, which in its turn is dual to $K_{q+1}(M_0, \partial M_0)$ by Lemma 14.14. Thus, there is an element $y \in K_{q+1}(M_0, \partial M_0)$ such that for $x^* = [M_0] \cap x$ we have $x^*(y) \neq 0$. Thus $\lambda(\partial x, \partial y) \neq 0$ contrary to the assumption that $\lambda(\partial x, \partial y) = 0$ for all $y \in G$. ☐

Exercise 14.17. Let (K, λ, μ) be an ε-quadratic form on a free finitely generated $\mathbb{Z}[\pi]$-module K. Show that the inclusion of any lagrangian $L \to K$ extends to an isomorphism of ε-quadratic forms $H_\varepsilon(L) \approx (K, \lambda, \mu)$. In particular, only hamiltonian forms admit lagrangians.

Proof. Let $p_1, ..., p_k$ be a basis for L. Since the form λ is non-degenerate, we may find elements $y_1, ..., y_k$ such that $\lambda(y_i, p_j) = \delta_{ij}$. We next use the Gramm-Schmidt process to replace $\{y_i\}$ with an orthonormal tuple

$\{x_i\}$. Namely, define $x_i = y_i - \varepsilon \sum p_j g_{ij}$, where the coefficients g_{ij} are such that

$$0 = \lambda(x_i, x_j) = \lambda_{ij} - \varepsilon\lambda(y_i, p_i g_{ji}) - \varepsilon\lambda(p_j g_{ij}, y_j) = \lambda_{ij} - \varepsilon g_{ji} - \bar{g}_{ij},$$

and

$$0 = \mu(x_i) = \mu(y_i - \varepsilon \sum p_j g_{ij}) = \mu(y_i) - \varepsilon g_{ii}.$$

Thus, we may put $g_{ii} = \varepsilon\mu(y_i)$, and $g_{ij} = 0$ for $i > j$ and $g_{ij} = \varepsilon\bar{\lambda}_{ij}$ for $i < j$.

It remains to show that $\{p_i, x_i\}$ is a hyperbolic basis for K. To begin with, the vectors $\{p_i, x_i\}$ are linearly independent.[16] Thus, the vectors $\{p_i, x_i\}$ form a basis for a free submodule K' of K. Suppose there is a vector $z \in K$ which is not in K'. Then the vector $z' = z - \varepsilon \sum x_i \lambda(p_i, z)$ is also in $K \setminus K'$. On the other hand, clearly $z' \in L^{\perp}$,[17] which implies that z' is an element in $L \subset K'$, contrary to the assumption. Thus $\{p_i, x_i\}$ is indeed a basis for K. □

[16] If $\sum p_i g_i + \sum x_i h_i = 0$, taking the λ-product of the left and right hand sides with p_j and x_j gives $h_j = g_j = 0$.

[17] For any basis element p_j in L we have

$$\lambda(p_j, z') = \lambda(p_j, z) - \varepsilon\lambda(p_j, x_j)\lambda(p_j, z),$$

which is zero since $\lambda(p_j, x_j) = \varepsilon$.

Corollary 14.18. *The quadratic form $(K, \lambda, \mu) \oplus (K, -\lambda, -\mu)$ is hyperbolic.*

Proof. Indeed, such a form admits a diagonal lagrangian $\{(x, x)\}$. □

Since $K_q U = K/F$,[18] the exact sequence of $\mathbb{Z}[\pi]$-modules

$$0 \to K_{q+1}M \to K_{q+1}(M, U) \xrightarrow{\partial_U} K_q U \to K_q M \to 0$$

implies[19] that $K_q M = K/(F + G)$ and $K_{q+1}M = F \cap G$. For example, when G is generated by the longitudes ℓ_i, we deduce that $K_q M = 0$, and therefore f is a homotopy equivalence.[20] More generally, an ε-quadratic formation (K, λ, μ, F, G) is *trivial* if $\mathbf{K} = (K, \lambda, \mu)$ is a hyperbolic ε-quadratic formation $(\mathbf{H}_\varepsilon(F), F, F^*)$.

[18] Indeed, $K = \Lambda\langle m_i, \ell_i\rangle$, $F = \Lambda\langle m_i\rangle$ and $K_q U = \Lambda\langle \ell_i\rangle$.

[19] Indeed, $K_{q+1}(M, U) = K_{q+1}(M_0, \partial M_0) \approx G$.

[20] **Ranicki explicitly requires here that $m \geq 5$.**

Given a normal degree one map $f\colon M \to X$, there are many choices of generators S_α of $K_q M$ which may lead to non-isomorphic formations. We say that formations $\mathbf{K}$ and $\mathbf{K}'$ are *stably isomorphic* if $\mathbf{K} \oplus \mathbf{T} = \mathbf{K}' \oplus \mathbf{T}'$ for some trivial formations $\mathbf{T}$ and $\mathbf{T}'$.

Exercise 14.19. Show that Heegaard splittings of f associated with different choices of generators S_α lead to stably isomorphic formations.

Our next step is to determine how a stable isomorphism class of formations of f changes under normal cobordism of f. Let (K, λ, μ) be a possibly singular $(-\varepsilon)$-quadratic form. Then the graph Γ_λ of the function $\lambda\colon K \to K^*$ is a submodule of $K \oplus K^*$ which consists of elements of the form $(x, \lambda(x))$. We say that the formation $(\mathbf{H}_\varepsilon(K), K, \Gamma_\lambda)$ is the boundary formation $\partial(K, \lambda, \mu)$.[21] We say that formations $\mathbf{K}$ and $\mathbf{K}'$ are *cobordant* if $\mathbf{K} \oplus \mathbf{B} = \mathbf{K}' \oplus \mathbf{B}'$ for some boundary formations $\mathbf{B}$ and $\mathbf{B}'$. It turns out that the cobordism classes of formations forms a group under the operation given by taking the direct sum of formations.

[21] It can be shown that a ε-quadratic formation $\mathbf{K}$ with lagrangians F and G is a *boundary* if and only if there is a lagrangian L in K which is a direct complement to both the lagrangians F and G.

<u>**Exercise**</u> **14.20.** Show that cobordism is an equivalence relation, and $\mathbf{K} \oplus \mathbf{K}'$ is cobordant to zero, where $\mathbf{K} = (K, \lambda, \mu, F, G)$ and $\mathbf{K}' = (K, -\lambda, -\mu, F, G)$.

Definition 14.21. The group $L_{2q+1}(\mathbb{Z}[\pi])$ is defined to be the cobordism group of ε-quadratic formations, where $\varepsilon = (-1)^q$.

We will show that for a normal, degree one, q-connected map $f\colon M \to X$ of manifold of dimension $2q+1$, the formation represents an element in L_{2q+1} which is trivial if and only if f is normally cobordant to a homotopy equivalence.

Theorem 14.22. *Let $f\colon W \to X \times [0,1]$ be a normal $(q+1)$-connected bordism between maps f_0 and f_1 of degree 1 of manifolds W_0 and W_1 of dimension $2q+1$. Then the formations $\mathbf{K}(f_0)$ and $\mathbf{K}(f_1)$ are cobordant.*

To prove Theorem 14.22 we will associate with the cobordism f an ε-quadratic form $\mathbf{K}(f)$ and show that $\mathbf{K}(f_0) \oplus (-\mathbf{K}(f_1)) = \partial\mathbf{K}(f)$. The ε-quadratic form $\mathbf{K}(f)$ is the kernel $(-\varepsilon)$-quadratic form $(K_{q+1}W, \lambda_W, \mu_W)$ of the normal degree one map $(W, \partial W) \to (X \times [0,1], X \times \partial[0,1])$. Note that the intersection form λ_W is defined by

$$K_{q+1}W \to K_{q+1}(W, \partial W) \to K^{q+1}W = (K_{q+1}W)^*.$$

Explicitly, the form $\partial\mathbf{K}(f)$ is the formation on a hyperbolic quadratic ε-form $\mathbf{H}_\varepsilon(F_W)$ with lagrangians $F_W = K_{q+1}W$, and $\Gamma = \{(x, \lambda_W(x))\}$.

Proof of Theorem **??**. It suffices to construct an isomorphism

$$(\mathbf{H}_\varepsilon(F_W), F_W, \Gamma) \to (\mathbf{H}_\varepsilon(F_0), F_0, G_0) \oplus (-\mathbf{H}_\varepsilon(F_1), F_1, G_1).$$

where the two summands on the left hand side are the formations $\mathbf{K}(f_0)$ and $\mathbf{K}(f_1)$. The map $F_W \to F_0 \oplus F_1$ is defined by

$$F_W = K_{q+1}W \to K_{q+1}(W, W_1) \oplus K_{q+1}(W, W_0) \approx F_0 \oplus F_1.$$

There is also a map $F_W^* \to F_0^* \oplus F_1^*$ given by

$$F_W^* = K^{q+1}W \to K^{q+1}(W, W_1) \oplus K^{q+1}(W, W_0) = F_0^* \oplus F_1^*.$$

These two maps define a map $\mathbf{H}_\varepsilon(F_W) \to \mathbf{H}_\varepsilon(F_0)$. By definition takes the lagrangian F_W to $F_0 \oplus F_1$. Next, let's find the image of the lagrangian, which is the graph of the map $\lambda_W \colon F_W \to F_W^*$.

□

Let $f \colon W \to X \times [0,1]$ be a normal $q+1$-connected cobordism of degree 1 between normal q-connected maps of degree 1 of manifolds W_0 of W_1 of dimension $m = 2q+1$. It follows that $K_{q+1}W$, and $K_{q+1}(W, W_i)$ are finitely generated free $\mathbb{Z}[\pi]$-modules. The normal cobordism f defines an ε-quadratic formation $(\mathbf{K}, F_W, G_W)$ on $f|W_0$ by

$$F_W = K_{q+1}(W, W_1), \quad G_W = K_{q+1}W, \quad \mathbf{K} = \mathbf{H}_\varepsilon(F_W),$$

where G is included into $K = F_W \oplus F_W^*$ by the sum of the map $G_W \to F_W$ induced by inclusion and the map $G_W \to F_W^*$ given by the composition

$$G_W \to K_{q+1}(W, W_0) \approx K^{q+1}(W, W_1) \approx K_{q+1}(W, W_1)^*.$$

Exercise 14.23. Show that when W is obtained by surgeries on M along the generators of K_qM in the beginning of the section, the resulting formation is the same as the formation in the beginning of the section.[22]

[22] We recall that W is obtained from $W_0 \times [0, \varepsilon]$ by attaching handles $D^{q+1} \times D^{q+1}$ along $S^q \times D^{q+1}$. Alternatively, we may say that W is obtained from $W_1 \times [1-\varepsilon, 1]$ by attaching handles $D^{q+1} \times D^{q+1}$ along $D^{q+1} \times S^q$. In particular, $K_{q+1}(W, W_1) = K_{q+1}(D^{q+1} \times D^{q+1}, D^{q+1} \times S^q)$. Thus, indeed, we can identify F with F_W. On the other hand, the lagrangian $G_W = K_{q+1}W$ is

$$K_{q+1}(W, D^{q+1} \times D^{q+1}) =$$
$$= K_{q+1}(M_0 \times [0,1], \partial M_0 \times [0,1]) = G.$$

Lemma 14.24. *A kernel formation of a degree 1, n-connected, normal map $f \colon M \to X$ is a boundary iff f is bordant to a homotopy equivalence.*

Proof. Suppose that $W \to X \times [0,1]$ is a normal bordism of f to a homotopy equivalence, we can make it $(q+1)$-connected by surgery below the middle range. It defines a formation with $G = K_{q+1}W \approx K_{q+1}(W, W_1) = F$, and the inclusion of $G \to F \oplus F^*$ given by the isomorphism $G \to F$ induced by the inclusion, and the composition

$$G \to K_{q+1}(W, M) = (K_{q+1}W)^*.$$

Consider the lagrangian $L = F^* \subset K$. Clearly, $F \oplus F^* = K$. We also claim that $G \oplus F^* = K$, which is obvious. Thus, the formation associated with f is a boundary.

On the other hand if a kernel formation of f is a boundary, then it is associated with a cobordism W of Exercise 14.23 to a homotopy equivalence. □

14.4 Further reading

According to S. P. Novikov, the odd dimensional groups $L_{2q+1}(\pi)$ were introduced by Wall in [Wall70] in 1968. The even dimensional groups $L_{2q}(\pi)$ has been introduced earlier by Novikov [No66], [No66a] and Wall [Wall66]. The original definition of groups L_* is related to the original definition of groups K_*.

For a ring A with the identity element, the group K_0A is the Grothendieck group of the monoid of isomorphism classes of finitely generated projective A-modules. The group K_1A is the abelianization of the general linear group $GL(A)/[GL(A), GL(A)]$. The definition of the groups L_{2q} is similar to the definition of the groups K_0, while the Wall's definition of L_{2q+1} is related to the definition of K_1.

14.5 Solutions to Exercises

Solution to Exercise 14.3. Let M be a closed manifold of dimension m, with fundamental class $[M]$. Then there is a well-defined pairing in cohomology $H^q\tilde{M} \otimes H^q\tilde{M} \longrightarrow \mathbb{Z}[\pi]$ which takes a pair (x, y) to $x([M] \cap y)$. The adjoint to the cohomology pairing is the composition $H^q\tilde{M} \approx H_q\tilde{M} \to (H^q\tilde{M})^*$ of the Poincaré duality and the map that takes z to the homomorphism $x \mapsto x(z)$. The homology intersection pairing

$$H_q\tilde{M} \otimes H_q\tilde{M} \longrightarrow \mathbb{Z}[\pi]$$

is defined by taking (u, v) to $U(v)$ where U is Poincaré dual to u.

Suppose now that $f : M \to X$ is a q-connected degree one map. The intersection form on K^qM is defined by restricting the intersection form on $H^q\tilde{M} = H^q\tilde{X} \oplus K^qM$. The adjoint to the pairing is the composition

$$K^qM \to H^q\tilde{X} \oplus K^qM \approx H_q\tilde{X} \oplus K_qM \to K_qM \to (K^qM)^*.$$

of the inclusion, the Poincaré duality, the projection, and the the homomorphism that takes x to the homomorphism $y \mapsto y(x)$. The composition of the first three homomorphisms is the Poincaré duality isomorphism, while the last map is an isomorphism by Lemma 13.14. Thus, the intersection pairing on K^qM (or, equivalently, the intersection pairing on K_qM) is non-degenerate. □

Solution to Exercise 14.5. To begin with let us prove the property:

$$\lambda(S_1, S_2) = (-1)^q \overline{\lambda(S_2, S_1)},$$

For $i = 1, 2$, let w_i denote the thread of the i-th sphere S_i, i.e., w_i is a chosen path from the base point in M to the basepoint in S_i. Given an intersection point x_j in $S_1 \cap S_2$, let u_{1j} and u_{2j} denote unique up to homotopy paths from the basepoints of S_1 and S_2 to x_j. Then

$$\lambda(S_1, S_2) = \sum \varepsilon_{S_1,S_2}(x_j) w_2 u_{2j} u_{1j}^{-1} w_1^{-1} = \sum \varepsilon_{S_1,S_2} g,$$

$$\lambda(S_2, S_1) = \sum \varepsilon_{S_2,S_1}(x_j) w_1 u_{1j} u_{2j}^{-1} w_2^{-1} = \sum \varepsilon_{S_2,S_1} g^{-1}.$$

The sign $\varepsilon_{S_1,S_2}(x_j)$ is positive if and only if the orientation of M given by the orientation of S_1 followed by the orientation of S_2 agrees with the orientation of $T_{x_j}M$ obtained from the orientation of the plane T_pM at a distinguished point $p \in M$ by transporting it along $u_{1j}w_1$. The sign $\varepsilon_{S_2,S_1}(x_j)$ is defined similarly, except now the orientation of S_2 is followed by the orientation of S_1, and the orientation on T_pM is transported along $u_{2j}w_2$. Thus, $\varepsilon_{S_2,S_1}(x_j)$ differs from $\varepsilon_{S_1,S_2}(x_j)$ by the factor $(-1)^q w(g)$. Consequently,

$$\lambda(S_1, S_2) = \sum \varepsilon_{S_1,S_2}(x_j) g = \sum (-1)^q \varepsilon_{S_2,S_1}(x_j) w(g) g = (-1)^q \overline{\lambda(S_2, S_1)},$$

where $\bar{g} = w(g)g^{-1}$.

Next, let us turn to the distributivity property:

$$\lambda(S, S_1 g_1 + S_2 g_2) = \lambda(S, S_1) g_1 + \lambda(S, S_2) g_2.$$

We clearly have the property $\lambda(S, S_1 + S_2) = \lambda(S, S_1) + \lambda(S, S_2)$, and therefore, it suffices to show that $\lambda(S_1, S_2 h) = \lambda(S_1, S_2) h$. We have

$$\lambda(S_1, S_2 h) = \sum \varepsilon_{S_1,S_2 h}(x_j) h^{-1} w_2 u_{2j} u_{1j}^{-1} w_1^{-1} = \sum \varepsilon_{S_1,S_2 h} h^{-1} g,$$

$$\lambda(S_1, S_2) h = \sum \varepsilon_{S_1,S_2}(x_j) h^{-1} w_2 u_{2j} u_{1j}^{-1} w_1^{-1} = \sum \varepsilon_{S_1,S_2} h^{-1} g.$$

It remains to observe that $\varepsilon_{S_1,S_2 h} = \varepsilon_{S_1,S_2}$, since the path $u_{1j}w_1$ along which the orientation from T_pM to $T_{x_j}M$ is transported is the same both for calculation of the sign $\varepsilon_{S_1,S_2 h}$, and the sign ε_{S_1,S_2}.

The proof of the following property is similar to one in the simply connected case:

$$\mu(S_1 + S_2) = \mu(S_1) + \mu(S_2) + \lambda(S_1, S_2),$$

Indeed, since λ is a well-defined map on $K \times K$, we may assume that S_1 and S_2 are Wall representatives of their homotopy classes. Then the self-intersection points contributing to $\mu(S_1 + S_2)$ are the self-intersections points contributing to $\mu(S_1)$ and $\mu(S_2)$ as well as the intersection points between S_1 and S_2.

The next property follows immediately from the definitions:

$$\mu(Sg) = \sum \varepsilon_{Sg}(x_j) g^{-1} w u_{2j} u_{1j}^{-1} w^{-1} g = \bar{g}\mu(S)g,$$

since $\varepsilon_{Sg} = \varepsilon_S \cdot w(g)$. Finally, let us show that

$$\lambda(S,S) = \mu(S) + (-1)^q \overline{\mu}(S) = 2\mu(S).$$

The latter equality is immediate since in Λ_ε we have $g = (-1)^q \bar{g}$. Next, since λ is a well-defined map, we may assume that S is a Wall representative of its homotopy class. In particular, its normal bundle in M is trivial. Therefore, each self-intersection of S contributes two intersection points between S and its close transverse copy S'. However, the "signs" of the two intersection points between S and S' are the same as one of the self-intersection point of S, which implies that $\lambda(S,S) = 2\mu(S)$. □

Solution to Exercise **??**. Consider the diagram of short exact sequences:

$$\begin{array}{ccccc} C^*(D^m, S^{m-1}) & \longrightarrow & C^*X & \longrightarrow & C^*X_0 \\ \downarrow & & \downarrow & & \downarrow \\ C_*D^m & \longrightarrow & C_*X & \longrightarrow & C_*(X_0, S^{m-1}), \end{array}$$

where vertical maps are given by taking the cap product with $[D^m]$, $[X]$ and the class $[X_0]$ defined from $[X]$ by the composition

$$H_m X \to H_m(X, D^m) \approx H_m(X_0, S^{m-1}).$$

where the homology groups are with coefficients in $\mathbb{Z}_\omega$. Similarly consider the diagram

$$\begin{array}{ccccc} C^*(X_0, S^{m-1}) & \longrightarrow & C^*X & \longrightarrow & C^*D^m \\ \downarrow & & \downarrow & & \downarrow \\ C_*X_0 & \longrightarrow & C_*X & \longrightarrow & C_*(D^m, S^{m-1}), \end{array}$$

□

99

Additional Topics

99.1 Topics for Chapter 1

99.1.1 Hopf fibration

There is an important map $S^3 \to S^2$ that even has its own name. It is called the *Hopf fibration*. In this section we will apply the Pontryagin construction to find the framed manifold M corresponding to the Hopf fibration. This explicit example may help the reader better understand the nuances in the general Pontryagin construction. Otherwise, the reader may skip this subsection as we will not need its results elsewhere.

The Hopf fibration is best defined in terms of complex numbers. To begin with, the sphere S^3 can be identified with the subset of $\mathbb{C} \times \mathbb{C}$ of complex vectors (u, v) of length 1. If we identify every point (u, v) of S^3 with any other point of the form $(\lambda u, \lambda v)$, where $\lambda \in \mathbb{C} \setminus \{0\}$, then we obtain the so-called complex projective line $\mathbb{C}P^1$ which is also a sphere S^2. We will denote the equivalence class of a point (u, v) in $\mathbb{C}P^1$ by $[u : v]$, and identify the point $[1 : 0]$ with the north pole of S^2.

By definition, the Hopf fibration f is a map $S^3 \to S^2$ defined by the projection $(u, v) \mapsto [u : v]$.[1]

[1] It is remarkable that the map f is not only smooth, but its differential is everywhere surjective. A map between closed manifolds with such a property is a *fiber bundle*, and, in particular, a *fibration*.

The Pontryagin theorem identifies the homotopy class in $\pi_3 S^2$ of the Hopf fibration with the cobordism class of a framed manifold M of dimension 1 in $\mathbb{R}^3$, where the manifold M is the inverse image of the north pole $[1 : 0]$, i.e., the manifold M is the circle $\{(e^{i\theta}, 0)\}$ in $\mathbb{C} \times \mathbb{C}$. To find the frame of the manifold M, let's choose a basis $\{e_1, e_2\}$ of the tangent space of S^2 at the north pole. For example, we may choose e_1 and e_2 to be the velocity vectors of the curves $[1 : t]$ and $[1 : it]$ respectively.

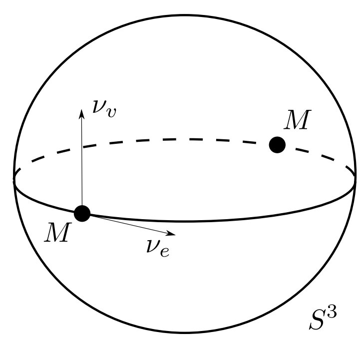


Figure 99.1: The sphere S^3 lies in $\mathbb{C}^2 = \mathbb{R}^3 \times \mathbb{R}$ with $\mathbb{R}^3$ being horizontal and $\mathbb{R}$ vertical. The framed manifold M is an equatorial circle S^1 in S in the horizontal plane spanned by e_1 and e_2. The normal equatorial direction ν_e is the remaining horizontal direction e_3, while the normal vertical direction ν_v is the vertical direction e_4.

For each point $x = (e^{i\theta}, 0)$ on the manifold M, we may use the differential $d_x f$ to pull the vectors e_1 and e_2 back to a basis $\{\tau_1, \tau_2\}$ of the perpendicular space $T_x^{\perp} M$. The easiest way to do that is by lifting the curves $[1 : t]$ and $[1 : it]$ to S^3. For example, the curve

$$r(t)(e^{i\theta}, te^{i\theta}), \qquad \text{where} \qquad r = (1 + t^2)^{-1/2},$$

is a lift of the curve $[1 : t]$, i.e., it maps by the Hopf fibration f precisely to the curve $[1 : t]$. Consequently, its velocity vector $(0, e^{i\theta})$ at x projects by $d_x f$ to the vector e_1. Clearly, the vector $(0, e^{i\theta})$ is orthogonal to the manifold M, and therefore it is the desired vector τ_1. Similarly, we find that τ_2 is the vector $(0, ie^{i\theta})$.

In particular, the perpendicular space $T_\theta^\perp M$ of M in S^3 is spanned by the perpendicular vectors e_3 and e_4, and, for example, the frame vector field τ_1 is written as $\tau_1(\theta) = \cos(\theta)e_3 + \sin(\theta)e_4$.

Finally, let us use the stereographic projection, see Figure 99.2, to identify $S^3 \setminus \{\infty\}$ with $\mathbb{R}^3$. To begin with, the stereographic projection takes the equatorial sphere S^2 of S^3 into the unit sphere S^2 in $\mathbb{R}^3$. Since M is an equatorial circle, it remains an equatorial circle

$$\{x_1^2 + x_2^2 = 1, \quad x_3 = 0\} \subset \mathbb{R}^3,$$

where $(u, v) = (x_1 + ix_2, x_3 + ix_4)$, and the equatorial normal direction e_3 stays put. On the other hand, up to scaling, the normal direction e_4 at $x \in M$ transforms by the stereographic projection into the vector $-x$. Thus, the framed manifold $M \subset \mathbb{R}^3$ of the Hopf fibration is the standard circle in $\mathbb{R}^3$ equipped with frame vector fields $\tau_1(x)$ and $\tau_2(x)$ that twist around the circle once as x traverses the circle, see Figure 99.3.

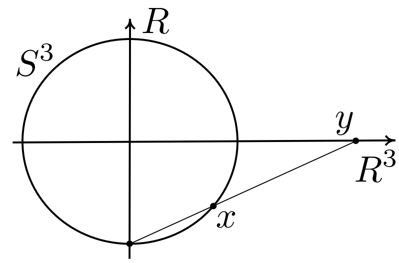


Figure 99.2: The stereographic projection: a point x in S^3 that is not the south pole corresponds to the point y in $\mathbb{R}^3$ that lies on the same line as the south pole of S^3 and x.

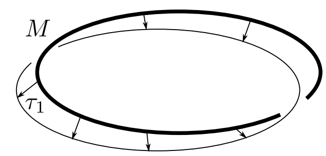


Figure 99.3: The end points of the vector field τ_1 form a closed curve that winds around the circle M. The linking number between the curve of end points of τ_1 and M is -1.

99.1.2 *Stable homotopy group π_1^S*

In this section we will sketch an argument that the unstable homotopy group $\pi_3 S^2$ is isomorphic to $\mathbb{Z}$, while the stable homotopy group π_1^S is isormorphic to $\mathbb{Z}_2$. Later we will establish these isomorphisms by a different argument.

The unstable group $\pi_3 S^2$.

We will only sketch here a calculation of $\pi_3 S^2$. A detailed proof will be given in section 3.4.

Let us identify S^2 with $\mathbb{R}^2 \cup \{\infty\}$ so that 0 is the north pole and $\{\infty\}$ is the south pole of S^2. Let $f : S^3 \to S^2$ be a map representing an element in $\pi_3(S^2)$. The Pontryagin construction identifies the homotopy class of f with the cobordism class of a framed manifold $M = f^{-1}(0)$ of dimension 1 in $\mathbb{R}^3$.

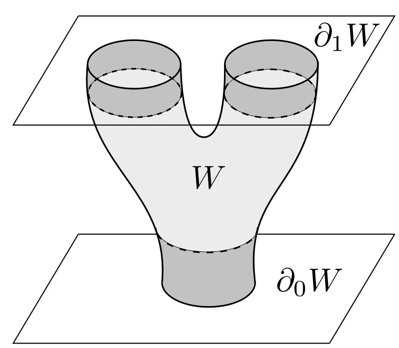


Figure 99.4: A pair of pants cobordism splits a circle into two circles. The inverse cobordism merges two circles into one.

By a framed cobordism (see Figure 99.1.2) the manifold M can be modified into the standard unit circle in $\mathbb{R}^2 \subset \mathbb{R}^3$ with a framing τ. As a point x runs over the standard unit circle, the framing $\tau(x)$ makes w full twists for some integer w. It turns out that the correspondence $[f] \mapsto w$ is a well-defined isomorphism $\pi_3 S^2 \approx \mathbb{Z}$. In particular, the Hopf fibration is a generator of $\pi_3 S^2$. [2]

The stable group π_1^S (Sketch of the Pontryagin calculation). Next, let us turn to investigating the stable homotopy group π_1^S. By the Freudenthal theorem, the group π_1^S can be identified with the group $\pi_4 S^3$. In view of the Pontryagin construction, its elements are represented by framed closed curves M in S^3. Furthermore, by the Freudenthal theorem, the group $\pi_4 S^3$ is a surjective image of the cyclic group $\pi_3 S^2 \approx \mathbb{Z}$, and therefore, the group $\pi_4 S^3$ is generated by the Freudenthal suspension of the Hopf element $[h]$. It turns out that the generator $[h]$ of π_1^S is non-trivial and satisfies $2[h] = 0$. This implies that $\pi_1^S \approx \mathbb{Z}_2$, see section 3.4. [3]

[2] It is worth mentioning that there is a short homotopy theoretic calculation of $\pi_3 S^2$ by means of the homotopy long exact sequence of the Hopf fibration:

$$\cdots \to 0 \to \pi_3 S^3 \to \pi_3 S^2 \to 0 \to \cdots .$$

Note that the middle homomorphism is defined by taking an element of $[f] \in \pi_3 S^3$ to the homotopy class of the composition $h \circ f$ of f with the Hopf fibration h. In particular, the generator [id] of the group $\pi_3 S^3$ that we have computed maps to the generator $h \circ \mathrm{id} = h$ of the group $\pi_3 S^2$. In other words, the group $\pi_3 S^2$ is generated by the class $[h]$ of the Hopf fibration h.

99.1.3 The Pontryagin-Thom construction

After the appearance of the Pontryagin construction, Rene Thom realized that its generalization can be used to compute the cobordism groups $\mathfrak{N}(m,k)$ of manifolds in $\mathbb{R}^{m+k}$. The generalization proposed by Thom is now known as the Pontryagin-Thom construction. We will sketch it here.

As a set the group $\mathfrak{N}(m,k)$ consists of the cobordism classes of manifolds of dimension m in $\mathbb{R}^{m+k}$. The sum of classes $[M_0]$ and $[M_1]$ represented by manifolds M_0 and M_1 is defined by first translating M_0 so that M_0 is disjoint from M_1, and then taking the cobordism class $[M_0 \sqcup M_1]$ of the disjoint union of manifolds. As in the case of the cobordism group of framed manifolds, it can be shown that $\mathfrak{N}(m,k)$ is an abelian group. Its zero is the cobordism class of an empty manifold. The inverse of an element $[M]$ is the element $[-M]$ constructed as in the case of the cobordism group of framed manifolds, except that now the manifolds under consideration are not framed. Note that the manifold $-M$ is a "mirror" reflection of M. In other words, it is diffeomorphic to M, but it is placed in $\mathbb{R}^{m+k}$ differently.[4]

[3] To show that π_1^S is non-trivial, Pontryagin identified the stable group π_1^S with the cobordism group of framed manifolds M of dimension 1 in $\mathbb{R}^k$ for $k > 3$, and introduced an invariant δ. Given a frame $(f_1(x), ..., f_{k-1}(x))$ at a point x of M, we may assume that it is orthonormal in view of the Gram-Schmidt process. There is a unique vector $f_k(x)$ at x such that $(f_1(x), ..., f_k(x))$ is an orthonormal basis for $\mathbb{R}^k$. Assigning to each point x an orthonormal basis, defines a map $\gamma\colon M \to SO(k)$. Pontryagin defined a homomorphism $\delta\colon \pi_1^S \to \mathbb{Z}_2$. When M is connected, the invariant δ equals zero when the curve γ is non-homotopic to the constant curve and 1 otherwise. When M consists of more than one component, the invariant δ is defined to be the sum of values of δ on components of M mod 2. It immediately follows that the value of the homomorphism δ on the framed manifold of the Hopf fibration is 1, which implies that the group π_1^S is non-trivial.

[4] We will discuss later the Whitney embedding theorem, which implies that if $k > m+1$, then M and $-M$ represent the same cobordism class, and therefore every element of $\mathfrak{N}(m,k)$ is of order 2 in this case.

For example, the group $\mathfrak{N}(0,k)$ for $k > 1$ is isomorphic to $\mathbb{Z}_2$. Indeed, let $\varphi\colon \mathfrak{N}(0,k) \to \mathbb{Z}_2$ be the homomorphism that associates with the class $[M]$ of a 0-dimensional manifold M, the parity of the number of points in M. If two manifolds M_0 and M_1 of dimension 0 are cobordant, then the number of points in M_0 has the same parity as the number of points in M_1. Thus, φ is well-defined. Furthermore, since two points bound a segment, φ is injective. Clearly, it is also surjective.

The group $\mathfrak{N}(1,k)$ for $k > 2$ is trivial since a closed manifold of dimension 1 is a finite union of circles bounding discs. The group $\mathfrak{N}(2,k)$

for $k > 3$ is isomorphic to $\mathbb{Z}_2$ with an isomorphism $\mathfrak{N}(2,k) \to \mathbb{Z}_2$ that evaluates the parity of the Euler characteristic of a manifold.[5] Higher order cobordism groups are hard to find at this point.

In order to calculate the group $\mathfrak{N}(m,k)$ in general, Rene Thom considered the topological space of pairs (L,v), where L is a vector subspace of $\mathbb{R}^{m+k}$ of dimension k and v a vector in L.[6] Let $MO(m,k)$ denote the one point compactification of this space.

[5] In general, the parity of the Euler characteristic of a manifold is an invariant of cobordism, i.e., if two manifolds M_0 and M_1 are cobordant, then the parities of their Euler characteristics are the same.

[6] The space $\mathrm{Gr}_m(\mathbb{R}^{m+k})$ of vector subspaces L of $\mathbb{R}^{m+k}$ of dimension m is called the *Grassmann* manifold. It is the quotient space $\mathrm{O}(m+k)/\,\mathrm{O}(m) \times \mathrm{O}(n)$ of the orthogonal group $\mathrm{O}(m+k)$. The topology on the space of pairs (L,v) is defined to be the coarsest topology for which the projections $(L,v) \to L$ to the Grassmann manifold and $(L,v) \to v$ to $\mathbb{R}^{m+k}$ are continuous.

Theorem 99.1 (Thom)**.** *There is an isomorphism of the cobordism group of manifolds of dimension m in $\mathbb{R}^{m+k}$ and the group $\pi_{m+k}MO(m,k)$.*

Sketch. To construct the isomorphism $\mathfrak{N}(m,k) \to \pi_{m+k}MO(m,k)$, let M be a manifold of dimension m in $\mathbb{R}^{m+k}$ representing an element in the cobordism group. Let U denote an ε-neighborhood of M; it consists of points $x + w$ where x ranges over points in M and w ranges over vectors of length $|w| < \varepsilon$ in $T_x^{\perp}M$. Define a map $f|U\colon U \to MO(m,k)$ by taking $x + w$ to $(L,v) = (T_x^{\perp}M, w/(\varepsilon - |w|))$, and extend it to a map of $S^{m+k} = \mathbb{R}^{m+k} \cup \{\infty\}$ by sending the complement to U onto the distinguished point of $MO(m,k)$. The map $f\colon S^{m+k} \to MO(m,k)$ represents an element in $\pi_{m+k}MO(m,k)$. Following the argument in the proof of the Pontryagin theorem, it can be shown that the correspondence $[M] \mapsto [f]$ is an isomorphism of groups. □

As in the case of framed manifolds, a manifold in $\mathbb{R}^{m+k}$ can be regarded as a manifold in $\mathbb{R}^{m+k+1} \supset \mathbb{R}^{m+k}$. This defines a homomorphism $\mathfrak{N}(m,k) \to \mathfrak{N}(m,k+1)$, which is an isomorphism for $k > m+1$. In particular, we may define a *cobordism group* $\mathfrak{N}^m$ of manifolds of dimension m as a group $\mathfrak{N}(m,k)$ for any $k > m+1$. By the Thom theorem, the group $\mathfrak{N}^m$ is isomorphic to $\pi_{m+k}MO(m,k)$ for any $k > m+1$.

99.1.4 The stable J-homomorphism

The Pontryagin construction identifies the stable homotopy group π_m^S of spheres with the cobordism group of framed manifolds M of dimension m. The simplest elements in the group π_m^S are those represented by a framed sphere $S^m \subset \mathbb{R}^{m+1} \subset \mathbb{R}^{m+k}$, where $k > m+1$.

There is a canonical frame of S^m given by the normal unit vector of S^m in $\mathbb{R}^{m+1}$ followed by the coordinate vectors $e_{m+2}, ..., e_{m+k}$ in $\mathbb{R}^{m+k}$. Let $v(x) = \{v_1(x), ..., v_k(x)\}$ be another frame of S^m. By using the

Gram-Schmidt orthogonalization process, we may assume that the frame $v(x)$ is orthonormal at each point x in S^m. Then, the frame v defines a continuous map $S^m \to \mathrm{SO}_k$ that associates with each point x the rotation of the canonical frame at x to the frame $v(x)$. In other words, there is a bijective correspondence between frames v and the elements of the homotopy group $\pi_m \mathrm{SO}_k$. [7]

The famous J-homomorphism is the map $J\colon \pi_m \mathrm{SO}_k \to \pi_{m+k}(S^k)$ that associates with each frame v of the sphere S^m the corresponding element in $\pi_{m+k} S^k$.

When $k > m+1$, the group $\pi_{m+k} S^k$ does not depend on the actual value of k, by the Freudenthal theorem. It is isomorphic to the stable homotopy group π_m^S. On the other hand, it is also well-known that in the dimensional range $k > m+2$ the group $\pi_m \mathrm{SO}_k$ is stable; it is denoted by $\pi_m \mathrm{SO}$.[8] Thus, there is a so-called *stable J-homomorphism*

$$J\colon \pi_m \mathrm{SO} \longrightarrow \pi_m^S.$$

The calculation of the J-homomorphism is a hard theorem established by Adams and Quillen. We will state their theorem without a proof. To begin with, by the so-called Bott periodicity, the group $\pi_m \mathrm{SO}$ for $m > 0$ only depends on the value of m mod 8. More precisely, when the value of m mod 8 is $0, 1, ..., 7$, the group π_m is isomorphic respectively to $\mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}$. By the Adams-Quillen theorem, when $m \neq 4q-1$, the J-homomorphism is injective provided that $m > 1$.[9] Suppose now that $m = 4q-1$. This is precisely the case when $\pi_m \mathrm{SO} \simeq \mathbb{Z}$. Then the image of the J-homomorphism is a finite subgroup of π_m^S of order given by the denominator of $B_q/4q$ where B_q is the q-th Bernoulli number.[10] For example,

$$B_1 = \frac{1}{6},\ B_2 = \frac{1}{30},\ B_3 = \frac{1}{42},\ B_4 = \frac{1}{30},\ B_5 = \frac{5}{66},\ B_6 = \frac{691}{2730}.$$

In general, the Bernoulli numbers are defined by the formula

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} - \sum_{q \geq 1} \frac{(-1)^{q+1} B_q}{(2q)!} z^{2q},$$

or, equivalently,[11] by

$$\frac{x}{\mathrm{th} x} = 1 + \frac{B_1}{2!}(2x)^2 - \frac{B_2}{4!}(2x)^4 + \frac{B_3}{6!}(2x)^6 - \cdots. \tag{99.1}$$

We will see later that the J-homomorphism is related to exotic smooth structures on spheres.

[7] Strictly speaking, in the definition of homotopy groups $\pi_m X$ of a space X we need to consider only based maps. However, since SO_k is an H-space, free homotopy classes of maps $S^m \to \mathrm{SO}_k$ are in bijective correspondence with based homotopy classes of based maps $S^m \to \mathrm{SO}_k$.

[8] Consider a map $\mathrm{SO}_k \to S^{k-1}$ that associates with a rotation of $\mathbb{R}^k$ the image of the first coordinate vector e_1. The long exact sequence of the fibration implies that $\pi_m \mathrm{SO}_{k-1} \approx \pi_m \mathrm{SO}_k$ provided that $k > m+2$.

[9] Note that in this case $\pi_m \mathrm{SO}$ is either trivial or $\mathbb{Z}_2$. Therefore, in this case the image of the J-homomorphism is either 0 or $\mathbb{Z}_2$.

[10] Of course, if the image of $J\colon \mathbb{Z} \to \pi_m^S$ is a finite group $\mathbb{Z}_d$ of order d, then the kernel of J is an infinite subgroup of $\mathbb{Z}$ generated by $d \in \mathbb{Z}$.

[11] To pass from one series to another one, observe that

$$\frac{x}{\mathrm{th} x} = \frac{2x}{e^{2x} - 1} + x.$$

99.2 Topics for Chapter 2

99.2.1 The Multicompression theorem

The Global Compression Theorem admits a number of improvements, which we will discuss in this section.

Relative Compression Theorem

Let Y be a closed subset in $\mathbb{R}^{m+k-1}$, and X the part of the manifold M in the cylinder $Y \times \mathbb{R}$.

Theorem 99.2 (Relative compression theorem)**.** *Let M be a compact manifold of dimension m in $\mathbb{R}^{m+k}$ with $k > 1$. Suppose that a unit perpendicular vector field v over M is already vertical up over X. Then an isotopy and deformation of v which straighten v up can be chosen so that the points of X are not displaced.*

Proof. As in the proof of the Compression Theorem, we may assume that the vector field v is nowhere vertical down over the manifold M, and rotate v so that it is at least a bit vertical up. Under the rotation the vector field v over X stays vertical up. Next we extend v over $Y \times \mathbb{R}$ by a vertical up vector field. We may also scale the vector field v if necessary so that at each point x of its domain the vector $v(x)$ is of unit length. Then the vector field v is a function $M \cup (Y \times \mathbb{R}) \to S^{m+k-1}$ which associates with each point x the vector $v(x)$. By construction, the image of the function v is in the upper hemisphere. Therefore the function v can be extended over $\mathbb{R}^{m+k}$ so that outside a neighborhood of M the vector field v is vertical up. Let F_t be the isotopy along v. Then the desired *relative isotopy* G_t is one[12] defined by $x \mapsto F_t(x) - te_{m+k}$. □

[12] For example, suppose that the vector field v is everywhere vertical up. Then at time t the isotopy F_t lifts everything up by t units. On the other hand, the isotopy G_t not only lifts everything up by t units (as it has the component $F_t(x)$) but it also translates everything down by t units (as it has the component $-te_{m+k}$), i.e., G_t keeps everything still.

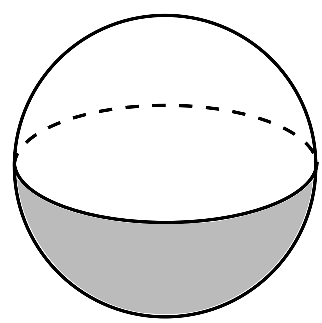


Figure 99.5: For the standard embedded sphere $S^2 \subset \mathbb{R}^3$, the horizontal set is the horizontal big circle, while the downset consists of the lower half hemisphere.

The Local Compression Theorem

We say that an isotopy of a manifold M is an ε-isotopy if it displaces each point $x \in M$ by a distance no more than ε.

Theorem 99.3. *[Local Compression Theorem] Given $\varepsilon > 0$, any unit perpendicular vector field v over a closed manifold M of dimension m in $\mathbb{R}^{m+k}$, with $k > 1$, can be made vertical up by an ε-isotopy of M and a deformation of the vector field v through normal vector fields.*

Recall that we say that in $\mathbb{R}^{m+k}$ the plane $\mathbb{R}^{m+k-1} \times \{0\}$ is horizontal, while the direction $\{0\} \times \mathbb{R}$ is vertical. The *horizontal* set H is a set of points x in M at which the perpendicular space $T_x^\perp M$ is horizontal. At each point x in M not in the horizontal set, the space $T_x^\perp M$ is slanted, and therefore there is a downmost directed unit vector $\psi(x)$. By definition, the *downset* $D = D(v)$ consists of points x at which the vector $v(x)$ is downmost, i.e., it coincides with $\psi(x)$, see Figure 99.5.

Proof of the Local Compression Theorem. Suppose that the downset D is empty. Then each vector $v(x)$ can be canonically deformed along the great circle passing through $v(x)$ and $\psi(x)$ to the upmost vector $-\psi(x)$. On the other hand, the upmost vector field can be canonically deformed along great circles to the vertical up vector field. Thus, in this case the compression theorem can be proven even without using an ε-isotopy.

When the downset D is not empty, this argument still allows us to assume that v is vertical up over the complement in M to a neighborhood of $\bar{D}$, e.g., see Figures 99.6 and 99.7. It turns out that if we construct now the relative isotopy G_t of the global compression theorem, then by appropriately choosing parameters in the construction of the isotopy, we may guarantee that the relative homotopy G_t is ε-small.[13] We will give more detail in §99.2.5. □

[13] Note that the vector field v is not vertical up only in a small neighborhood of the downset. We will show in §99.2.5 that we may assume that the downset D and the horizontal set H are manifolds such that $\partial\bar{D}$ coincides with H. It will follow that D (as well as H) is relatively small, and therefore even though v is not vertical up near D, the relative isotopy G_t is small.

The Multicompression Theorem

For applications, the multicompression theorem is most useful.

Theorem 99.4. *Suppose that the manifold M of dimension m in $\mathbb{R}^{m+k}$ is equipped with $n < k$ linearly independent perpendicular vector fields $v_1, ..., v_n$. Then there is an ambient isotopy F_t that straightens the vectors $v_1, ..., v_n$ up in the sense that $F_0 = \mathrm{id}$ while $dF_1(v_i) = e_i$ for $i = 1, ..., n$.*

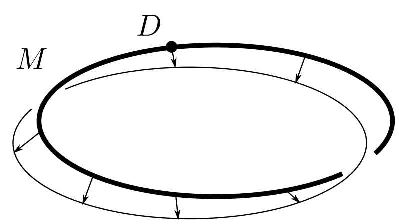


Figure 99.6: A sphere M with a perpendicular vector field v in $\mathbb{R}^3$. The downset D consists of one point x where $v(x)$ is downmost.

Proof. We will prove the multicompression theorem by induction on the number n of vector fields. The base of induction is the Compression Theorem. Suppose that the Multicompression Theorem is true for manifolds with less than n vector fields. Let us show that the theorem is also true when a manifold is equipped with n vector fields. By induction, we may assume that the vectors $v_1, ..., v_{n-1}$ are already vertical up. By the Gramm-Schmidt orthogonalization process, we may also assume that each $v_n(x)$ is perpendicular to $T_x M$ and to $v_i(x)$ for $i < n$. We say that the directions parallel to the plane spanned by $e_1, ..., e_{n-1}$ are vertical, while the directions parallel to the

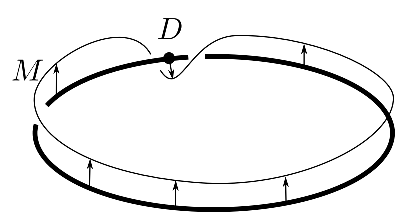


Figure 99.7: We may always rotate the vector field v so that it is vertical up everywhere except a small neighborhood of D.

plane spanned by $e_n, ..., e_{n+k}$ are horizontal. Then the vector $v_n(x)$ at each $x \in M$ is in the affine plane H_x of horizontal vectors at x perpendicular to $T_x M$.

<u>Exercise</u> 99.5. Show that for a sufficiently small real number $\varepsilon > 0$, open ε-neighborhoods in H_x of each point $x \in M$ form a new manifold W of dimension $m + k - (n-1)$ such that the vector fields $e_1, ..., e_{n-1}$ over W are normal.

Next we apply the construction in the local compression theorem except that now we deform and extend the vector field v_n only over W in such a way that v_n stays horizontal and tangent to W. The local isotopy of M along v_n straightens up the vector field v_n over M, and displaces each point x of M by a distance at most ε.[14] Since the manifold M stays within W, the isotopy of M is well-defined and the vectors $e_1, ..., e_{n-1}$ remain normal to M. This completes the proof of the Multicompression Theorem. □

[14] To visualize this isotopy, we note that near a point $x \in W$, the projection π of a small neighborhood of $x \in W$ to the horizontal plane passing through the point x is a diffeomorphism. Furthermore, the projection of the constructed isotopy of the manifold M near x is the isotopy of $\pi(M)$ near $\pi(x)$ along the horizontal plane as prescribed by the Compression Theorem.

99.2.2 The Smale-Hirsch Theorem

The Compression theorem has an important application to Immersion Theory. An *immersion* of a manifold M of dimension m to $\mathbb{R}^{m+k}$ is a smooth map f such that for every point $x \in M$ the differential df of f at x is of rank m.[15] If an immersion is a homeomorphism onto image, then it is an *embedding*. An immersion of a compact manifold is an embedding if and only if it is injective. An immersion of M to $\mathbb{R}^{m+k}$ is usually denoted by $M \looparrowright \mathbb{R}^{m+k}$.

[15] That is to say, if we write the map f in local coordinates on M and any coordinates on $\mathbb{R}^{m+k}$, then the matrix of the first derivatives of f at x is of rank m.

We note that the differential df of an immersion is a continuous (and even smooth) family[16] of injective linear maps $d_x f : T_x M \to \mathbb{R}^{m+k}$. A *formal immersion* of a manifold M into $\mathbb{R}^{m+k}$ is a continuous family of injective linear maps $F_x : T_x M \to \mathbb{R}^{m+k}$ parametrized by the points x in M. Note that though each immersion f defines a formal immersion F with $F_x = d_x f$, not every formal immersion F is of the form $F_x = d_x f$.

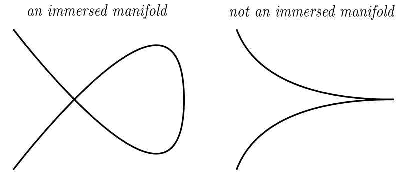


Figure 99.8: Immersion vs. non-immersion.

Example 99.6. Let S^1 be the standard unit circle in $\mathbb{R}^2$. Let v be the unit tangent vector field v in the counterclockwise direction. Then for each point x in S^1, the vector $v(x)$ is a basis of the tangent plane $T_x S^1$. Define a formal immersion of S^1 to $\mathbb{R}^2$ by

$$\begin{aligned} F_x : T_x S^1 &\longrightarrow \mathbb{R}^2, \\ F_x : \alpha v &\mapsto \alpha e_1, \end{aligned} \tag{99.2}$$

[16] What does it mean that the family $\{d_x f\}$ is smooth? Let $v(x)$ be a vector field on M. Then the correspondence $x \mapsto d_x f(v(x))$ is a function $M \to \mathbb{R}^{m+k}$. We say that df is smooth if the constructed function is smooth for every smooth vector field $v(x)$.

where $\alpha \in \mathbb{R}$. If there is an immersion f satisfying $F_x = d_x f$ for each x, then the tangent line to the immersed circle $f(S^1)$ is horizontal at each point of $f(S^1)$, which is impossible. Thus F is a formal immersion which is not the differential of a genuine immersion.

We are now in position to formulate and prove a version of the celebrated Smale-Hirsch theorem.

Theorem 99.7 (Smale-Hirsch theorem)**.** *Suppose that there is a formal immersion F of a manifold M of dimension m into $\mathbb{R}^{m+k}$ with $k > 0$. Then M admits an immersion f into the Eulcidean space $\mathbb{R}^{m+k}$.*[17]

[17] **Strong Smale-Hirsch theorem.** A stronger version of the Smale-Hirsch theorem is true. Namely, there are topological spaces $\mathrm{Imm}(M,N)$ of immersions $M \to N$ and formal immersions $\mathrm{hImm}(M,N)$ as well as a map

$$d\colon \mathrm{Imm}(M,N) \to \mathrm{hImm}(M,N)$$

that associates with each immersion its differential. The strong version of the Smale-Hirsch theorem asserts that the map d is a weak-homotopy equivalent. In other words, the parametric and relative versions of Theorem 99.7 are also true.

Proof. Suppose that M is a submanifold in $\mathbb{R}^n$. We will place M into an even bigger space $\mathbb{R}^{m+k} \times \mathbb{R}^n$, but with a set of n linearly independent normal vector fields. The multicompression theorem then will imply the existence of an immersion $M \to \mathbb{R}^{m+k}$.

We place M with all its tangent vectors into $\mathbb{R}^{m+k} \times \mathbb{R}^n$ by mapping a tangent vector v at a point $x \in M$ into the pair $(F_x(v), x)$. For each x, there are now two copies of the plane T_xM. A vertical copy V_x embedded into $\mathbb{R}^{m+k} \times \{x\}$ by F_x and a horizontal copy H_x embedded into $\{0\} \times \mathbb{R}^n$ by the inclusion of M into $\mathbb{R}^n$. Note that at this moment we have a manifold M embedded into $\mathbb{R}^{m+k} \times \mathbb{R}^n$ and n linearly independent horizontal vector fields $e_1(x), ..., e_n(x)$ over M. However, the vector fields $e_1(x), ..., e_n(x)$ are not normal, so before we apply the multicompression theorem we need to deform the vectors e_i into the normal ones.

There is a linear map of $V_x \oplus H_x$ that sends each vector $v \in V_x$ into the corresponding vector $v \in H_x$ and each vector $v \in H_x$ into the vector $-v \in V_x$. We may extend this linear map to a linear map θ_x of $\mathbb{R}^{m+k} \times \mathbb{R}^n$ by requiring that θ_x is trivial on the orthogonal complement to $V_x \oplus H_x$.

Let $e_i(x)$ denote the i-th standard basis vector of $\{0\} \times \mathbb{R}^n$ at the point $x \in \{0\} \times M$. Since $e_i(x)$ is orthogonal to V_x and the linear map θ_x takes V_x into H_x, we conclude that θ_x takes $e_i(x)$ to a vector perpendicular to T_xM. In other words, we constructed a placement of M into $\mathbb{R}^{m+k} \times \mathbb{R}^n$ with n linearly independent perpendicular vector fields $\theta(e_i)$ over M. Finally, we use the Multicompression Theorem to get an immersion of M into $\mathbb{R}^{m+k}$. □

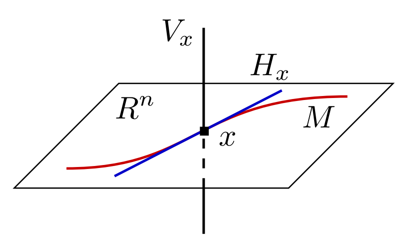


Figure 99.9: The vertical and horizontal copies of T_xM.

It is remarkable that the relative and parametric versions of the Smale-

Hirsch theorem are also true. In particular, Theorem 99.8, which is an important version of the Smale-Hirsch theorem, takes place. To formulate Theorem 99.8, we need two new notions. A deformation of an immersion through immersions is called a *regular homotopy*, while a deformation of a formal immersion through formal immersions is called a *homotopy*.

Theorem 99.8. *Let f_0, f_1 be two immersions of a manifold M of dimension m into $\mathbb{R}^{m+k}$ with $k > 0$. Suppose that the formal immersions $F_0 = df_0$ and $F_1 = df_1$ are homotopic through formal immersions F_t, with $t \in [0,1]$, of the manifold M to $\mathbb{R}^{m+k}$. Then f_0 is regularly homotopic to f_1. In fact, the set of regular homotopy classes of immersions $M \to \mathbb{R}^{m+k}$ of a manifold of dimension m is isomorphic to the set of homotopy classes of formal immersions of M into $\mathbb{R}^{m+k}$.*

In the following section we will see that Theorem 99.8 considerably simplifies the study of immersions.

99.2.3 *The Smale Paradox*

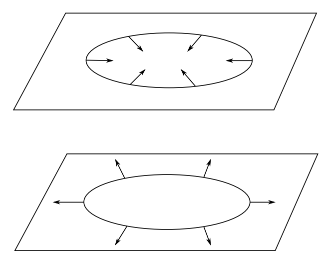


Figure 99.10: The standard sphere S with the unit outward perpendicular vector field v and the everted sphere $f_1(S)$ with the unit inward perpendicular vector field $df_1(v)$.

Let S denote the standard unit sphere in $\mathbb{R}^3$ centered at the origin. The inclusion f_0 of S to $\mathbb{R}^3$ is an embedding. There is another embedding f_1 of S to $\mathbb{R}^3$ obtained from f_0 by postcomposing it with the reflection $r\colon (x,y,z) \mapsto (-x,y,z)$ of $\mathbb{R}^3$.

Theorem 99.9 (Smale Paradox)**.** *There is a regular homotopy f_t, with $t \in [0,1]$, of the standard embedding f_0 to the everted embedding f_1.*

The regular homotopy f_t, whose existence was proved by Smale, everts the sphere S inside out. Indeed, extend f to a regular homotopy of a neighborhood of S. Let v be the unit outward perpendicular vector field over S. Then $df_t(v)$ is a normal vector field over $f_t(S)$ for any t. At time $t = 1$, the normal vector field $df_1(v)$ is inward directed over $f_1(S) = S$.[18] In other words, the outside directed normal vector field transforms under f_t into the inside directed normal vector field.

[18] To prove that $df_1(v)$ as an inward directed vector field, note that under regular homotopy an orientation of the regular neighborhood of S is preserved. At the north pole the vectors $(\partial_x, \partial_y, v)$ define a positive orientation. We have $df_1(\partial_x) = \partial_x$, and $df_1(\partial_y) = \partial_y$. Therefore, $df_1(v)$ should be an inward directed vector field.

The existence of Smale's homotopy is not obvious. For example, simply pushing the northern hemisphere down and the southern hemisphere up does not accomplish anything. Such a deformation produces a ridge that is not allowed under a regular homotopy, see Figure 99.11.

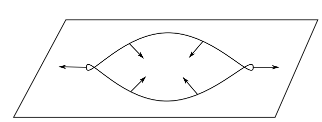


Figure 99.11: Pushing the southern hemisphere up and northern hemisphere down creates a ridge over the equator.

Proof of the Smale paradox. By Theorem 99.8, the set of regular homotopy classes of immersions $S^2 \to \mathbb{R}^3$ is isomorphic to the set of homotopy classes of formal immersions, which are continuous families $F_x : T_x S^2 \to \mathbb{R}^3$ of injective homomorphisms parametrized by $x \in S^2$. In view of the Gram-Schmidt orthogonalization process, we may assume that each F_x preserves the lengths of vectors and angles between the vectors.

Let v denote the unit outward perpendicular vector field over the standard unit sphere S^2 in $\mathbb{R}^3$. Given a point $x \in S^2$, note that any two orthonormal vectors v_1, v_2 in $T_x S^2$ together with the vector $v(x)$ form a basis of $\mathbb{R}^3$ at x. Therefore each F_x can be uniquely extended to a linear map $\tilde{F}_x : \mathbb{R}^3 \to \mathbb{R}^3$ in such a way that $(\tilde{F}_x v_1, \tilde{F}_x v_2, \tilde{F}_x v)$ is the positive orthonormal basis for $\mathbb{R}^3$. It follows, that the set of homotopy classes of formal immersions $\{F_x\}$ is isomorphic to the set of homotopy classes of families $\{\tilde{F}_x\}$ of rotations of $\mathbb{R}^3$. By taking x to $\tilde{F}_x$, the latter family defines a continuous map $S^2 \to \mathrm{SO}_3$ to the space of rotations.

To summarise, we have shown that the set of regular homotopy classes of immersions $S^2 \to \mathbb{R}^3$ is isomorphic to $\pi_2(\mathrm{SO}_3) = 1$. Thus, any two immersions of S^2 to $\mathbb{R}^3$ are regularly homotopic. In particular, the standard embedding f_0 is regularly homotopic to the everted embedding f_1. □

99.2.4 Cobordisms of oriented immersed manifolds

The Compression Theorem suggests yet another geometric interpretation of stable homotopy groups of spheres. In this section we will briefly describe the construction.

Let M be a manifold with a frame $\tau_1, ..., \tau_k$ representing an element in $\pi_{m+k} S^k$. By the Multicompression Theorem, we may assume that $\tau_i = e_i$ for $i = 1, ..., k-1$. Let $\pi : \mathbb{R}^{m+k} \to \mathbb{R}^k$ denote the linear projection with kernel spanned by the vectors $e_1, ..., e_{k-1}$. Since at each point $x \in M$ the kernel of $d_x \pi$ is normal to the manifold M, the projection π restricts to an immersion of M. Furthermore, the vector field $v = d\pi(e_k)$ is normal over an *immersed manifold* $\pi(M)$ in $\mathbb{R}^{m+1}$. For each $x \in M$, by projecting the vector $v(x)$ at $\pi(x)$ to the perpendicular space $\pi_*(T_x^\perp M)$, we may assume that the vector field v is perpendicular over $\pi(M)$. We emphasize that the vector field v is a function $M \to \mathbb{R}^{m+1}$ associating to each point x in M a vector $v(x)$ in $\mathbb{R}^{m+1}$. In particular, at a double point $\pi(x) = \pi(y)$ on the immersed manifold $\pi(M)$ there are vectors $v(x)$ and $v(y)$ which may not coincide.

In general, an immersed manifold of dimension m in $\mathbb{R}^{m+1}$ together with a perpendicular vector field is said to be an *oriented immersed manifold*. The cobordism group of oriented immersed manifolds is defined similarly to the cobordism group of framed manifolds.

Theorem 99.10. *The group π_m^S is isomorphic to the cobordism group of oriented immersed manifolds $M^m \looparrowright \mathbb{R}^{m+1}$.*

Proof. The Pontryagin construction together with the Multicompression Theorem defines a homomorphism h of the stable homotopy group π_m^S to the cobordism group of framed immersed manifolds M^m in $\mathbb{R}^{m+1}$. The homomorphism h is well-defined since, as we have seen, cobordant framed embedded manifolds compress to framed immersed manifolds.[19]

Let us show that the homomorphism h is surjective. Given a framed immersed manifold $M^m \in \mathbb{R}^{m+1}$, choose an integer $k > m+1$, and lift the immersed manifold M to an embedding of M into $\mathbb{R}^{m+1} \times \mathbb{R}^{k-1}$. We will denote the embedded copy of M in $\mathbb{R}^{m+k}$ by M'. The normal frame vector of the immersed manifold M in $\mathbb{R}^{m+1}$ lifts to a normal vector field v over M'. Let $e_1, ..., e_{k-1}$ be the standard basis vectors of the vertical (second) factor in $\mathbb{R}^{m+1} \times \mathbb{R}^{k-1}$. Then $e_1, ..., e_{k-1}, v$ form a frame of the embedded manifold M'. The Pontryagin construction identifies M' with an element α of the stable homotopy group π_m^S such that $h(\alpha)$ is represented by the framed immersed manifold M in $\mathbb{R}^{m+1}$. Thus h is surjective. The injectivity of the homomorphism h is proved similarly. □

[19] The Pontryagin construction identifies a map $f: S^{m+k} \to S^k$ with a framed manifold M_f of dimension m in $\mathbb{R}^{m+k}$. We then compress M_f to a framed immersed manifold $\pi(M_f)$ in $\mathbb{R}^{m+k}$ and declare that $h([f])$ is the cobordism class of $\pi(M_f)$. A priory, it may happen that when we choose another representative g in the class $[f]$ of the homotopy group π_m^S, the Pontrjagin construction together with the compression theorem could result in a framed immersed manifold $\pi(M_g)$ that is not cobordant to $\pi(M_f)$. However, since g and f represent the same element of π_m^S, the Pontrjagin construction yields cobordant framed manifolds M_g and M_f, and, furthermore, the cobordism between M_g and M_f can be compressed to a cobordism of immersed manifolds $\pi(M_g)$ and $\pi(M_f)$. Thus, the homomorphism h is well-defined.

Thus, by Theorem 99.10, the group π_0^S is isomorphic to the cobordism group of signed points in $\mathbb{R}$, the group π_1^S is isomorphic to the cobordism group of oriented closed immersed curves in $\mathbb{R}^2$, while π_2^S is isomorphic to the cobordism group of oriented immersed surfaces in $\mathbb{R}^3$.

99.2.5 The Local Compression Theorem

In this section we complete the proof of the Local Compression Theorem. We will break the proof into three steps.

Case 1. Suppose that the downset D (as well as the horizontal set H) is empty. We have seen that in this case we can straighten up the vector field v by a deformation of the vector field v.

Case 2. Suppose now that the downset D is not empty, while the horizontal set H is still empty. We will also assume that the vector field v is generic.

Lemma 99.11. *Under the above conditions, D is a closed manifold.*

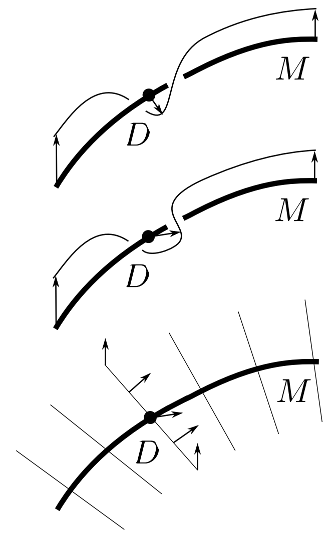


Figure 99.12: Only near the downset D the vector field v is not vertical up (upper figure). We can rotate each vector $v(x)$ in the plane $\langle v(x), e_{k+m}$ so that $v(x)$ has positive last component and its angle with the horizontal plane is at least μ (middle figure). Finally we can extend v so that it is vertical up outside a δ-neighborhood U of M (lower figure). The isotopy F_t is one along the vector field v.

Proof. Let x be a point on D, and O a neighborhood of x in M. We may use an exponential map $\exp\colon O \times D^k \to \mathbb{R}^{m+k}$ to identify a neighborhood of O in $\mathbb{R}^{m+k}$ with $O \times D^k$. Then a perpendicular vector $v(y)$ at $y \in O$ is a unit vector in $\{y\} \times D^k$ with a tip at $S^{k-1} = \partial D^k$. In other words, in view of the exponential map, essentially the vector field v over O is a function $O \to S^{k-1}$. Furthermore, by appropriately choosing the exponential map, we may assume that the map $O \to S^{k-1}$ similarly defined by ψ is the constant map to the south pole $\{\infty\}$. If v is generic in the sense that the south pole $\{\infty\}$ is a regular value v, then $D \cap O = v^{-1}(\infty)$ is a manifold. Clearly if $y \in \bar{D}$, then $y \in D$ since H is empty, i.e., D is a closed set in $\mathbb{R}^{m+k}$. □

First, we apply the argument in Case 1 to modify the vector field v so that it is vertical up everywhere except a small neighborhood of D. Next, we follow the argument in the proof of the Compression Theorem 2.1 and obtain an isotopy F_t, see Figure 99.12. The desired isotopy is the isotopy $G_t(x) = F_t(x) - te_{m+k}$ of the relative compression theorem.

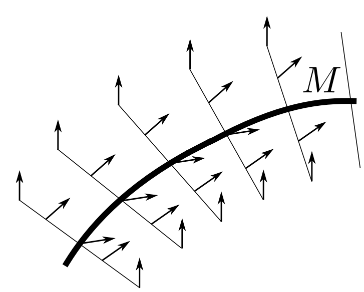


Figure 99.13: The isotopy G_t constructed by means of the Global Compression Theorem may move points far. For example, consider a manifold $M = \mathbb{R}^1$ as well as a vector field v on a vertical plane $L \subset \mathbb{R}^3$ as on Figure 99.13. The isotopy F_t along v moves every point of L along L. Furthermore, for $x \in M$ the trace $F_t x$ stays below M. The point x also can not escape a neighborhood of M since outside a neighborhood of M the vector field v is vertical up. Thus x is forced to move far up along the manifold M.

Lemma 99.12. *By appropriately choosing all parameters in the construction, we may guarantee that the isotopy G_t is arbitrarily C^0-small.*

Before turning to the proof of Lemma 99.12, let's observe that the straightening isotopy $G_t = F_t - te_{m+k}$ constructed by means of of the Global Compression Theorem may move points far along the manifold M since near the manifold M the vector field v may differ from e_{m+k}, see Figure 99.13. The isotopy G_t constructed by means of the Local Compression Theorem moves points along M only for a short period of time, as the vector field v differs from e_{m+k} only in a neighborhood of a relatively small set D; compare the isotopy on Figure 99.13 with that on Figure 99.12. This allows us to conclude that the latter isotopy does not displace points of M much.

Proof. Let x be a point in D. Choose local coordinates near x so that x_1 is the coordinate along the gradient flow of the hight function h on M[20], $x_M = (x_2, ..., x_m)$ are coordinates on M perpendicular to x_1, $x_h = (x_{m+1}, ..., x_{m+k-1})$ the horizontal coordinates perpendicular to M, and x_{m+k} is the vertical coordinate.

[20] In other words, the x_1-direction on the manifold M is the direction of fastest growth of h. Since M has only finitely many critical points, without loss of generality we may assume that the gradient vector field of h is nowhere zero near D.

The initial vector field v on M near x is tangent to the coordinate surfaces $L = (x_1, x_h, x_{m+k})$ since v is perpendicular to M. The deformation of $v(y)$ at the point y near x—constructed in the proof of the Global Compression Theorem—is performed in the vector space $P = \langle v(y), e_{m+k} \rangle$ which is in $T_y L$. Then v is linearly extended over a neighborhood U of M. In particular, each extended vector is a linear combination of v and e_{m+k}, i.e., we may still assume that the extended vector field is tangent to coordinate surfaces L. This implies that each point $y \in M$ in a neighborhood of x flows over its coordinate surface L. Thus, at least for a short period of time each point y stays in $L \cap V$ for a small neighborhood V of D outside of which v is vertical up.

Recall that $h\colon M \to \mathbb{R}$ denotes the hight function on M. By choosing v generic, we may assume that each flow curve of the gradient vector field ∇h intersects D at a finite number of points. Then L intersects D at finitely many points, and therefore the path components of $L \cap V$ are of small diameter.

Now let us follow the trace of a point $x \in M$ along the vector field v, i.e., we consider the trace of the point x under the isotopy F_t rather than the isotopy G_t. Since the vector field v is everywhere at least slightly up, the path of the point x enters and exits the neighborhood V finitely many times. Outside V the path is vertical up. Such segments of the path of x correspond to stationary intervals of the trace of x under the isotopy G_t. When the isotopy F_t brings the point x into the neighborhood V, the point x travels within a small path component of $L \cap V$ untill it leaves V. This part of the trace of x corresponds to a very small displacement of x under the isotopy G_t. Therefore, indeed, the isotopy G_t displaces points of M slightly. □

Case 3. Suppose now that the horizontal set H is not empty.

Exercise 99.13. Show that if H is not empty, then, without loss of generality we may assume that the horizontal set H is a closed manifold of dimension $m - k$, while the downset D is a manifold of dimension $m - k + 1$ with $\partial \bar{D} = H$.

The argument in Case 3 is only slightly different from the one in Case 2. Let $\tilde{D}$ denote a manifold obtained from $\bar{D}$ by attaching a thin collar $\partial\bar{D} \times [0, \varepsilon)$ to $\bar{D}$ inside M in such a way that $\tilde{D}$ is a smooth manifold in $M \subset \mathbb{R}^{m+k}$ containing the manifold with boundary $\bar{D}$. The union of open perpendicular discs of radius ε at points in $\tilde{D}$ is a neighborhood V of $\tilde{D}$. We may assume that the rotated and extended vector field v in the compression theorem is vertical up outside V.

As in Case 2, in a neighborhood of every point $x \in M \setminus H$ near $\tilde{D}$ there are coordinates (x_1, x_M, x_h, x_{m+k}) in $\mathbb{R}^{m+k}$ such that the isotopy of the Compression Theorem floats x along the coordinate surface $L = (x_1, x_h, x_{m+k})$. Its tangent space at x is generated by $\partial/\partial x_1, \partial/\partial x_h$ and $\partial/\partial x_{m+k}$. On the other hand, the vector $\partial/\partial x_1$ is a scalar multiple of the projection of $\partial/\partial x_{m+k}$ to $T_x M$. Therefore the space $T_x L$ is generated by $\partial/\partial x_1$ as well as $T_x^{\perp} M$. By continuity, the points x in H are also floated only in the directions in $\langle \partial/\partial x_1 \rangle \oplus T_x^{\perp} M$.

Without loss of generality we may assume that each flow curve of ∇h on M, i.e., each curve in the x_1-direction, intersects H at finitely many points.[21] Furthermore, we may assume that each flow curve of ∇h on M intersects $\tilde{D}$ at finitely may points. Therefore each coordinate surface L intersects $\tilde{D}$ at finitely many points, which means that each path component of $L \cap V$ has small diameter. Since each point $x \in D$ floating along L stays in V, this implies that x floats a small distance. By continuity, this is true for points in H as well, i.e., for all points in M.

[21] This intuitively obvious claim can be rigorously proved by means of singularity theory. In fact, the set H is the set of singular points of the projection $p\colon M \to \mathbb{R}^{m+k-1}$ to the horizontal plane, while the points at which ∇h is tangent to H is the set of singular points of $p|H$; the latter set is discrete in general.

99.2.6 Solutions to Exercises

Solution to Exercise 2.2. To begin with, let us show that any normal vector field v over M is isotopic to a perpendicular vector field. To this end, let w be the perpendicular vector field over M obtained from v by projecting each vector $v(x)$ to the perpendicular vector space $T_x^{\perp} M$. Let n_x denote the vector subspace of $T_x^{\perp} M$ perpendicular to the vector space $\langle w(x) \rangle$ spanned by $w(x)$, and put $V_x = n_x \oplus \langle v(x) \rangle$, $W_x = n_x \oplus \langle w(x) \rangle$. For sufficiently small $\varepsilon > 0$, let V_x^{ε} denote the open ε-disc in V_x centered at 0. Then the union $V^{\varepsilon} = \cup V_x^{\varepsilon}$ is a neighborhood of M, and each point in V^{ε} corresponds to a uniquely defined pair (x, u) of a point x in M and a vector α in V_x^{ε}. We may similarly define a neighborhood W^{δ} of M for a small real number $\delta > 0$. Choose ε and δ so that $\varepsilon \ll \delta$. Then $V^{\varepsilon} \subset W^{\delta}$. There is an isotopy F_t of V^{ε} in W^{δ} which takes each disc V_x^{ε} into the corresponding disc W_x^{δ}.[22] We will

[22] This statement is known as the Uniqueness of the Tubular Neighborhood Theorem.

discuss the construction of such an isotopy in a moment, but for now assume that F_t has been constructed. Then the time dependent vector field of the isotopy F_t can be extended over all $\mathbb{R}^{m+k}$ and thus define a desired ambient isotopy that preserves M and takes the vector field v to a perpendicular one.

Now let us prove the statement of Exercise 2.2. We may now assume that the vector fields v_0 and v_1 are perpendicular. Then the homotopy v_t of the vector field v_0 to v_1 can be realized by a continuous rotation of each vector $v_0(x)$ to $v_1(x)$ in $T_x^{\perp}M$.[23] We smooth the obtained isotopy of a neighborhood of $\mathbb{R}^{m+k}$ and extend it over all $\mathbb{R}^{m+k}$. The resulting isotopy preserves M and takes the vector field v_0 to the vector field v_1, which completes the proof of the statement in Exercise 2.2.

[23] The proof of this intuitively obvious fact requires fibrations. Let G denote the topological space of pairs (x, t, g) where x ranges over points in M, $t \in [0, 1]$,and g ranges over rotations of $T_x^{\perp}M$ such that $g(v_0(x)) = v_t(x)$. then the projection $(x, t, g) \to (x, t)$ is a fibration $G \to M \times [0, 1]$. The inclusion $M \to M \times \{0\}$ into the base space of the fibration has an obvious lift $x \mapsto (x, 0, \mathrm{id})$, and therefore, the homotopy of inclusions $M \to M \times \{t\}$ lifts to a homotopy of maps $M \to G$. The last inclusion $M \to M \times \{1\}$ lifts to a map $x \mapsto (x, 1, g_x)$ which produces the desired rotations g_x of the fibers $T_x^{\perp}M$.

Now let us prove the existence of the isotopy F_t of V^{ε} in W^{δ} which takes each disc V_x^{ε} to a disc in W_x^{δ}. Suppose that each point (x, α) in V^{ε} corresponds to a point $(y(x, \alpha), \beta(x, \alpha))$ in W^{δ}. Then we define an isotopy F_t of V^{ε} by $F_t(x, \alpha) = (y(x, t\alpha), \frac{1}{t}\beta(x, t\alpha))$ for $t \in (0, 1]$ and $F_0(x, \alpha) = \lim_{t\to 0} F_t(x, \alpha)$.[24] It is not immediately clear that the map F_t is a well-defined diffeomorphism onto image for all t. However, since $\beta(x, 0) = 0$, by the Morse Lemma there is a function valued matrix $A(x, \alpha)$ such that $\beta(x, \alpha) = A(x, \alpha)\alpha$. Therefore, for $t \in (0, 1]$,

$$F_t(x, \alpha) = (y(x, t\alpha), \frac{1}{t}\beta(x, t\alpha)) = (y(x, t\alpha), A(x, \alpha)\alpha). \qquad (99.3)$$

[24] In other words, given a vector α in V_x^{ε}, first shrink it to the vector $t\alpha$ in V_x^{ε}, then find its components $(y(x, t\alpha), \beta(x, t\alpha))$ in W^{δ}, and finally shrink the vector $\beta(x, t\alpha)$ to $\frac{1}{t}\beta(x, t\alpha)$ in W_y^{δ}.

The equation (99.3) gives a new definition of F_t valid for $t \in [0, 1]$; the new definition agrees with the old one. On the other hand, the map F_t defined by (99.3) is a smooth map. Since we can similarly define a map from $F_t(V^{\varepsilon})$ to V^{ε}, the map F_t is a diffeomorphism. □

Solution to Exercise 99.5. Since M is a smooth manifold in $\mathbb{R}^{m+k}$ of dimension m, there is a diffeomorphism Ψ of a neighborhood V of the origin in $\mathbb{R}^{m+k}$ into a neighborhood $U \subset \mathbb{R}^{m+k}$ of any prescribed point $x \in M$ such that $\Psi^{-1}(M \cap U)$ is the intersection of V and the coordinate plane $\mathbb{R}^m$. Recall that the restriction $\psi = \Psi|(V \cap \mathbb{R}^m)$ is called a coordinate chart on M.

In order to show that W is a smooth manifold in $\mathbb{R}^{m+k}$ we need to construct maps $\tilde{\Psi}$ that restrict to coordinate charts $\tilde{\psi}$ on W. The domain of $\tilde{\Psi}$ will be a small neighborhood $\tilde{V}$ of $V \cap \mathbb{R}^m$ in $\mathbb{R}^{m+k}$ such that for each point $(x_1, ..., x_{m+k})$ in $\tilde{V}$, the point $(x_1, ..., x_m, 0, ..., 0)$ is in $V \cap \mathbb{R}^m$. Let $\tau_{m+1}, ..., \tau_s$, with $s = m + k - (n-1)$ be vector fields over $M \cap U$ at each point $x \in M \cap U$ spanning the plane H_x. Put

$$\begin{aligned} \tilde{\Psi}(x_1, ..., x_{m+k}) &= \psi(x_1, ..., x_m) \\ &+ x_{m+1}\tau_{m+1} + \cdots + x_s\tau_s \\ &+ x_{s+1}e_1 + \cdots x_{s+n-1}e_{n-1}. \end{aligned}$$

Then $\tilde{\Psi}^{-1}(W)$ is the intersection of $\tilde{V}$ and the coordinate plane $\mathbb{R}^s$ of points $(x_1, ..., x_s, 0, ..., 0)$. The restriction $\tilde{\psi}$ of $\tilde{\Psi}$ to the intersection $\tilde{V} \cap \mathbb{R}^s$ is a coordinate chart on W. If ε is sufficiently small, then the coordinate charts $\tilde{\psi}$ constructed for all points $x \in M$ cover the set W, and therefore the set W is a manifold.

By construction, the dimension of the manifold W is $s = m + k - (n - 1)$, while the vector fields $e_1, ..., e_{n-1}$ over W are normal. □

Solution to Exercise 99.13. The proof of this fact relies on consideration of Grassmann manifolds. We will only sketch the argument. The Grassmann manifold $\mathrm{Gr}_k \mathbb{R}^{m+k}$ is the space of vector subspaces of dimension k in $\mathbb{R}^{m+k}$. There is an obvious inclusion $\mathrm{Gr}_k \mathbb{R}^{m+k-1} \to \mathrm{Gr}_k \mathbb{R}^{m+k}$ which associates an m-plane in the horizontal space the same plane in the total space $\mathbb{R}^{m+k}$. Consdier the Gauss map $\mathfrak{G}\colon M \to \mathrm{Gr}_x \mathbb{R}^{m+k}$ defined by $x \mapsto T_x^\perp M$. The horizontal set H is precisely the inverse image under $\mathfrak{G}$ of the set $\mathrm{Gr}_m \mathbb{R}^{m+k-1}$. By the inverse function theorem, this implies that H is a closed manifold of dimension $m - k$ provided that $M \subset \mathbb{R}^{m+k}$ is generic.

It is helpful to visualize the normal disc of $\mathrm{Gr}_m \mathbb{R}^{m+k-1}$ in $\mathrm{Gr}_x \mathbb{R}^{m+k}$. To this end, pick a horizontal vector space $\mathcal{H}$ representing a point in $\mathrm{Gr}_x \mathbb{R}^{m+k}$. Then any other plane $\mathcal{H}_w$ in the normal disc is parametrized by a vector $w \in \mathcal{H}$. The plane $\mathcal{H}_w$ is spanned by $w - |w|e_{m+k}$ and the subspace in $\mathcal{H}$ perpendicular to w. In $\mathcal{H}_w$ the downmost vector is $w - |w|e_{m+k}$. The downmost vector maps to w under the projection along e_{m+k} from $\mathcal{H}_w$ to $\mathcal{H}$.

Let x be a point in H, and let O be the perpendicular disc of H in M at x; it is of dimension k. Since the Gauss map $\mathfrak{G}$ is generic, it maps the disc O isomorphically onto a normal disc of $\mathrm{Gr}_k \mathbb{R}^{m+k-1}$ in $\mathrm{Gr}_k \mathbb{R}^{m+k}$. For each y in O the orthogonal projection $T_y^\perp M \to T_x^\perp M$ is an isomorphism. We may modify v so that under this isomorphism $v(y)$ maps to a positive multiple of $v(x)$ for all y. Then near H the set D consists of points y such that $\mathfrak{G}(y)$ is the plane $\mathcal{H}_w$ where w is a non-negative multiple of $v(x)$. Clearly, $\partial \bar{D} = H$. □

99.3 Topics for Chapter 3

99.3.1 Example: Cobordisms

Let us describe a cobordism W from an empty manifold to an empty manifold with W diffeomorphic to a torus. The simplest cobordism W has four points $p_0, ..., p_3$ at which the tangent plane is horizontal. These are critical points of the height function f, see Figure 99.14.

When $t < 0$, the level W_t of the cobordism is empty. As t passes critical values of the height function, we observe modifications of W_t. At $t = f(p_0)$ a circle is born. Then, the level W_t splits into two circle, merges into one circle, and finally turns into an empty set at $t = f(p_3)$.

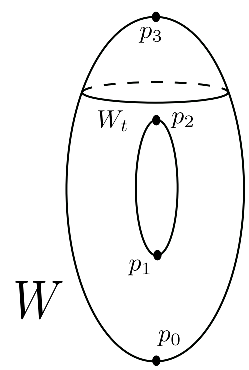


Figure 99.14: The cobordism W with four spherical surgeries.

More precisely, at the time $f(p_0)$, the spherical surgery of index $i = 0$ is performed on the empty level set $W_{-\varepsilon}$. The surgery removes an empty handle h_i from the empty level set and attaches a new handle $h_j = D^0 \times S^1$. Thus, after passing the critical level $t = f(p_0)$, the level set W_t turns into a circle. At the time $f(p_1)$, the level set is undergoing a surgery of index $i = 1$. Two segments $h_i = S^0 \times D^1$ on the level set are replaced with two segments $h_j = D^1 \times S^0$. The level set turns into a pair of circles. At the time $f(p_2)$, another surgery of index $i = 1$ takes place. Again a pair of segments h_i on the level set is replaced with another pair of segments h_j. The resulting level set is a circle. Finally, at the time $f(p_3)$, the level set is undergoing a surgery of index $i = 2$. A handle $h_i = S^1 \times D^0$ is replaced with an empty handle h_j. Thus, the level set W_t becomes empty.

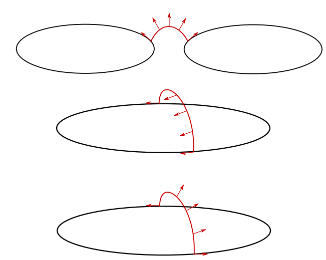


Figure 99.15: Spherical surgeries of index 1 that respectively decrease, increase, and does not change the number of components of W_t.

Note that the spherical surgery both at the level $f(p_1)$ and at the level $f(p_2)$ is of index 1. However, the former surgery results in increasing the number of path components of W_t, while the latter results in decreasing. It is also possible that the number of path components of W_t does not change when a spherical surgery of index 1 is performed, see Figure 99.15.

99.3.2 Cairns-Whitehead technique of smoothing manifolds

Recall that a smooth submanifold M of $\mathbb{R}^{m+k}$ is defined by means of smooth local charts Ψ. A *locally flat topological manifold* M is defined similarly, except that in its definition the local chart Ψ is only required to be continuous, not necessarily smooth. [25]

[25] In other words, a subset M of $\mathbb{R}^{m+k}$ is said to be a *locally flat topological manifold* of dimension m if for each point $x \in M$ there is a coordinate neighborhood U of x in $\mathbb{R}^{m+k}$ with continuous coordinates $(x_1, ..., x_{m+k})$ on U such that $U \cap M$ is the set $(x_1, ..., x_m, 0, ..., 0)$.

In this section we will discuss a method of smoothing (locally flat) topological manifolds, i.e., approximating topological manifolds by their smooth counterparts. We will use standard mollifiers.

Standard mollifiers.

By definition, the *standard mollifier* on $\mathbb{R}^n$ is a smooth function η on $\mathbb{R}^n$ with support in the unit ball $|x| < 1$, defined on the unit ball by $\eta(x) = c \exp \frac{1}{|x|^2 - 1}$, where c is the positive constant such that $\int_{\mathbb{R}^n} \eta dx = 1$, see Figure 99.16. For each $\varepsilon > 0$, there is a closely related smooth function $\eta_\varepsilon(x) = \frac{1}{\varepsilon^n} \eta(\frac{x}{\varepsilon})$ with support in the ε-ball $|x| < \varepsilon$. Still $\int_{\mathbb{R}^n} \eta_\varepsilon dx = 1$, see Figure 99.16. Mollifiers η_ε allow us to approximate any integrable function f on $\mathbb{R}^n$ by a smooth function, namely,

$$f^\varepsilon(x) = \int_{\mathbb{R}^n} \eta_\varepsilon(x - y) f(y) dy.$$

In fact, if the original function f is continuous, then the functions f^ε approach f uniformly on compact subsets of $\mathbb{R}^n$ as $\varepsilon \to 0$.

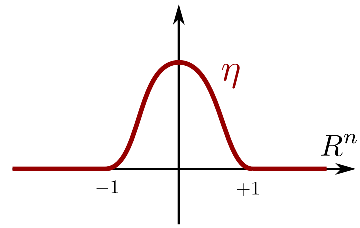


Figure 99.16: The standard mollifier η.

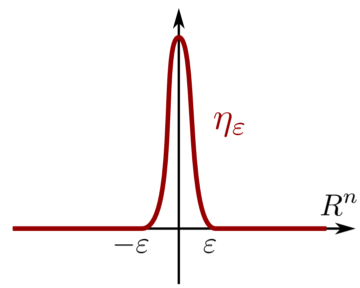


Figure 99.17: The mollifier η_ε.

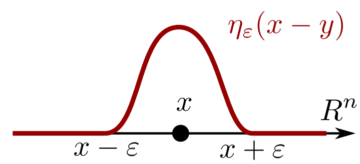


Figure 99.18: The function $\eta_\varepsilon(x - y)$

To make sense of the definition of the approximating function f^ε, we note that the function $\eta_\varepsilon(x - y)$ is trivial outside the set $|x - y| < \varepsilon$, see Figure 99.18. Thus, for each point x, the integral in the definition of $f^\varepsilon(x)$ is essentially taken over the disc $|x - y| < \varepsilon$ of radius ε centered at x. Furthermore, since $\int_{\mathbb{R}^n} \eta_\varepsilon dx = 1$, the value $f^\varepsilon(x)$ is the weighted average of the values $f(y)$ where y ranges over points in the ε-disc centered at x.

Lipschitz functions.

Another important ingredient needed here is the Lipschitz function. Given two open subsets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$, a map $f : U \to V$ is said to be Lipschitz if $|f(x) - f(y)| \leq C|x - y|$ for all x and y in U and for some constant C that does not depend on x and y.[26] In other words, a Lipschitz function may increase distances between points, but only by a bounded factor. In particular, every Lipschitz function is continuous. In fact, by the Rademacher theorem: Every Lipschitz function $f : U \to \mathbb{R}^m$ from an open subset U of $\mathbb{R}^n$ is differentiable almost everywhere in U, i.e., f is differentiable in the complement to a set of measure zero. The same is true for *locally Lipschitz functions*, i.e., functions f that are Lipschitz in sufficiently small neighborhood of each point x of the domain. To visualize the non-differentiable behavior of a Lipschitz function f, let $\Sigma \subset U$ denote the set of measure zero in the complement to which the function f is differentiable. Then the graph of the restriction of f to $U \setminus \Sigma$ is a differentiable manifold in $\mathbb{R}^m \times \mathbb{R}^n$, and, in particular, it has a tangent plane L at each point. Let now G

[26] Lipschitz functions $f : \mathbb{R} \to \mathbb{R}$ are easy to visualize. These are functions whose rate of change $|f(x) - f(y)|/|x - y|$ is bounded by a constant C.

denote the graph of the function f in $\mathbb{R}^m \times \mathbb{R}^n$. We have seen that in the complement to $f(\Sigma)$ it has a field of tangent planes $L(z)$. It turns out that if the plane field $L(z)$ can be extended to a continuous plane field over G, then f is differentiable.

Let now M be a locally flat topological manifold of dimension m in $\mathbb{R}^{m+k}$. To smooth the manifold M near a point p, suppose that there is an affine plane $N \subset \mathbb{R}^{m+k}$ of dimension k passing through the point p such that for a small disc D of dimension m perpendicular to N and centered at p, locally the manifold M is the graph of a Lipschitz function $f : D \to N$.[27] Then the graph of the mollified function f^ε near the point p is a smooth manifold M_p locally approximating the manifold M. The affine plane N in the local construction is called a *transverse plane*. Informally, it gives us a direction in which we may perturb the manifold M to smooth it in a neighborhood of the point p.

[27] The statement that M is locally the graph of a Lipschitz function is equivalent to the following elementary assumption. Suppose that there is a neighborhood O of the points p in M, and a positive number $\gamma < \pi/2$ such that for any points p', p'' in the neighborhood O the angle between N and the line through the points p', p'' is at least γ.

We aim to turn this local construction into a global one. To this end, suppose that at each point p of M there is chosen a transverse plane $N(p)$ continuously depending on p. Then we say that N is a *transverse field* over M.

Theorem 99.14 (Cairns-Whitehead)**.** *Given a transverse field over a compact topological manifold M in $\mathbb{R}^{m+k}$, the manifold M is Lipeomorphic to a nearby smooth submanifold in $\mathbb{R}^{m+k}$.*

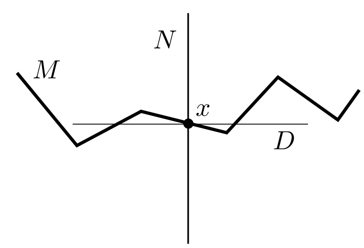


Figure 99.19: A transverse plane N of dimension k at a point $x \in M$ satisfies the defining property that near x the manifold M is the graph of a Lipschitz function $f : D \to N$ on a small disc D of dimension m centered at x and perpendicular to N.

We note that a transverse field over a manifold M may not exist. Indeed, Kervaire constructed a topological manifold of dimension 10, which can be placed in $\mathbb{R}^{21}$, and which admits no smoothing. Thus, the Kervaire manifold admits transverse fields only locally, not globally.

On the other hand, the transverse field N determines the smoothing of the manifold M uniquely up to a diffeomorphism. Different transverse fields may lead to smooth manifolds that are not diffeomorphic.

We will explain the idea behind the proof of Theorem 99.14 following the argument of Pugh.

Sketch of the proof. To begin with we extend the transverse field N over a neighborhood of M in $\mathbb{R}^{m+k}$, and approximate the transverse field N by a smooth one. Its restriction to M is still a transverse field. Now, for each point p, choose a small δ-neighborhood $N_\delta(p)$ of p in the affine plane $N(p)$. If δ is small enough, then the discs $N_\delta(p)$ do not intersect for different p, and form a neighborhood U of M. There is a projection π of U to the manifold M which projects each disc $N_\delta(p)$ to the point p. Of course, the map π is not smooth (unless the manifold M is smooth), but its mollified approximation $\pi^\varepsilon: U \to U$ is smooth.

In a neighborhood of each point $p \in M$, the set M is the graph of a Lipschitz function $f_p: D(p) \to N(p)$ with Lipschitz constant $C(p)$. Let S denote the supremum of all $C(p)$ where p ranges over points in M. We say then that a set Y in U is *almost horizontal*, if for every point $y \in Y$ a neighborhood of y in Y is the graph of a Lipschitz function $D(p) \to N_\delta(p)$ with Lipschitz constant $\leq S$. In particular, the set M itself is almost horizontal in U.

We will perturb the manifold M by means of a *section* $\sigma: M \to U$, i.e., by means of a continuous map which displaces each point p within the disc $N_\delta(p)$.[28] Let Σ denote the set of all sections σ which produce almost horizontal displacements $\sigma(M)$ of the manifold M, see Figure 99.20. It is a complete metric space with metric

$$d(\sigma_1, \sigma_2) = \sup_{p \in M} \{|\sigma_1(p) - \sigma_2(p)|\}.$$

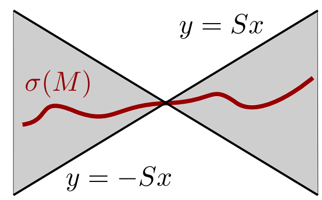


Figure 99.20: The sections σ in Σ satisfy the property that their graphs are far from being vertical. Their graphs are within a cone $-Sx \leq y \leq Sx$, where x is the coordinate on $D(p)$ and y is the coordinate on $N(p)$.

Suppose that the approximation π^ε takes each section $\sigma \in \Sigma$ to another section $\pi^\varepsilon \circ \sigma$. In other word, the map π^ε defines a map $\pi^\varepsilon_\sharp: \Sigma \to \Sigma$. Then $\pi^\varepsilon_\sharp$ is a contraction in the sense that it decreases the distances between sections[29], and therefore, there is a unique fixed section σ_0, i.e., a section σ_0 that is sent to itself by $\pi^\varepsilon_\sharp$. It can be shown that $\sigma_0(M)$ is a differentiable submanifold approximating M, which completes the proof modulo two statements which we will discuss next in more detail.

[28] More precisely, a map $\sigma: M \to U$ is a section if the composition $\pi: \sigma: M \to M$ is the identity map.

[29] i.e., $d(\pi^\varepsilon_\sharp(\sigma_1), \pi^\varepsilon_\sharp(\sigma_2)) \leq \alpha d(\sigma_1, \sigma_2)$ for some constant $0 < \alpha < 1$.

First, we assumed that the approximation π^ε takes each section $\sigma \in \Sigma$ to another section $\pi^\varepsilon \circ \sigma$. In general this is not true; some care is necessary. Fix a point p_0 in M and consider a cube neighborhood $D \times N_\delta$ about p_0 where $D = D(p_0)$ and $N_\delta = N_\delta(p_0)$. Take a point (x, y) in the cube neighborhood, where x is the coordinate on D, and y is the coordinate on N_δ with $(0,0)$ corresponding to p_0. We claim that π^ε takes (x, y) to a point (x', y') such that

$$x' = x + o(x,y), \qquad \textit{and} \qquad y' = f(x) + o(x,y),$$

where $f: D \to N_\delta$ is a function whose graph is a neighborhood of p_0 is M. Indeed, since the field of planes N is smooth, all planes in the field are almost parallel to $N(p_0)$. Therefore

$$(x', y') = \pi^\varepsilon(x,y) = \pi(x,y) + o(x,y) = (x, f(x)) + o(x,y).$$

The formulas for x' and y' show that π^ε takes almost horizontal lines to almost horizontal lines.[30] Since the plane field N is smooth (and therefore nearly vertical in a neighborhood of p_0), we deduce that indeed π^ε takes sections σ to graphs of other sections.[31] In other words, we define $\pi^\varepsilon_\sharp(\sigma)$ to be the section that takes M to the set $\pi^\varepsilon \circ \sigma$.

[30] If we write $(x_1', y_1') = \pi^\varepsilon(x_1, y_1)$ and $(x_2', y_2') = \pi^\varepsilon(x_2, y_2)$, then $|y_2' - y_1'| \approx |f(x_2) - f(x_1)| < S|x_2 - x_1| \approx S|x_2' - x_1'|$.

[31] It is not true that $\pi(\pi^\varepsilon(\sigma(x))) = \pi(\sigma(x))$ for $x \in M$ as π^ε may compress a section vertically, as well as slightly shift it horizontally.

Second, we need to show that if σ_0 is a fixed point of $\pi^\varepsilon_\sharp$, then the manifold $\sigma_0(M)$ is indeed smooth. To begin with $\sigma_0(M)$ is a Lipschitz manifold, and therefore, by the Rademacher theorem it is differentiable almost everywhere. In fact, there is a subset $W \subset \sigma_0(M)$ of full measure such that W consists of points at which $\sigma_0 M$ is differentiable and $\pi^\varepsilon|W : W \to W$ is a homeomorphism.[32] Let $L(z)$ be the tangent plane field to $\sigma_0 M$ over W. It remains to show that the plane field $L(z)$ admits a continuous extension to a plane field over $\sigma_0(M)$.

[32] We note that the restriction of π^ε to $\sigma_0 M$ is a homeomorphism, say g. Let K denote the set of points in $\sigma_0 M$ at which $\sigma_0 M$ is non-differentiable. The set W is the complement in $\sigma_0(M)$ to all the sets $g^i K$ of measure 0 for $i \in \mathbb{Z}$.

To extend $L(z)$ over $\sigma_0(M)$, let $\mathcal{H}$ denote the complete metric set of all almost horizontal continuous plane fields over $\sigma_0(M)$. The map π^ε acts on $\mathcal{H}$ and defines a contraction $\pi^\varepsilon_\sharp : \mathcal{H} \to \mathcal{H}$. Therefore, there is a unique plane field H such that $\pi^\varepsilon(H) = H$. Since H coincides with $L(z)$ over W, it follows that H is a desired extension.

Thus the manifold $\sigma_0 M$ is differentiable. It remains to show that every differentiable manifold M in $\mathbb{R}^{m+k}$ can be approximated by a smooth manifold. To begin with, by the Whitney Smoothing Theorem, there exists a smooth manifold $M' \subset \mathbb{R}^{m+k'}$ and a C^1-homeomorphism $g : M' \to M$.[33] We may regard g as a C^1-embedding into $\mathbb{R}^{m+k}$. The mollified approximation g^ε of g is a smooth map $M' \to \mathbb{R}^{m+k}$. The first partial derivatives of g^ε are the corresponding mollified derivatives of g. Since g is an immersion, this implies that g^ε is also an immersion. In particular, locally, in a neighborhood of each point, g^ε is an embedding. Therefore, if ε is sufficiently small, then g^ε is an embedding. □

[33] The Whitney Smoothing Theorem for abstract manifolds (not necessarily submanifolds of $\mathbb{R}^{m+k}$) asserts that the C^1-atlas of every manifold contains a C^∞-subatlas.

99.4 Topics for Chapter 5

99.4.1 Bilinear forms on free modules

Let R be a commutative ring with 1. In our applications, we will only consider rings $\mathbb{Z}, \mathbb{R}, \mathbb{Q}$ and $\mathbb{Z}_2$. Let V be a free finitely generated R-module with basis $e_1, ..., e_n$, i.e., V consists of elements of the form

$$v = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n,$$

where the coefficients $\alpha_1, ..., \alpha_n$ are from the ring R; elements of V can be added, and scaled. Bilinear, symmetric, skew-symmetric, symplectic, and non-degenerate forms, as well as quadratic forms on the module V are defined the same way as for vector spaces.

A bilinear form φ on the module V is completely determined by its values $\varphi_{ij} = \varphi(e_i, e_j)$ on the basis vectors. In particular, all properties of the bilinear form φ can be reformulated in terms of the matrix $B = (\varphi_{ij})$. The form φ is symmetric, if its matrix is symmetric, i.e., $B = B^t$, where B^t stands for the transpose of the matrix B. The form φ is skew-symmetric if $B = -B^t$. Similarly, the form φ is symplectic, if $B = -B^t$, and all terms on the diagonal of B^t are trivial. Finally, the form φ is non-degenerate if and only if B has an inverse.[34]

[34] Indeed, we need to show that for every linear function $f: V \to R$ there are unique vectors v and w such that $f(x) = v \cdot x = x \cdot w$ for all x in V. Of course, it suffices to require that the identities hold only for the basis vectors e_i. Thus, in order to verify $f(x) = x \cdot w$ we need to find an element w such that

$$(f(e_1), ..., f(e_n))^t = (e_1 \cdot w, ..., e_n \cdot w)^t$$

On the left hand side we are given an arbitrary vector F, while on the right we have a vector Bw. The equation $F = Bw$ can be solved for any vector F if and only if B has a right inverse, i.e., there is a matrix W such that $I = BW$. Similarly, an element v exists if and only if B has a left inverse. We also note that B has a left inverse if and only if B has a right inverse.

99.5 Topics for Chapter 6

99.5.1 Heegaard splittings

Let P and Q be two manifolds with boundary ∂P and ∂Q. The boundary connected sum $P\#_b Q$ of the manifolds P and Q is a manifold with boundary obtained from $P \sqcup Q$ by attaching a 1-handle H^1 along an attaching sphere S^0 which consists of a point on P and a point on Q, and then smoothing the corners.

We call the boundary connected sum of g copies of the solid torus $S^q \times D^{q+1}$ a *ball with g handles*. For example, when $q = 1$, every surface of genus g in $\mathbb{R}^3$ bounds a ball with g handles.

Theorem 99.15. *Let M be a $(q-1)$-connected framed manifold of dimension $m = 2q + 1$. Then there is a ball with g handles W such that M is diffeomorphic to $W \cup_\varphi W$, where φ is a diffeomorphism between the boundaries of the two copies of W.*

Proof. Choose a handle decomposition of the manifold M. Since M is $(q-1)$-connected we may choose a handle decomposition with only $0, q, q+1$ and m-handles. Let W denote the union of handles of indices 0 and q, and W' the union of handles of indices $q+1$ and m.

In particular, the manifold W is obtained from a disc H^0 by attaching handles H_i^q of index q along attaching spheres $S^{q-1} \subset \partial H^0$. Let g denote the number of handles of index q. Since the dimension of the sphere ∂H^0 is $2q$, any set of g attaching spheres is isotopic to any other set of g attaching spheres. For this reason we may choose any position for the attaching spheres of handles of index q. In particular, we may represent the disc H^0 as the boundary connected sum of g copies of $D^q \times D^{q+1}$. We may assume that the attaching sphere of the handle H_i^q is the sphere $\partial D^q \times \{0\}$ in the i-th summand of H^0. Furthermore, we may assume that the handle H_i^q is attached along the thickening $\partial D^q \times D^{q+1}$ of the attaching sphere.

Next we note that the manifold obtained from $D^q \times D^{q+1}$ by attaching the handle $H_i^q = D^q \times D^{q+1}$ is the total space E of a vector bundle over S^q. In fact, since M is framed, the space E is diffeomorphic to $S^q \times D^{q+1}$.[35] Therefore, the manifold W is a ball with g handles. The same argument shows that W' is a ball with g' handles. Since ∂W coincides with $\partial W'$ we conclude that $g' = g$, and therefore W' is diffeomorphic

[35] Let ε denote a trivial vector bundle of dimension 1 over any space X. Let $S \subset M$ denote the zero section of the vector bundle E. Then E is the total space of the normal vector bundle ν of S in M. The isomorphism $TM \oplus T^\perp M = \varepsilon^{m+k}$ restricted over S produces

$$TS \oplus \nu \oplus T^\perp M = \varepsilon^{m+k}.$$

Since M is framed, we have $T^\perp M = \varepsilon^k$. Also $TS \oplus \varepsilon^k = \varepsilon^{q+k}$. Finally, since ν is a vector bundle of dimension $q+1$ over a manifold of dimension q, and it is stably trivial (i.e., $\nu \oplus \varepsilon^{q+k} = \varepsilon^{m+k}$), it follows that ν is trivial.

to W. □

99.5.2 *Alternative proof of the surgery theorem for framed manifolds of odd dimension*

Let M be a closed $(q-1)$-connected framed manifold of dimension $m = 2q+1$. We have seen that the manifold M is a union $W \cup_\varphi W'$ where $W \approx W'$ is a ball with g handles, and φ is a diffeomorphism identifying the boundaries of W and W'. Suppose that g is minimal in the sense that for any framed manifold M' which is cobordant to M, and any handle decomposition of M', the corresponding number g' for M' is at least g. We will show then that M is a homotopy sphere.

To prove the claim, let us study the homomorphism φ_* of homology groups.

The homology group $H_q(\partial W) = H_q(\#_g S^q \times S^q)$ is a free abelian group generated by meridian classes m_i and longitude classes ℓ_i, where $i = 1, ..., g$. We may choose the generators in such a way that $m_i \cdot \ell_i = 1$. Similarly, the homology group $H_q(\partial W')$ is a free abelian group generated by meridian classes m_i' and longitude classes ℓ_i' such that $m_i' \cdot \ell_i' = 1$. Since the diffeomorphism φ identifies ∂W with $\partial W'$ we may actually assume that $\{m_i, \ell_i\}$ and $\{m_i', \ell_i'\}$ are two bases of the group $H_q(\partial W)$. In particular, the classes m_i' and ℓ_i' can be written as linear combinations

$$m_j' = \sum a_{ij} m_i + \sum b_{ij} \ell_i,$$
$$\ell_j' = \sum c_{ij} m_i + \sum d_{ij} \ell_i.$$

If we multiply the first equation by ℓ_i, we get $m_j' \cdot \ell_i = a_{ij}$. We similarly find that $m_i \cdot m_j' = b_{ij}$, $\ell_j' \cdot \ell_i = c_{ij}$ and $m_i \cdot \ell_j' = d_{ij}$. It follows that the matrix of the differential φ_* of the map $\varphi \colon \partial W' \to \partial W$ is given by

$$\begin{bmatrix} a_{ij} & c_{ij} \\ b_{ij} & d_{ij} \end{bmatrix}$$

We aim to simplify the matrix. Recall that since the dimension of M is $2q+1$, a framed surgery is possible along any embedded sphere in M of dimension q. In particular, take the core S of a boundary connected summand $S^q \times D^{q+1}$ in W. There exists a framed surgery of M along S. Without loss of generality we may assume that the thickening of the attaching sphere S of the framed surgery is the connected summand $S^q \times D^{q+1}$; if necessary, we may reparametrize the connected summand $S^q \times D^{q+1}$ (this results in modifying the basis $\{m_i, \ell_i\}$). Then the framed surgery replaces the connected summand $S^q \times D^{q+1}$ in W with a connected summand $D^{q+1} \times S^{q+1}$. Such a surgery results in a change of bases $m_i \mapsto \ell_i$ and $\ell_i \mapsto \pm m_i$. In other words, the i-th row of the matrix $[\varphi_*]$ is exchanged with the $(g+i)$-th row of the matrix $[\varphi_*]$. We may also change the parametrization of a connected summand $S^q \times D^{q+1}$ so that it still remains a thickening of the attaching sphere of a framed surgery. This corresponds to a homomorphism $m_i \mapsto m_i$ and $\ell_i \mapsto \ell_i + \alpha m_i$ where α is any even integer.

Another observation is that since $m'_1 \cdot \ell'_j = 1$, we have $\sum a_{i1} d_{i1} \pm b_{i1} c_{i1} = 1$, which we can state as the following lemma.

Lemma 99.16. *The elements in the first column of the matrix $[\varphi_*]$ are relatively prime.*

Theorem 99.17. *The manifold M is a homotopy sphere.*

Proof. To begin with we observe that the meridian classes m_i are represented by the belt spheres of handles H^q_i, while the meridian classes m'_j are represented by the attaching spheres of handles H^{q+1}_j. Therefore, sliding handles, and renumbering the handles allows us to turn the submatrix $[b_{ij}]$ into a diagonal one. By Lemma 99.16, the numbers $\{b_{11}, a_{11}, a_{12}, ..., a_{1g}\}$ are relatively prime.

Suppose now that $|b_{11}| < |a_{11}|$. Then we can perform a surgery along the first summand $S^q \times D^{q+1}$ which results in switching a_{11} with b_{11}. Thus we may assume that $|b_{11}| \geq |a_{11}|$.

If $|b_{11}| > |a_{11}| > 0$, then we can reduce $|b_{11}|$ without changing all other numbers in the set $\{b_{11}, a_{11}, a_{12}, ..., a_{1g}\}$. Indeed, we can perform a surgery on the first summand $S^q \times D^{q+1}$ of W, which corresponds to $m_1 \mapsto \ell_1$ and $\ell_1 \mapsto m_1$, and then choose a reparametrization which corresponds to $m_1 \mapsto m_1$ and $\ell_1 \mapsto \ell_1 \pm 2m_1$, and then perform again a surgery along the first summand. The resulting map is

$$m_1 \mapsto \ell_1 \mapsto \ell_1 \pm 2m_1 \mapsto m_1 \pm 2\ell_1$$

and $\ell_1 \mapsto \ell_1$. Therefore the only term that is modified in the first column is b_{11}. Thus we can modify b_{11} by $\pm 2a_{11}$,[36] and, consequently, we can make $|b_{11}| \leq |a_{11}|$. If $b_{11} = \pm a_{11}$, then $\{a_{11}, ..., a_{1g}\}$ are relatively prime. Let's perform a surgery on all summands $S^q \times D^{q+1}$ of W. Then $\{b_{11}, ..., b_{1g}\}$ become relatively prime. Therefore by handle slides and renumeration we can get $b_{11} = \pm 1$. By the Handle Cancellation Lemma, a pair of handles in the handle decomposition of M can be cancelled, which contradicts the choice of the handle decomposition.

[36] We have

$$\begin{aligned} b_{11} &= m_1 \cdot m'_1 \\ &\mapsto (m_1 \pm 2\ell_1) \cdot m'_1 \\ &= b_{11} \pm (\pm 2a_{11}). \end{aligned}$$

If $a_{11} = 0$, then we can perform a surgery along the first summand $S^q \times D^{q+1}$ which results in switching a_{11} with b_{11}, and we again obtain that $\{a_{11}, ..., a_{1g}\}$ are relatively prime. Finally, if $|b_{11}| = |a_{11}|$, then $\{a_{11}, ..., a_{1g}\}$ are already relatively prime. $\square$

99.6 *Topics for Chapter 14*

99.6.1 *The Whitehead torsion*

Let $W \subset \mathbb{R}^{m+k} \times [0,1]$ be an h-cobordism between two manifolds M_0 and M_1. In particular, the manifolds M_0, M_1 as well as W have the same fundamental group π. We may assume that the hight function f is a Morse function, with critical points of each index ordered in some way. Choose a distinguished point $*$ in W, and for each critical point p of f choose a path γ_p in W from the point $*$ to the critical point p. If p is a critical point of index n, and q a critical point of index $n-1$, then each intersection point of the attaching sphere of the handle H^n corresponding to p with the belt sphere of the handle H^{n-1} corresponding to q lies on a *special trajectory* $\gamma_{p,q}$ from p to q along the gradient vector field of f. The incidence $\langle p, q\rangle$ is an element in $\mathbb{Z}[\pi]$ given by the linear combination $\sum \pm\gamma_p * \gamma_{p,q} * \gamma_q^{-1}$, where $\gamma_{p,q}$ ranges over all special trajectories, and the sign for terms in the linear combination is taken according to the orientations.

We will next define a complex of modules over $\mathbb{Z}[\pi]$ associated with the h-cobordism W. To this end, let C_n denote a free $\mathbb{Z}[\pi]$-module whose generators $[p]$ are in bijective correspondence with critical points p of f of index n. The differential $d_n\colon C_n \to C_{n-1}$ is a homomorphism of $\mathbb{Z}[\pi]$-modules. It takes $[p]$ to $\sum\langle p, q\rangle[q]$. The $\mathbb{Z}[\pi]$-modules C_n together with differentials d_n form a chain complex

$$\cdots \to C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \to \cdots .$$

We note that each C_n is a free $\mathbb{Z}[\pi]$-module with an ordered set of generators, there are only finitely many non-zero terms C_n, and the chain complex is *acyclic*, i.e., its homology groups are trivial. Handle rearrangements correspond to the following modifications of the chain complex:

- changing the order of generators of C_n (corresponds to reordering handles of index n),
- replacing one generator with the sum of the generator and a multiple of another generator (corresponds to handle slides),
- replacing a generator $[q]$ with $\pm g[q]$ where $g \in \pi$ (corresponds to replacing γ_q with $\gamma_q * g^{-1}$)
- replacing $d_n\colon C_n \to C_{n-1}$ with $d_n \otimes \mathrm{id}\colon C_n \oplus \mathbb{Z}[\pi] \to C_{n-1} \oplus \mathbb{Z}[\pi]$ (corresponds to creating a pair of handles).

We can study the cobordism W algebraically. To this end, consider the set of all acyclic chain complexes of free $\mathbb{Z}[\pi]$-modules C_n with ordered sets of generators and such that only finitely many terms C_n are non-zero. There is an operation on such chain complexes given by term-wise sum operation. Furthermore, the above four modifications of chain complexes is an equivalence relation. It turns out that the set of equivalence classes of chain complexes with term-wise sum operation is a group $\mathrm{Wh}(\pi)$, called the *Whitehead group*. In particular, every h-cobordism W gives rise to an element $\tau(W)$ in the Whitehead group of $\pi = \pi_1 W$. This element is called the *Whitehead torsion* of the h-cobordism W.

99.6.2 The s-cobordism theorem

By definition, an h-cobordism W is an s-cobordism if the Whitehead torsion $\tau(W)$ is trivial.

Theorem 99.18. *An h-cobordism W of a manifold M of dimension $m \geq 1$ is trivial if and only if it is an s-cobordism.*

Sketch of the proof. Handle rearrangements of a handle decomposition of W do not change the Whitehead torsion $\tau(W)$. Therefore if W is a trivial h-cobordism, then $\tau(W) = 0$. Suppose now that W is an s-cobordism. Let us show that W is a trivial cobordism.

To begin with since the inclusion of the manifold M into the cobordism manifold W induces an isomorphism $\pi_0 M \to \pi_0 W$, we may pair all 0-handles of the cobordism W with some of the 1-handles that each pair is canceling. In other words, we may assume that W has no 0-handles. All 1-handles can be traded for 3-handles, and so on. Similarly, we may flip the cobordism W up side down and repeat the argument. As a result, we obtain a new handle decomposition of W with only handles in the middle degrees q and $q+1$. Under these handle rearrangements, the chain complex turn into one with only non-trivial modules C_q and C_{q+1}, i.e., the new chain complex is of the form

$$\cdots \to 0 \to C_{q+1} \xrightarrow{d_{q+1}} C_q \to 0 \to \cdots,$$

where C_{q+1} and C_q are free $\mathbb{Z}[\pi]$ modules with ordered bases. Since $\tau(W) = 0$, there is a sequence of further modifications of the chain complex that turn d_{q+1} into the identity map. Each elementary modification of the chain complex corresponds to a handle rearrangement of W. Therefore, there is a sequence of handle rearrangements, after which the q and $q+1$-handles of W are paired, and, hence, could be cancelled.[37] □

[37] For details, see Proposition 8.31 and Theorem 8.33 in the book **Algebraic and Geometric Surgery** by A. Ranicki.

Exercise 99.19. Suppose that in an acyclic chain complex C_* the modules $C_i = 0$ for $i < q$. Show that trading q-handles for $(q+2)$-handles corresponds to replacing the chain complex

$$\to C_{q+3} \to C_{q+2} \to C_{q+1} \to C_q \to 0$$

with the chain complex

$$\to C_{q+3} \xrightarrow{d'_{q+3}} C_{q+2} \oplus C_q \xrightarrow{d'_{q+2}} C_{q+1} \to 0 \to 0$$

where d'_{q+3} is the map $(d_{q+3}, 0)$, while the map d'_{q+2} is the sum of d_{q+2} and $\Gamma\colon C_q \to C_{q+1}$, where Γ is a right inverse of d_{q+1}. More generally, let $\Gamma\colon C \to C$ be a chain contraction. Then there is a sequence of rearrangements of handles in the decomposition of W which turn the chain complex C into a chain complex of the form

$$\cdots \to 0 \to C_{odd} \xrightarrow{d+\Gamma} C_{even} \to 0 \to \cdots$$

where $C_{odd} = \oplus C_{q+i}$ with i ranging over all odd integers, while $C_{even} = \oplus C_{q+j}$ with j ranging over all even integers. Consequently, the group $\mathrm{Wh}(\pi)$ can be defined as a group of equivalence classes of matrices.

Bibliography

[BM51] A. L. Blakers and W. S. Massey, The homotopy groups of a triad. I, Ann. of Math., vol. 53 (1951),161-205.

[Bra69] W. Browder, The Kervaire invariant of framed manifolds and its generalization. Ann of Math. 90 (1969), 157-186.

[Br93] G. E. Bredon, Topology and geometry, Graduate texts in Mathematics, Springer-Verlag New York Inc., 1993.

[Br72] W. Browder, Surgery on simply-connected manifolds. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 65. Springer-Verlag, New York-Heidelberg, 1972. ix+132 pp.

[Ca40] S. Cairns, Homeomorphisms between topological manifolds and analytic manifolds. Ann. of Math. (2) 41 (1940), 796-808.

[Ce32] E. Čech, Höherdimensionale Homotopiegruppen, Verhandlungen des Internationalen Mathematikerkongress, Zürich, Orell Füssli.

[EM02] Y. Eliashberg, N. Mishachev, Introduction to the H-principle, American Mathematical Soc., 2002.

[FF16] A. Fomenko, D. Fuchs, Homotopical Topology, Graduated Texts in Mathematics, Springer International Publishing AG Switzerland, 2016.

[Fr38] H. Freudenthal, Über die Klassen der Sphärenabbildungen. I. Große Dimensionen, Compositio Mathematica, 5 (1938), 299-314.

[Gr86] M. Gromov, Partial differential relations, Springer, 1986.

[Ha02] A. Hatcher, Algebraic topology. (English summary) Cambridge University Press, Cambridge, 2002. xii+544 pp.

[HHR16] M. A. Hill, M. J. Hopkins, D. C. Ravenel, On the nonexistence of elements of Kervaire invariant one. Ann. of Math. (2) 184 (2016), no. 1, 1-262.

[Hi59] M.W. Hirsch, Immersions of manifolds, Trans. Amer. Math. Soc. 93 (1959), 242-276.

[Ho30] H. Hopf, Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche, Mathematische Annalen v. 104 (1930), 637 - 665.

[Hu35] W. Hurewicz. Beiträge zur Topologie der Deformationen. Nederl. Akad. Wetensch. Proc. Ser. A 38(1935), 112-119, 521-528; 39(1936), 117-126, 213-224.

[J54] I. James, On the iterated suspension, Quart. J. Math., Oxford Ser. 2, 5 (1954), 1-10.

[Ke60] M. Kervaire, A manifold which does not admit any differentiable structure. Comment. Math. Helv. 34 (1960), 257-270.

[KM63] M. Kervaire, J. Milnor, Groups of homotopy spheres. I. Ann. of Math. (2) 77 (1963), 504-537.

[KS77] R. Kirby; L. Siebenmann, Foundational essays on topological manifolds, smoothings, and triangulations. With notes by John Milnor and Michael Atiyah. Annals of Mathematics Studies, No. 88. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1977. vii+355 pp.

[Ko07] A. Kosinksi, Differential manifolds, Dover Publ., Inc., 2007.

[KS77] U. Koschorke, B. Sanderson, Geometric interpretations of the generalized Hopf invariant, Math. Scand. 41 (1977), 199-217.

[Lee13] John M. Lee, Introduction to Smooth Manifolds. Second edition. Graduate Texts in Mathematics, 218. Springer, New York, 2013. xvi+708 pp.

[Ly] David W. Lyons, An elementary introduction to the Hopf fibration, nilesjohnson.net/hopf-articles/Lyons_Elem-intro-Hopf-fibration.pdf.

[Mi56] J. Milnor, On manifolds homeomorphic to the 7-sphere, Ann. of Math. (2) 64, 399–405 (1956).

[Mi59] J. Milnor, Differentiable manifolds which are homotopy spheres. Unpublished notes, 1959.

[Mi61] J. Milnor, A procedure for killing homotopy groups of differentiable manifolds, Proc. Sympos. Pure Math. 3, Amer. Math. Soc., Providence, Rhode Island, 1963.

[Mi65] J. Milnor, Lectures on the h-cobordism theorem. Notes by L. Siebenmann and J. Sondow. Princeton University Press, Princeton, N.J. 1965.

[MiSt74] J. Milnor, J. Stasheff, Characteristic classes, Annals of mathematical studies 76, Prinston university press, 1974.

[Mo25] M. Morse, Relations between the critical points of a real function of n independent variables, Trans. Amer. Math. Soc. 27 (1925), 345-396.

[Moo96] J. Moore, Lecture Notes on Seiberg-Witten Invariants, Lecture Notes in Mathematics 1629, Springer-Verlag Berlin Heidelberg, 1996.

[No66] S. P. Novikov. On manifolds with free abelian fundamental group and their application. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 30 (1966), 207-246.

[No66a] S. P. Novikov. Pontrjagin classes, the fundamental group and some problems of stable algebra. 1968 Amer. Math. Soc. Translations Ser. 2, Vol. 70: 31 Invited Addresses (8 in Abstract) at the Internat. Congr. Math. (Moscow, 1966) pp. 172-179 Amer. Math. Soc., Providence, R.I.

[Po50] L. S. Pontryagin, Homotopy classification of the mappings of an $(n+2)$-dimensional sphere on an n-dimensional one. (Russian) Doklady Akad. Nauk SSSR (N.S.) 70, (1950). 957?959.

[Po85] L. S. Pontryagin, Smooth manifolds and their applications in homotopy theory. Third edition. "Nauka”, Moscow, 1985. 175 pp.

[Pu02] Ch. Pugh, Smoothing a topological manifold. Topology Appl. 124 (2002), no. 3, 487-503.

[RS01] C. Rourke, B. Sanderson, The compression theorem. I. Geom. Topol. 5 (2001), 399-429.

[RS01a] C. Rourke, B. Sanderson, The compression theorem. II. Directed embeddings. Geom. Topol. 5 (2001), 431-440.

[RS03] C. Rourke, B. Sanderson, The compression theorem. III. Applications. Algebr. Geom. Topol. 3 (2003), 857-872.

[Sm58] S. Smale, A classification of immersions of the two-sphere, Trans. Amer. Math. Soc. 90 (1958), 281-290.

[Sm59] S. Smale, The classification of immersions of spheres in Euclidean spaces, Ann. Math. 69 (1959), 27-344.

[Sm60] S. Smale, The generalized Poincaré conjecture in higher dimensions, Bull. Amer. Math. Soc. 66 (1960), 373-375.

[Wall62] C. T. C. Wall, Killing the middle homotopy groups of odd dimensional manifolds. Trans. Amer. Math. Soc. 103 (1962), 421-433.

[Wall66] C. T. C. Wall. Surgery of non-simply-connected manifolds. Ann. of Math. (2) 84 (1966), 217-276.

[Wall70] C. T. C. Wall. Surgery on compact manifolds. Second edition. Edited and with a foreword by A. A. Ranicki. Mathematical Surveys and Monographs, 69. American Mathematical Society, Providence, RI, 1999. xvi+302 pp.

[Wa60] A. H. Wallace, Modifications and cobounding manifolds, Canad. J. Math. 12 (1960), 503-528.

[Wa62] A. H. Wallace, A geometric method in differential topology, Bull. Amer. Math. Soc. 68 (1962), 533-542.

[Wh35] H. Whitney, Differentiable manifolds in Euclidean space, Proceedings of the National Academy of Sciences of the United States of America, Vol. 21, No. 7 (1935), pp. 462-464.

[Wh53] G. W. Whitehead, On the Freudenthal theorem, Ann. of Math.

57, No. 2, (1953), 209-228.

[Wh61] J. H. Whitehead, Manifolds with transverse fields in euclidean space. Ann. of Math. (2) 73 (1961), 154-212.

[Wh78] G. W. Whitehead, Elements of homotopy theory, Graduate Texts in Mathematics, Springer-Verlag, New York, Hiedelberg, Berlin, 1978.