UNEXPECTED RELATIONS OF COBORDISM CATEGORIES WITH ANOTHER SUBJECTS IN MATHEMATICS

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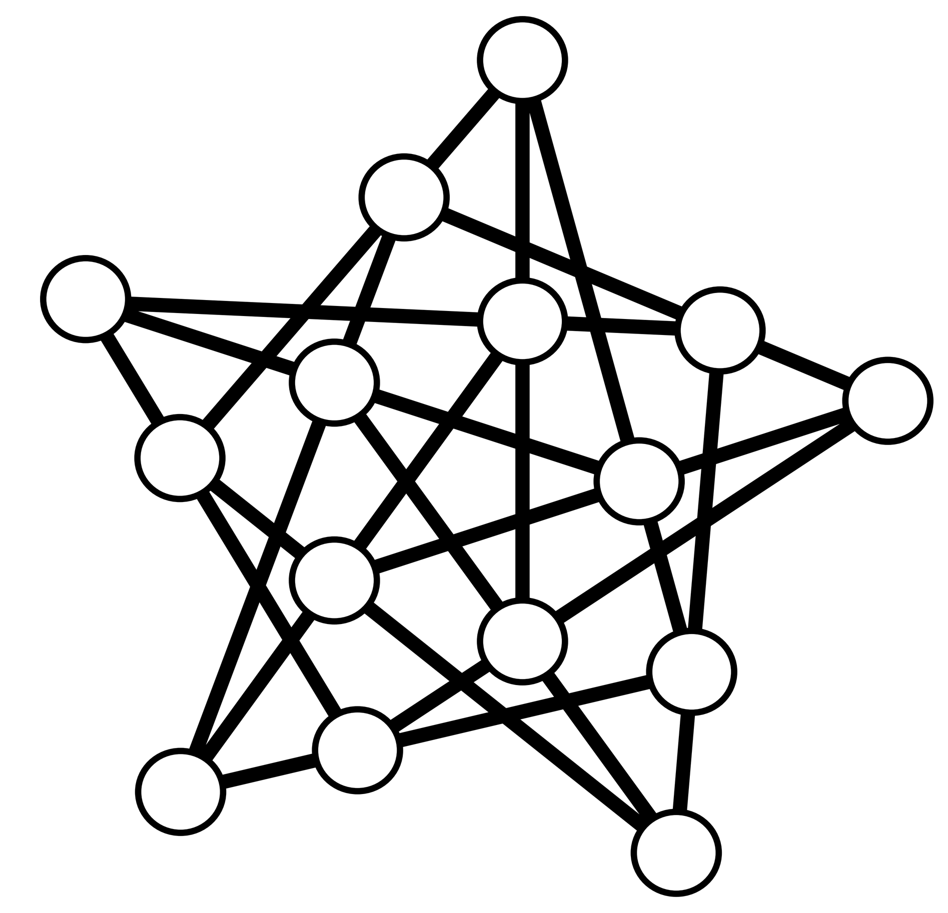


Cremona-Richmond configuration

1. A-B-C
2. A-K-L
3. D-A-E
4. D-G-F
5. E-I-H
6. J-D-M
7. E-O-N
8. B-H-F
9. J-B-O
10. C-G-I
11. C-M-N
12. F-N-K
13. M-H-L
14. G-O-L
15. J-I-K

A	0001 0010	B	0001 1000	C	0001 1010	D	0010 0100	E	0010 0101
F	0100 1000	G	0100 1010	H	0101 1000	I	0101 1010	J	0110 1001
K	0011 1100	L	0011 1101	M	0110 1011	N	0111 1011	O	0111 1001

Fill the crossword

1. The dimension of the universal embedding of the symplectic polar space.
2. The density of a language with four letters
3. The rank of the $\mathbb{Z}_2^n$-cobordism category in dimension $1+1$.

1, 2, 5, 15, 51, 187, 715, 2795, 11051, 43947, 175275, 700075, 2798251, 11188907, 44747435, 178973355, 715860651, 2863377067, 11453377195, 45813246635, 183252462251, 733008800427, 2932033104555, 11728128223915

$$g_2(n) = \frac{(2^n+1)(2^{n-1}+1)}{3} \qquad g_p(n) = \frac{p^{2n-1}+p^{n+1}-p^{n-1}+p^2-p-1}{p^2-1}$$

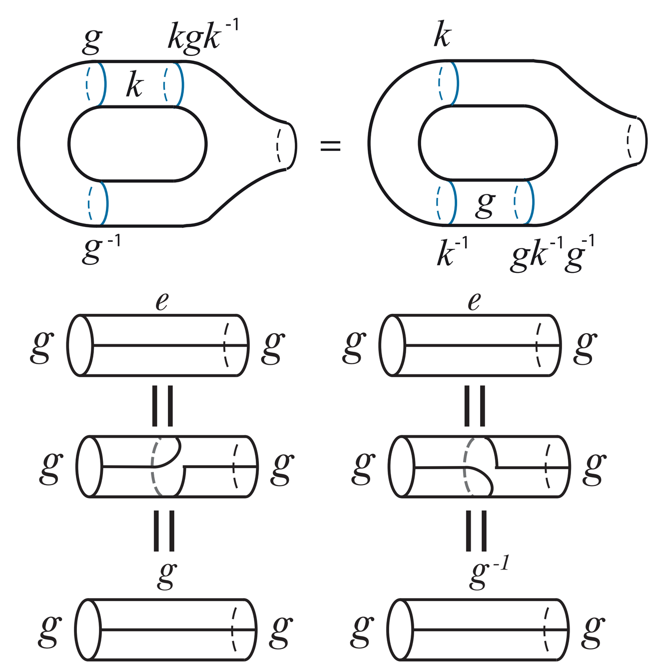
1. The dimension of the universal embedding of the symplectic polar space

Consider a $\mathbb{Z}_2$-vector space of dimension $2n$ with a symplectic form ω. Consider the geometry with lines of three elements defined as follows. The points are the maximal totally isotropic subspaces of dimension n, i.e. $\omega(V)=0$ for a V a subspace. The lines are given bye the totally isotropic subspaces of dimension $n-1$. Denote X and $\mathcal{L}$ the sets of points and lines respectively. We consider the linear map $\sigma : \mathbb{Z}_2\mathcal{L} \longrightarrow \mathbb{Z}_2X$ sending each line to the sum of its three elements. The dimension of the universal embedding of the symplectic polar space is the dimension of the module $\mathbb{Z}_2X/\sigma(\mathbb{Z}_2\mathcal{L})$. For example for $n=2$ we have $X=\{(0,1),(1,0),(1,1)\}$ with only one line. For $n=3$ the geometry gives the Cremona-Richmond configuration.

3. The rank of the $\mathbb{Z}_2^n$-cobordism category in dimension $1+1$

We considers the cardinality of the quotient of $\mathbb{Z}_2^n \times \mathbb{Z}_2^n$ under the action of the special linear group $\mathrm{SL}(2,\mathbb{Z})$. This group is generated by two matrices which produce essentially two basic equations $(g,k)\sim(k,-g)$ and $(g,k)\sim(g,k+mg)$. The orbits of this quotient gives a set of generators for the monoid of principal $\mathbb{Z}_2^n$-bundles over closed surfaces with two boundary circles up to a homeomorphism identification. For $n=1$, we get two orbits $(0,0)$ and $(0,1)\sim(1,0)\sim(1,1)$. For $n=2$, we get 5 orbits

1. $\begin{pmatrix}0&0\\0&0\end{pmatrix}$
2. $\begin{pmatrix}0&0\\0&1\end{pmatrix}\sim\begin{pmatrix}0&0\\1&0\end{pmatrix}\sim\begin{pmatrix}0&0\\1&1\end{pmatrix}$
3. $\begin{pmatrix}0&1\\0&0\end{pmatrix}\sim\begin{pmatrix}1&0\\0&0\end{pmatrix}\sim\begin{pmatrix}1&1\\0&0\end{pmatrix}$
4. $\begin{pmatrix}0&1\\0&1\end{pmatrix}\sim\begin{pmatrix}1&0\\1&0\end{pmatrix}\sim\begin{pmatrix}1&1\\1&1\end{pmatrix}$
5. $\begin{pmatrix}0&1\\1&1\end{pmatrix}\sim\begin{pmatrix}1&1\\1&0\end{pmatrix}\sim\begin{pmatrix}1&0\\1&1\end{pmatrix}\sim\begin{pmatrix}1&1\\0&1\end{pmatrix}\sim\begin{pmatrix}0&1\\1&0\end{pmatrix}$.



No.	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7
1	111**1**	111**2**	21**1**2	21**2**2	21**3**2	23**4**2	**1**122
2	112**1**	112**3**	23**1**2	23**2**2	23**3**2	23**4**3	12**33**
3	121**1**	121**3**	12**1**2	12**2**2	12**3**2		
4	122**1**	122**3**	22**1**2	22**2**2	22**3**2		
5	123**1**	123**4**	23**1**3	23**2**3	23**3**3		
6	211**1**	211**3**					
7	212**1**	212**3**					
8	213**1**	213**4**					
9	221**1**	221**3**					
10	222**1**	222**3**					
11	223**1**	223**4**					
12	231**1**	231**4**					
13	232**1**	232**4**					
14	233**1**	233**4**					
15	234**1**	234**4**					

(N1) $\mathrm{wt}(v_i) \le 2$ for every $i \in \{1,\cdots,k\}$.
(N2) If $v_i \succ v_j$ (i.e., $i<j$) and $\mathrm{wt}(v_i)=\mathrm{wt}(v_j)=2$, then $\beta(v_i)\le\beta(v_j)$.
(N3) If $v_i \succ v_j \succ v_k$, $\mathrm{wt}(v_i)=\mathrm{wt}(v_j)=\mathrm{wt}(v_k)=2$, and $\beta(v_i)=\beta(v_j)<\beta(v_k)$, then $\alpha(v_k)>\beta(v_i)$.
(N4) There do not exist $v_i \succ v_j \succ v_k \succ v_l$ such that $\mathrm{wt}(v_i)=\mathrm{wt}(v_j)=\mathrm{wt}(v_k)=\mathrm{wt}(v_l)=2$ and $\beta(v_i)=\beta(v_j)=\beta(v_k)<\beta(v_l)$.

Reference

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2. The density of a language with four letters

The density of a language with four letters is defined by counting words as follows. Take words with letters 1,2,3,4 of length n with the property that from left to right each letter satisfies $0 \le a_i \le \max_{j\le i}\{a_j\}+1$. Thus the first letter is always 1, so we can dismiss it. Consequently, for $n=2$ there are two words 1 and 2, for $n=3$ the words are 11, 12, 21, 22, 23 while for $n=4$ we have 15 words

111	112	121	122	123
211	212	213	221	222
223	231	232	233	234

2 ⇔ 3

As an illustration for $n=2$

$$\begin{matrix}1\\1\end{matrix}\longmapsto\begin{pmatrix}0&0\\0&0\end{pmatrix}\ \begin{matrix}1\\2\end{matrix}\longmapsto\begin{pmatrix}0&0\\1&0\end{pmatrix}\ \begin{matrix}2\\1\end{matrix}\longmapsto\begin{pmatrix}1&0\\0&0\end{pmatrix}$$

$$\begin{matrix}2\\2\end{matrix}\longmapsto\begin{pmatrix}1&0\\1&0\end{pmatrix}\ \begin{matrix}2\\3\end{matrix}\longmapsto\begin{pmatrix}1&0\\1&1\end{pmatrix}.$$

When we have words with a letter with value 4 we forget the first two zeros as follows

$$\begin{matrix}2\\3\\4\end{matrix}\longmapsto\begin{matrix}1\ 0\\1\ 1\\0\ 1\ 0\ 0\end{matrix}\longmapsto\begin{pmatrix}1&0\\1&1\\0&1\end{pmatrix}.$$

No.	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7
1	0000	0001	1001	$\begin{bmatrix}1001\\0010\end{bmatrix}$	$\begin{bmatrix}1001\\0011\end{bmatrix}$	$\begin{bmatrix}1010\\0101\end{bmatrix}$ 23**4**2	0011
2	0010	$\begin{bmatrix}0010\\0001\end{bmatrix}$	$\begin{bmatrix}1001\\0100\end{bmatrix}$	$\begin{bmatrix}1001\\0100\\0010\end{bmatrix}$	$\begin{bmatrix}1001\\0100\\0011\end{bmatrix}$	$\begin{bmatrix}1010\\0011\end{bmatrix}$ 23**4**3	$\begin{bmatrix}0100\\0011\end{bmatrix}$
3	0100	$\begin{bmatrix}0100\\0001\end{bmatrix}$	0101	$\begin{bmatrix}0101\\0010\end{bmatrix}$	$\begin{bmatrix}0101\\0011\end{bmatrix}$		
4	0110	$\begin{bmatrix}0110\\0001\end{bmatrix}$	$\begin{bmatrix}1100\\0101\end{bmatrix}$	$\begin{bmatrix}1100\\0101\\0010\end{bmatrix}$	$\begin{bmatrix}1100\\0101\\0011\end{bmatrix}$		
5	$\begin{bmatrix}0110\\0010\end{bmatrix}$	$\begin{bmatrix}0100\\0010\\0001\end{bmatrix}$	$\begin{bmatrix}1000\\0101\end{bmatrix}$	$\begin{bmatrix}1000\\0101\\0010\end{bmatrix}$	$\begin{bmatrix}1000\\0101\\0011\end{bmatrix}$		
6	1000	$\begin{bmatrix}1000\\0001\end{bmatrix}$					
7	1010	$\begin{bmatrix}1010\\0001\end{bmatrix}$					
8	$\begin{bmatrix}1000\\0010\end{bmatrix}$	$\begin{bmatrix}1000\\0010\\0001\end{bmatrix}$					
9	1100	$\begin{bmatrix}1100\\0001\end{bmatrix}$					
10	$\begin{bmatrix}1100\\0110\end{bmatrix}$	$\begin{bmatrix}1100\\0110\\0001\end{bmatrix}$					
11	$\begin{bmatrix}1100\\0110\end{bmatrix}$	$\begin{bmatrix}1100\\0110\\0001\end{bmatrix}$					
12	$\begin{bmatrix}1100\\0100\end{bmatrix}$	$\begin{bmatrix}1100\\0100\\0001\end{bmatrix}$					
13	$\begin{bmatrix}1010\\0100\end{bmatrix}$	$\begin{bmatrix}1010\\0100\\0001\end{bmatrix}$					
14	$\begin{bmatrix}1000\\0110\end{bmatrix}$	$\begin{bmatrix}1000\\0110\\0001\end{bmatrix}$					
15	$\begin{bmatrix}1000\\0100\\0010\end{bmatrix}$	$\begin{bmatrix}1000\\0100\\0010\\0001\end{bmatrix}$					