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Configuration spaces and peak representations

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Abstract. *Eulerian* idempotents of types A and B generate representations with topological interpretations, as the cohomology of configuration spaces of types A and B. We provide an analogous cohomological interpretation for the representations generated by idempotents in the *peak algebra*, called the *peak representations*. We describe the peak representations as sums of *Thrall's higher Lie characters*, give Hilbert series and branching rule recursions for them, and discuss connections to Jordan algebras.

Keywords: Peak algebra, configuration spaces, Solomon's descent algebra, higher lie characters, hyperplane arrangements, Varchenko-Gelfand ring, Type A, Type B

1 Introduction

This abstract concerns the cohomology $H^*X = H^*(X, \mathbf{k})$ with coefficients in a field $\mathbf{k}$ for three different topological configuration spaces $X = X_n, Y_n, Z_n$ having large symmetry groups W. For each, the (ungraded) cohomology carries the regular representation of W, that is, $H^*X \cong \mathbf{k}\,W$. Our goal is to study and exploit the following surprising fact: for $\mathbf{k}$ of characteristic zero, the decomposition into H^iX matches a combinatorial direct sum decomposition for certain complete families $\{E_i\}$ of *orthogonal idempotents* in $\mathbf{k}\,W$:

$$H^*X = \bigoplus_i H^iX \qquad \cong \qquad \bigoplus_i (\mathbf{k}\,W)E_i = \mathbf{k}\,W. \tag{1.1}$$

The first two spaces X_n, Y_n are well-studied: X_n is the *ordered configuration space* of n points in $\mathbb{R}^3$ while Y_n is the $\mathbb{Z}_2$*-orbit configuration spaces* for the $\mathbb{Z}_2$-action via $\mathbf{x} \mapsto -\mathbf{x}$:

$$X_n := \mathrm{Conf}_n\,\mathbb{R}^3 = \{\mathbf{x} \in (\mathbb{R}^3)^n : x_i \neq x_j \text{ for } 1 \leq i < j \leq n\},$$
$$Y_n := \mathrm{Conf}_n^{\mathbb{Z}_2}\,\mathbb{R}^3 = \{\mathbf{x} \in (\mathbb{R}^3)^n : x_i \neq \pm x_j \text{ for } 1 \leq i < j \leq n, \text{ and } x_i \neq 0 \text{ for } 1 \leq i \leq n\}$$

Note that X_n has an action of the *symmetric group* $W = \mathfrak{S}_n$ permuting the coordinates of $\mathbf{x}$, while Y_n carries an action of the *hyperoctahedral group* $W = \mathfrak{S}_n^{\pm}$ by permuting and negating coordinates. Both spaces have cohomology concentrated only in even degrees and total cohomology carrying the regular representation $\mathbf{k}\,W$ for $W = \mathfrak{S}_n, \mathfrak{S}_n^{\pm}$.

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The idempotent decompositions of $\mathbf{k}\,\mathfrak{S}_n$ and $\mathbf{k}\,\mathfrak{S}_n^{\pm}$ will come from the *type A and B Eulerian idempotents* $\{E_k^{\mathfrak{S}_n}\}_{k=0,1,\dots,n-1}$ in $\mathbf{k}\,\mathfrak{S}_n$ and $\{E_k^{\mathfrak{S}_n^{\pm}}\}_{k=0,1,\dots,n}$ in $\mathbf{k}\,\mathfrak{S}_n^{\pm}$, defined in work of Reutenauer [13], Gerstenhaber–Schack [10], and F. Bergeron and N. Bergeron [4].

The Eulerian idempotents lie within the subalgebras of the group algebras $\mathbf{k}\,W$ known as *Solomon's descent algebra* $\mathrm{Sol}(W)$, meaning that when expressed as $\sum_{w\in W} c_w w$, their coefficients c_w depend only upon the Coxeter group *descent set* of w. Work of Hanlon [11], Sundaram-Welker [16] and Brauner [6] gives a correspondence between these objects:

$$H^{2k}X_n \cong (\mathbf{k}\,\mathfrak{S}_n)\, E_{n-1-k}^{\mathfrak{S}_n} \text{ for } k = 0, 1, \dots, n-1, \tag{1.2}$$

$$H^{2k}Y_n \cong \left(\mathbf{k}\,\mathfrak{S}_n^{\pm}\right) E_{n-k}^{\mathfrak{S}_n^{\pm}} \quad \text{for } k = 0, 1, \dots, n. \tag{1.3}$$

In this abstract, we use (1.2) and (1.3) as the starting point to give a third correspondence of the form (1.1) for the space $Z_n := Y_n / \mathbb{Z}_2^n \cong \mathrm{Conf}_n(\mathbb{RP}^2 \times (0,\infty))$, where $\mathbb{Z}_2^n$ is the normal subgroup of $\mathfrak{S}_n^{\pm}$ consisting of sign changes; thus $\mathfrak{S}_n \cong \mathfrak{S}_n^{\pm} / \mathbb{Z}_2^n$ acts on Z_n.

The idempotents $\{E_k^{\mathcal{P}_n}\}$ in this new correspondence lie inside the *peak algebra* $\mathcal{P}_n$, which is the further subalgebra of $\mathrm{Sol}(\mathfrak{S}_n)$ inside $\mathbf{k}\,\mathfrak{S}_n$ whose elements $\sum_{w\in W} c_w w$ have coefficients c_w depending only upon the *peak set* of $w = (w_0 := 0, w_1, \dots, w_n)$

$$\mathrm{Peak}(w) := \{i : 1 \le i \le n-1 \text{ and } w_{i-1} < w_i > w_{i+1}\}.$$

Our main contribution is to relate the *peak representations* $(\mathbf{k}\,\mathfrak{S}_n)\, E_{n-k}^{\mathcal{P}_n}$ to the cohomology ring H^*Z_n, and to explicitly describe these families of representations in terms of Thrall's famed *higher Lie characters* Lie_λ for λ an integer partition of n.

Theorem 1.1. *Let $\mathbf{k}$ be a field of characteristic zero.*

(i) The peak idempotent $E_k^{\mathcal{P}_n}$ in $\mathbf{k}\,\mathfrak{S}_n$ vanishes unless $k \equiv n \bmod 2$.

(ii) The cohomology $H^iZ_n = H^i(Z_n, \mathbf{k})$ vanishes unless $i \equiv 0 \bmod 4$.

(iii) As a $\mathfrak{S}_n$-representation, the total cohomology carries the regular representation:

$$H^*Z_n \cong \mathbf{k}\,\mathfrak{S}_n.$$

(iv) For $0 \le k \le n$ with k even, one has $\mathfrak{S}_n$-representation isomorphisms

$$(\mathbf{k}\,\mathfrak{S}_n)\, E_{n-k}^{\mathcal{P}_n} \cong H^{2k}Z_n \cong \bigoplus_{\substack{\lambda \vdash n: \\ \mathrm{odd}(\lambda) = n-k}} \mathrm{Lie}_\lambda,$$

where $\mathrm{odd}(\lambda)$ is the number of odd parts of λ.

In fact, we refine Theorem 1.1 (see Theorems 4.4 and 4.6) by introducing several (compatible) decompositions of H^*Z_n and a family of primitive idempotents in $\mathcal{P}_n$.

Although $\mathcal{P}_n$ is a well-known subalgebra of $\mathrm{Sol}(\mathfrak{S}_n)$, it is in general difficult to directly relate the two algebras. Our work offers a step in this direction. The novelty of our approach is to avoid computations in the algebras themselves, and instead develop and utilize concrete combinatorial descriptions of the rings H^*X_n, H^*Y_n, and H^*Z_n.

The remainder of the abstract proceeds as follows. Section 2 gives necessary background on the Type A and B stories. We then develop properties of H^*Y_n in Section 3, which will be instrumental in proving our main results on the peak representations in Section 4. In Section 5 we provide generating function formulae and branching rule recursions for the peak representations, and relate this story to the free Jordan algebra.

2 Background

We review here in more detail the spaces X_n, Y_n, their cohomology rings, and their relationship to the Eulerian idempotents and Lie characters Lie_λ discussed in Section 1.

2.1 The (associated graded) Varchenko-Gelfand ring

The cohomology rings $\mathcal{X}_n := H^*X_n$ and $\mathcal{Y}_n := H^*Y_n$ are closely related to the *reflection hyperplane arrangements* $\mathcal{A}_W \subset V = \mathbb{R}^n$ associated to the groups $W = \mathfrak{S}_n, \mathfrak{S}_n^{\pm}$:

$$\mathcal{A}_{\mathfrak{S}_n} = \{x_i = x_j\}_{1\le i<j\le n} \qquad \mathcal{A}_{\mathfrak{S}_n^{\pm}} = \{x_i = 0\}_{1\le i\le n} \sqcup \{x_i = \pm x_j\}_{1\le i<j\le n}.$$

In particular, Moseley [12] proved there are algebra isomorphisms

$$\mathcal{X}_n \cong \mathcal{VG}(\mathcal{A}_{\mathfrak{S}_n}) \qquad \mathcal{Y}_n \cong \mathcal{VG}(\mathcal{A}_{\mathfrak{S}_n^{\pm}}),$$

where $\mathcal{VG}(\mathcal{A})$ is the *(associated graded) Varchenko-Gelfand ring*, defined for any real hyperplane arrangement $\mathcal{A} \subset \mathbb{R}^n$ as the quotient of $\mathbf{k}[u_i]_{H_i\in\mathcal{A}}$ by an ideal[1]

$$\mathcal{J}_A = \langle u_i^2,\ \sum_{j=1}^{c} \epsilon(C, i_j)\cdot u_{i_1}u_{i_2}\cdots\widehat{u_{i_j}}\cdots u_{i_{c-1}}u_{i_c} \text{ for all } C\subset\mathcal{A}\rangle.$$

Here $C = (C_+, C_-)$ is an *oriented matroid* signed circuit of $\mathcal{A}$, with $\epsilon(C, i_j) = \pm 1$, depending on whether i_j lies in C_+ or C_-.

Example 2.1. When $\mathcal{A} = \mathcal{A}_{\mathfrak{S}_n}$, work of Arnol'd [2] and Cohen [8] shows that $\mathcal{X}_n$ has presentation given by

$$\mathcal{X}_n \cong \mathcal{VG}(\mathcal{A}_{\mathfrak{S}_n}) = \mathbf{k}[u_{ij}]_{1\le i<j\le n}/\langle u_{ij}^2,\ \ u_{ij}u_{ik} - u_{ij}u_{jk} + u_{ik}u_{jk}\rangle.$$

Barcelo [3] constructed an elegant *non-broken circuit* monomial basis for $\mathcal{X}_n$, obtained by taking products with at most one element from each set U_i below:

$$U_1 = \{u_{12}\},\ U_2 = \{u_{13},\ u_{23}\}, \cdots,\ U_{n-1} = \{u_{1n},\ u_{2n},\ \cdots,\ u_{(n-1),n}\}.$$

[1] In fact, one can take coefficients in $\mathbb{Z}$ rather than $\mathbf{k}$. However, in what follows, we will want $\mathbf{k}$ to be a field with characteristic not dividing 2.

In [6], the second author showed that $\mathcal{VG}(\mathcal{A})$ admits a decomposition by intersection subspaces (i.e. flats) in $\mathcal{A}$. The component of $\mathcal{VG}(\mathcal{A})_X$ indexed by X is the $\mathbb{Z}$-span of all monomials $\{u_{i_1} \cdots u_{i_\ell}\}$ for which $H_{i_1} \cap \cdots \cap H_{i_\ell} = X$.

In the case of a reflection arrangement $\mathcal{A}_W$, we can group flats by their W-orbits $[X]$, which gives a coarser decomposition of $\mathcal{VG}(\mathcal{A}_W) = \bigoplus \mathcal{VG}(\mathcal{A}_W)_{[X]}$. The flats and flat orbits in $\mathcal{A}_{\mathfrak{S}_n}$ and $\mathcal{A}_{\mathfrak{S}_n^\pm}$ have elegant (and useful!) combinatorial descriptions.

Famously, the flats of $\mathcal{A}_{\mathfrak{S}_n}$ biject with set partitions of $[n]$. This isomorphism identifies a flat X with the set partition $\pi_X = \{B_1, \cdots, B_k\}$ where i and j are in the same block B_ℓ if and only if $x_i = x_j$ in X. The $\mathfrak{S}_n$-orbits of these flats biject with integer partitions of n: the orbit of π_X corresponds to the partition $\lambda_X = \{|B_1|, \cdots, |B_k|\}$.

Similarly, the flats in $\mathcal{A}_{\mathfrak{S}_n^\pm}$ can be identified with a set partition on a *subset* S of $[n]^\pm := \{\bar{1}, \bar{2}, \cdots, \bar{n}, 1, 2, \cdots n\}$, where S does not contain both i and $\bar{i}$. Given a flat X, identify $\bar{i}$ with $-x_i$ and let $\tau_X = \{C_1, \cdots C_k\}$ where for $i, j \in [n]$, indices i and j (resp. i and $\bar{j}$) appear in the same block C_ℓ if and only if $x_i = x_j \neq 0$ (resp. if and only if $x_i = -x_j \neq 0$) in X. Note that two set partitions related by $i \mapsto \bar{i}$ correspond to the same flat. The $\mathfrak{S}_n^\pm$ orbit of τ_X is indexed by a partition $\mu_X = \{|C_i|, \cdots, |C_k|\}$ of $0 \leq m \leq n$.

We write $\mathcal{X}_{\lambda_X}^{(n)} := \mathcal{VG}(\mathcal{A}_{\mathfrak{S}_n})_{[\pi_X]}$ and $\mathcal{Y}_{\mu_X}^{(n)} := \mathcal{VG}(\mathcal{A}_{\mathfrak{S}_n^\pm})_{[\tau_X]}$, giving the decompositions

$$\mathcal{X}_n = \bigoplus_{\lambda \vdash n} \mathcal{X}_\lambda^{(n)} \qquad \mathcal{Y}_n = \bigoplus_{\mu \vdash 0 \leq m \leq n} \mathcal{Y}_\mu^{(n)}.$$

2.2 The Eulerian idempotents and higher Lie characters

The idempotents $\{E_k^{\mathfrak{S}_n}\}$ and $\{E_k^{\mathfrak{S}_n^\pm}\}$ from Section 1 can be defined via the formula in [6]:

$$\sum_{k=0}^{r} t^k E_k^W = \frac{1}{|W|} \sum_{w \in W} \left(\prod_{i=1}^{\mathrm{des}(w)} (t - e_i) \prod_{i=1}^{r-\mathrm{des}(w)} (t + e_i) \right) \cdot w,$$

which recovers work of Garsia–Reutenauer [9] for $W = \mathfrak{S}_n$ and Bergeron–Bergeron [4] for $W = \mathfrak{S}_n^\pm$. Here, r is the *rank* of $\mathcal{A}_W$ ($r = n-1$ for $W = \mathfrak{S}_n$ and $r = n$ for $W = \mathfrak{S}_n^\pm$) and the e_i are the *exponents* of W ($e_i = i$ for $W = \mathfrak{S}_n$ and $e_i = 2i-1$ for $W = \mathfrak{S}_n^\pm$). The *descent number*, $\mathrm{des}(w)$ is the number of simple reflections s of W with $\ell(ws) < \ell(w)$.

The E_k^W have a refinement due to Bergeron–Bergeron–Howlett–Taylor [5], who introduced families of complete, primitive orthogonal idempotents in $\mathrm{Sol}(W)$ for any finite Coxeter group W. These idempotents, which we will call the *BBHT idempotents*, are indexed by W-flat orbits. We omit the technical definitions, but note that by the discussion in §2.1, for $W = \mathfrak{S}_n, \mathfrak{S}_n^\pm$ they can be indexed as $\{E_\lambda^{\mathfrak{S}_n} : \lambda \vdash n\}$ and $\{E_\mu^{\mathfrak{S}_n^\pm} : \mu \vdash m, m \leq n\}$.

To recover the $\{E_k^{\mathfrak{S}_n}\}$ and $\{E_k^{\mathfrak{S}_n^\pm}\}$, group $\{E_\lambda^{\mathfrak{S}_n}\}$ and $\{E_\mu^{\mathfrak{S}_n^\pm}\}$ by partition *length* ℓ:

$$E_k^{\mathfrak{S}_n} = \sum_{\lambda:\ \ell(\lambda)=k} E_\lambda^{\mathfrak{S}_n} \qquad E_k^{\mathfrak{S}_n^\pm} = \sum_{\mu:\ \ell(\mu)=k} E_\mu^{\mathfrak{S}_n^\pm}. \tag{2.1}$$

We can also refine the isomorphisms in (1.2) and (1.3) using the BBHT idempotents:

Theorem 2.2 (Brauner, [6]). *There are $\mathfrak{S}_n$ and $\mathfrak{S}_n^{\pm}$ representation isomorphisms*

$$\mathcal{X}_\lambda^{(n)} \cong (\mathbf{k}\,\mathfrak{S}_n)\, E_\lambda^{\mathfrak{S}_n} \qquad\qquad \mathcal{Y}_\mu^{(n)} \cong \left(\mathbf{k}\,\mathfrak{S}_n^{\pm}\right) E_\mu^{\mathfrak{S}_n^{\pm}}.$$

In fact, there is more to say in the case of $W = \mathfrak{S}_n$, relating to the *higher Lie representations* $\{\mathrm{Lie}_\lambda\}$ of Thrall [17]. Let $\mathcal{C}_\lambda$ be the conjugacy class of $\mathfrak{S}_n$ indexed by the partition $\lambda = (1^{m_1}, 2^{m_2}, \cdots n^{m_n})$. The centralizer Z_λ of an element of $\mathcal{C}_\lambda$ has isomorphism type

$$Z_\lambda \cong \prod_{j=1}^{n} \mathfrak{S}_{m_j}[\mathbb{Z}_j],$$

where $\mathbb{Z}_j$ is the cyclic group of order j, and $\mathfrak{S}_{m_j}[\mathbb{Z}_j]$ is the wreath product. Specifically, the action of $\mathfrak{S}_{m_j}$ in this wreath product swaps the m_j blocks of λ of size j.

We will be interested in a linear character ω_λ on Z_λ obtained from extending faithful characters on each $\mathbb{Z}_j$ to Z_λ, where ω_λ restricts trivially on the wreath factors $\mathfrak{S}_{m_j}$ of Z_λ.

Write $\uparrow_H^G$ to be the representation induction from a subgroup H of G to G.

Definition 2.3. Give a partition $\lambda \vdash n$, define $\mathrm{Lie}_\lambda := \omega_\lambda \uparrow_{Z_\lambda}^{\mathfrak{S}_n}$.

Thrall proved that $\mathbf{k}\,\mathfrak{S}_n \cong \bigoplus_{\lambda \vdash n} \mathrm{Lie}_\lambda$. A beautiful result of Hanlon [11] then shows that $\mathrm{Lie}_\lambda \cong (\mathbf{k}\,\mathfrak{S}_n) E_\lambda^{\mathfrak{S}_n}$. Using (2.1), we can thus conclude

$$(\mathbf{k}\,\mathfrak{S}_n)\, E_{n-1-k}^{\mathfrak{S}_n} \cong \bigoplus_{\substack{\lambda \vdash n: \\ \ell(\lambda) = n-k}} \mathrm{Lie}_\lambda \cong H^{2k} X_n.$$

Example 2.4. When $\lambda = (n)$, the representation $\mathrm{Lie}_n := \mathrm{Lie}_{(n)}$ is isomorphic to the multilinear component of the free Lie algebra, defined and generalized in §5.1.

3 Presentations, Filtrations, and Decompositions of $H^* Y_n$

Our first task is to study the ring $\mathcal{Y}_n := H^* Y_n$ in greater detail. It will be important for the remainder of this section to assume that **the field k has characteristic larger than** n, so that $2 \in \mathbf{k}^\times$ and $\mathbf{k}[\mathfrak{S}_n^{\pm}]$ is semisimple. This allows us to make an invertible change-of-variables that diagonalizes the action of the normal subgroup $\mathbb{Z}_2^n$ within $\mathfrak{S}_n^{\pm}$.

The presentation of $\mathcal{Y}_n \cong \mathcal{VG}(\mathcal{A}_{\mathfrak{S}_n^{\pm}})$ was first given by Xicotencatl [18]; it is isomorphic to $\mathbf{k}[u_{ij}^{+}, u_{ij}^{-}, u_i]/J_{\mathfrak{S}_n^{\pm}}$ for $1 \leq i < j \leq n$, with generators corresponding to

$$u_{ij}^{+} \longleftrightarrow \{x_i = x_j\} \qquad u_{ij}^{-} \longleftrightarrow \{x_i = -x_j\} \qquad u_i \longleftrightarrow \{x_i = 0\}$$

respectively. The generating relations for $\mathcal{J}_{\mathfrak{S}_n^{\pm}}$ are given in Table 1.

We will introduce a new basis for $\mathcal{Y}_n$, a filtration using that basis, and a corresponding associated graded ring. Along the way, we will see several useful decompositions of $\mathcal{Y}_n$.

Definition 3.1. For $1 \leq i < j \leq n$, define an isomorphism of graded $\mathbf{k}$-algebras $\mathcal{B}$ by

$$u_i \longmapsto u_i \qquad v_{ij} \longmapsto u_{ij}^+ + u_{ij}^- \qquad w_{ij} \longmapsto u_{ij}^+ - u_{ij}^-$$

with inverse given by $\mathcal{B}^{-1}(u_i) = u_i, \mathcal{B}^{-1}(u_{ij}^+) = \frac{1}{2}(v_{ij} + w_{ij}), \mathcal{B}^{-1}(u_{ij}^-) = \frac{1}{2}(v_{ij} - w_{ij})$.

We wish to rewrite the presentation $\mathcal{Y}_n := \mathbf{k}[u_{ij}^+, u_{ij}^-, u_i] / \mathcal{J}_{\mathfrak{S}_n^\pm}$ in terms of these new variables v_{ij}, w_{ij}, using a Gröbner basis argument. Introduce a lexicographic monomial ordering $\prec$ on $\mathbf{k}[v_{ij}, w_{ij}, u_i]$, in which the variables u_i, v_{ij}, w_{ij} are ordered as follows:

$$u_1 < u_2 < \cdots u_n < v_{12} < w_{12} < v_{13} < w_{13} < \cdots < v_{(n-1)1} < w_{(n-1)n}. \tag{3.1}$$

Theorem 3.2. *The isomorphism $\mathcal{B} : \mathbf{k}[v_{ij}, w_{ij}, u_i] \longrightarrow \mathbf{k}[u_{ij}^+, u_{ij}^-, u_i]$ induces a graded $\mathbf{k}$-algebra isomorphism, where $\mathcal{I}$ is generated by the relations $\mathcal{G}$ listed in Table 1 below:*

$$\mathbf{k}[v_{ij}, w_{ij}, u_i] / \mathcal{I} \xrightarrow{\sim} \mathbf{k}[u_{ij}^+, u_{ij}^-, u_i] / \mathcal{J}_{\mathfrak{S}_n^\pm} =: \mathcal{Y}_n,$$

Moreover, $\mathcal{G}$ gives a Gröbner basis for the ideal $\mathcal{I}$ with respect to $\prec$, in which the standard monomial $\mathbf{k}$-basis for the quotient $\mathbf{k}[v_{ij}, w_{ij}, u_i] / \mathcal{I}$ is the set of monomials $\mathcal{V}$ obtained from taking products with at most one element from each of these sets V_i:

$$V_1 = \{u_1\},\ V_2 = \{u_2,\ v_{12},\ w_{12}\},\ \cdots,\ V_n = \{u_n,\ v_{1n},\ w_{1n}, \cdots, v_{(n-1)n},\ w_{(n-1)n}\}.$$

We make two observations about the $\mathfrak{S}_n^\pm$ action on $\mathcal{Y}_n$. First, elements of $\mathbb{Z}_2^n \subset \mathfrak{S}_n^\pm$ scale all of u_i, v_{ij}, w_{ij} via ± 1; thus Theorem 3.2 will allow us to construct a monomial basis for $H^* Z_n \cong (\mathcal{Y}_n)^{\mathbb{Z}_2^n}$ in §4. Second, the generators segregate into two $\mathfrak{S}_n^\pm$-orbits: $\{u_i\}_{1 \leq i \leq n}$ and $\{v_{ij}, w_{ij}\}_{1 \leq i < j \leq n}$. This leads to a helpful *filtration*, as follows.

For $q \in \mathbf{k}[v_{ij}, w_{ij}, u_i]$, let $\deg(q)$ be the polynomial degree of q, $\deg_{\mathcal{V}}(q)$ to be the degree of q in the v_{ij} and w_{ij} variables, and $\deg_u(q)$ be the degree in the u_i variables. Our key insight is that $\mathcal{Y}_n$ admits a filtration by $\deg_u$. In particular, define the ideal

$$P^{(i)} := \{q \in \mathcal{Y}_n \subset \mathbf{k}[u_i, v_{ij}, w_{ij}] : \deg_u(q) \geq i\}.$$

For example, when $n = 2$ the ideal $P^{(1)}$ is the $\mathbf{k}$-span of $\{u_1,\ u_2,\ u_1 v_{12},\ u_1 w_{12},\ u_1 u_2\}$.

Proposition 3.3. *There are $\mathfrak{S}_n^\pm$-stable ascending filtrations on $\mathcal{Y}_n$ given by*

$$P^{(n)} \subset P^{(n-1)} \subset \cdots \subset P^{(1)} \subset P^{(0)}.$$

The associated graded ring $\overline{\mathcal{Y}_n} = \bigoplus_{i=0}^n P^{(i)} / P^{(i+1)}$ has presentation $\mathbf{k}[v_{ij}, w_{ij}, u_i] / \mathfrak{gr}(\mathcal{I})$ for $1 \leq i < j \leq n$, where the relations generating $\mathfrak{gr}(\mathcal{I})$ are given in Table 1.

The motivation for introducing and studying the associated graded ring $\overline{\mathcal{Y}_n}$ is that in our context (i.e. $\mathbf{k}\,\mathfrak{S}_n^\pm$ being a semisimple algebra), we have $\overline{\mathcal{Y}_n} \cong \mathcal{Y}_n$ as $\mathfrak{S}_n^\pm$-modules. Hence, it suffices to study the basis and representations on $\overline{\mathcal{Y}_n}$.

We will see that $\overline{\mathcal{Y}_n}$ has several useful decompositions that make studying the representations on $\mathcal{Y}_n$ (and eventually $H^* Z_n$) far more tractable.

Relations for $\mathcal{J}_{\mathfrak{S}_n^\pm}$	Relations for $\mathcal{I}$	Relations for $\mathfrak{gr}(\mathcal{I})$
u_i^2	u_i^2	u_i^2
$u_iu_{ij}^+ - u_iu_{ij}^- - u_{ij}^+u_{ij}^-$	$v_{ij}w_{ij}$	$v_{ij}w_{ij}$
$u_iu_j - u_iu_{ij}^- - u_ju_{ij}^-$	$u_iw_{ij} - u_jv_{ij}$	$u_iw_{ij} - u_jv_{ij}$
$(u_{ij}^+)^2$	$v_{ij}^2 - 2u_iw_{ij}$	v_{ij}^2
$(u_{ij}^-)^2$	$w_{ij}^2 + 2u_iw_{ij}$	w_{ij}^2
$u_iu_j - u_iu_{ij}^- - u_ju_{ij}^-$	$u_iv_{ij} - 2u_iu_j - u_jw_{ij}$	$u_iv_{ij} - u_jw_{ij}$
$u_{ij}^+u_{jk}^+ - u_{ij}^+u_{ik}^+ - u_{ik}^+u_{jk}^+$	$v_{ij}w_{jk} - w_{ij}w_{ik} - v_{ik}v_{jk}$	$v_{ij}w_{jk} - w_{ij}w_{ik} - v_{ik}v_{jk}$
$u_{ij}^-u_{jk}^+ - u_{ij}^-u_{ik}^- - u_{ik}^-u_{jk}^+$	$w_{ij}w_{jk} - v_{ij}w_{ik} - w_{ik}w_{jk}$	$w_{ij}w_{jk} - v_{ij}w_{ik} - w_{ik}w_{jk}$
$-u_{ij}^-u_{jk}^- + u_{ij}^-u_{ik}^+ - u_{ik}^+u_{jk}^-$	$v_{ij}v_{jk} - v_{ij}v_{ik} - v_{ik}w_{jk}$	$v_{ij}v_{jk} - v_{ij}v_{ik} - v_{ik}w_{jk}$
$-u_{ij}^+u_{jk}^- + u_{ij}^+u_{ik}^- - u_{ik}^-u_{jk}^-$	$w_{ij}v_{jk} - w_{ij}v_{ik} - w_{ik}v_{jk}$	$w_{ij}v_{jk} - w_{ij}v_{ik} - w_{ik}v_{jk}$

Table 1: Generating relations for the ideals $\mathcal{J}_{\mathfrak{S}_n}, \mathcal{I}$ and $\mathfrak{gr}(\mathcal{I})$.

First, one can show that the flat orbit decomposition from §2.1 persists in $\overline{\mathcal{Y}_n}$; we will abuse notation and write $\mathcal{Y}_\mu^{(n)}$ instead of $\overline{\mathcal{Y}_\mu}^{(n)}$ since they are isomorphic.

The second useful decomposition is the following bi-grading:

$$\mathcal{Y}_{k,\ell}^{(n)} := \text{span}_{\mathbf{k}}\{q \in \overline{\mathcal{Y}_n} : \deg(q) = k \quad \deg_{\mathcal{V}}(q) = \ell\}.$$

In fact, this bi-grading can be refined to a third decomposition by signed partitions, which are pairs of partitions (λ^+, λ^-) such that $|\lambda^+| + |\lambda^-| = n$.

Definition 3.4. Given a monomial in $q \in \mathbb{Q}[u_i, v_{ij}, w_{ij}]$, associate to q a signed partition $(\lambda^+_{(q)}, \lambda^-_{(q)})$ as follows:

1. Construct a graph $\mathcal{G}(q)$ with vertex set $[n] = \{1, 2, \cdots, n\}$ by drawing an edge between i and j if v_{ij} or w_{ij} occurs in q, and drawing a loop at i if u_i occurs in q;

2. Let $\mathcal{G}_1 = (E_1, V_1), \cdots, \mathcal{G}_k = (E_k, V_k)$ be the connected components of $\mathcal{G}(q)$. Then

$$\lambda^+_{(q)} := \{|V_\ell| : \mathcal{G}_\ell \text{ has no loops}\} \quad \lambda^-_{(q)} := \{|V_\ell| : \mathcal{G}_\ell \text{ has loops}\}.$$

Proposition 3.5. *There is a decomposition of $\overline{\mathcal{Y}_n}$ by signed partitions $\overline{\mathcal{Y}_n} = \bigoplus_{(\lambda^+,\lambda^-)} \mathcal{Y}^{(n)}_{(\lambda^+,\lambda^-)}$, where*

$$\mathcal{Y}^{(n)}_{(\lambda^+,\lambda^-)} := \text{span}_{\mathbf{k}}\{\textit{monomials } q \in \overline{\mathcal{Y}_n} : (\lambda^+_{(q)}, \lambda^-_{(q)}) = (\lambda^+, \lambda^-)\}.$$

This decomposition is compatible with the other decompositions of $\overline{\mathcal{Y}_n}$, in the sense that:

$$\mathcal{Y}^{(n)}_{\mu} = \bigoplus_{(\lambda^+,\lambda^-):\ \lambda^+=\mu} \mathcal{Y}^{(n)}_{(\lambda^+,\lambda^-)} \qquad \mathcal{Y}^{(n)}_{k,\ell} = \bigoplus_{\substack{(\lambda^+,\lambda^-):\ \ell(\lambda^+)=n-k \\ \ell(\lambda^+)+\ell(\lambda^-)=n-\ell}} \mathcal{Y}^{(n)}_{(\lambda^+,\lambda^-)}$$

For example, suppose $n = 8$ and $q = w_{12} \cdot u_5 \cdot v_{56} \cdot u_7 \cdot v_{24}$. Then q is in the bi-graded piece $\mathcal{Y}^{(8)}_{5,3}$ and we have $\lambda^+_{(q)} = \{3, 1, 1\}$ and $\lambda^-_{(q)} = \{2, 1\}$. Thus $q \in \mathcal{Y}^{(8)}_{((3,1,1),(2,1))} \subset \mathcal{Y}^{(8)}_{(3,1,1)}$.

Theorem 3.6. *There is a well-defined, $\mathfrak{S}_n$-equivariant surjection of* $\mathbf{k}$*-vector spaces*

$$\gamma : \overline{\mathcal{Y}_n} \longrightarrow \mathcal{X}_n = \mathbf{k}[u_{ij}]_{1\le i<j\le n} / \langle u_{ij}^2,\ \ u_{ij}u_{ik} - u_{ij}u_{jk} + u_{ik}u_{jk} \rangle$$
$$\mathcal{Y}^{(n)}_{(\lambda^+,\lambda^-)} \longmapsto \mathcal{X}^{(n)}_{(\lambda^+\cup\lambda^-)},$$

defined by sending $\qquad \gamma(u_i) = 1, \qquad \gamma(w_{ij}) = u_{ij} \qquad \gamma(v_{ij}) = u_{ij}.$

Proof idea. The key observation is that the relations $u_i w_{ij} - u_j v_{ij}$ and $u_i v_{ij} - u_j w_{ij}$ in $\mathfrak{gr}(\mathcal{I})$ mean that one can give a presentation of $\overline{\mathcal{Y}_n}$ as a quotient of a subring of $\mathbf{k}[v_{ij}, w_{ij}, u_i]$, by an ideal $\tilde{\mathcal{I}} \subset \mathfrak{gr}(\mathcal{I})$ that omits the relation u_i^2. From this, one can define a surjection of *vector spaces*; note however that γ cannot be extended to a map of algebras. □

4 Main Results

At last, we are ready to analyze the peak representations. Our investigations began from an observation of Aguiar, Bergeron and Nyman [1] relating the descent algebras $\mathrm{Sol}(\mathfrak{S}_n)$ and $\mathrm{Sol}(\mathfrak{S}^{\pm}_n)$ to the *peak algebra* $\mathcal{P}_n$.

Recall that one can express the hyperoctahedral group of all signed permutations as $\mathfrak{S}^{\pm}_n = \mathfrak{S}_n \ltimes \mathbb{Z}_2^n$ where $\mathbb{Z}_2^n$ is the normal subgroup performing arbitrary sign changes in the coordinates. The quotient map $\mathfrak{S}^{\pm}_n \twoheadrightarrow \mathfrak{S}^{\pm}_n / \mathbb{Z}_2^n \cong \mathfrak{S}_n$ of groups, which forgets the signs in a signed permutation, gives rise to a surjective $\mathbf{k}$-algebra map $\varphi : \mathbf{k}\,\mathfrak{S}^{\pm}_n \twoheadrightarrow \mathbf{k}\,\mathfrak{S}_n$. In [1], it was shown that the peak subalgebra $\mathcal{P}_n$ is exactly the image under φ of $\mathrm{Sol}(\mathfrak{S}^{\pm}_n)$, that is, φ restricts to an algebra surjection $\mathrm{Sol}(\mathfrak{S}^{\pm}_n) \overset{\varphi}{\twoheadrightarrow} \mathcal{P}_n$.

As a consequence, one can define a family of *peak idempotents* inside $\mathcal{P}_n \subset \mathbf{k}\,\mathfrak{S}_n$ via

$$E^{\mathcal{P}_n}_k := \varphi(E^{\mathfrak{S}^{\pm}_n}_k) \text{ for } k = 0, 1, \cdots, n \qquad E^{\mathcal{P}_n}_{\mu} := \varphi(E^{\mathfrak{S}^{\pm}_n}_{\mu}) \text{ for } \mu \vdash m \le n.$$

Both families inherit from $\{E^{\mathfrak{S}^{\pm}_n}_k\}$ and $\{E^{\mathfrak{S}^{\pm}_n}_{\mu}\}$ the property of being a complete system of orthogonal idempotents in $\mathbf{k}\,\mathfrak{S}_n$, and the $\{E^{\mathcal{P}_n}_{\mu}\}$ are also primitive if nonzero. Note that some of the $E^{\mathcal{P}_n}_k$ and $E^{\mathcal{P}_n}_{\mu}$ will be zero, which we characterize in Theorems 1.1 and 4.6. By construction, one recovers $E^{\mathcal{P}_n}_k$ from the $E^{\mathcal{P}_n}_{\mu}$ by summing over all μ of length k.

Our goal is to relate the peak idempotents to the ring $\mathcal{Z}_n := H^* Z_n$, where

$$Z_n := Y_n / \mathbb{Z}_2^n = \mathrm{Conf}_n(\left(\mathbb{R}^3 \setminus \{\mathbf{0}\}\right) / \mathbb{Z}_2) = \mathrm{Conf}_n(\mathbb{RP}^2 \times (0,\infty))$$

is the configuration space of n ordered points within the quotient $\mathbb{R}^3 \setminus \{\mathbf{0}\}$ under the $\mathbb{Z}_2$-action via $\mathbf{x} \mapsto -\mathbf{x}$, so that $\left(\mathbb{R}^3 \setminus \{\mathbf{0}\}\right) / \mathbb{Z}_2 \;\cong\; \mathbb{RP}^2 \times (0,\infty)$.

Note that $(\mathcal{Y}_n)^{\mathbb{Z}_2^n} \cong \mathcal{Z}_n$. The filtration, bigrading, and finer decompositions (by flat orbits and signed partitions) on $\mathcal{Y}_n$ from Section 3 persist when one takes $\mathbb{Z}_2^n$-fixed spaces, giving a bigraded $\mathfrak{S}_n$-representation on an associated graded ring $\overline{\mathcal{Z}_n}$:

$$\mathcal{Z}_{k,\ell}^{(n)} := (\mathcal{Y}_{k,\ell}^{(n)})^{\mathbb{Z}_2^n}, \qquad \mathcal{Z}_{\mu}^{(n)} := (\mathcal{Y}_{\mu}^{(n)})^{\mathbb{Z}_2^n}, \qquad \mathcal{Z}_{(\lambda^+,\lambda^-)}^{(n)} := (\mathcal{Y}_{(\lambda^+,\lambda^-)}^{(n)})^{\mathbb{Z}_2^n}.$$

We first construct monomial a basis for $\mathcal{Z}_n$, using the fact that by Theorem 3.2, the basis $\mathcal{V}$ of $\mathcal{Y}_n$ diagonalizes the action of the normal subgroup $\mathbb{Z}_2^n \leq \mathfrak{S}_n^{\pm}$ on $\mathcal{Y}_n$.

Definition 4.1. For $1 \leq i < j < k \leq n$, let $\mathcal{I}_1 := \{u_i w_{ij}\}$, $\mathcal{I}_2 := \{w_{ij} w_{ik}\}$, $\mathcal{I}_3 := \{v_{ij} w_{jk}\}$. Let $\tilde{\mathcal{V}}$ be the monomials obtained from products in $\mathcal{I}_j$ for $j = 1,2,3$ that are also in $\mathcal{V}$.

Theorem 4.2. *The set $\tilde{\mathcal{V}}$ is a basis for $\mathcal{Z}_n$ and $\overline{\mathcal{Z}_n}$ that is compatible with the decomposition by signed partitions: $\overline{\mathcal{Z}_n} = \bigoplus \mathcal{Z}_{(\lambda^+,\lambda^-)}^{(n)}$.*

Proof idea. We construct a bijection from $\tilde{\mathcal{V}}$ to the monomial basis of $\mathcal{X}_n$ from Example 2.1. This involves defining a "pairing lemma" to group quadratic terms appearing in $q \in \tilde{\mathcal{V}}$ and then mapping: $u_i w_{ij}$ to u_{ij}, $w_{ij} w_{ik}$ to $u_{ij} u_{ik}$, and $v_{ij} w_{jk}$ to $u_{ij} u_{jk}$. □

Example 4.3. The basis for $\mathcal{Z}_{4,2}^{(4)}$ is $\{(u_1 w_{12})(u_3 w_{34}),\ (u_1 w_{13})(u_2 w_{24}),\ (u_1 w_{14})(u_2 w_{23})\}$.

Given a partition λ of n, recall that $\ell(\lambda)$ is its number of parts and $|\lambda|$ is its size. Let $\mathrm{Odd}(\lambda)$ (resp. $\mathrm{Even}(\lambda)$) be the partition obtained by taking only the odd (resp. even) parts of λ. We call λ an *odd partition* if $\mathrm{Odd}(\lambda) = \lambda$ and an *even partition* if $\mathrm{Even}(\lambda) = \lambda$. Write $\mathrm{odd}(\lambda) = \ell(\mathrm{Odd}(\lambda))$ and $\mathrm{even}(\lambda) = \ell(\mathrm{Even}(\lambda))$.

Theorem 4.4. *The space $\mathcal{Z}_{(\lambda^+,\lambda^-)}^{(n)}$ vanishes unless λ^+ is an odd partition and λ^- is an even partition, while $\mathcal{Z}_{\mu}^{(n)}$ vanishes unless μ is an odd partition and $n - |\mu|$ is even.*
Moreover, the map γ restricts to an $\mathfrak{S}_n$-equivariant vector-space isomorphism $\gamma : \mathcal{Z}_n \longrightarrow \mathcal{X}_n$:

$$\gamma(\mathcal{Z}_{(\lambda^+,\lambda^-)}^{(n)}) = \mathcal{X}_{(\lambda^+ \cup \lambda^-)}^{(n)} \qquad \gamma^{-1}(\mathcal{X}_{\lambda}^{(n)}) = \mathcal{Z}_{(\mathrm{Odd}(\lambda),\mathrm{Even}(\lambda))}^{(n)}.$$

Thus, for non-vanishing $\mathcal{Z}_{(\lambda^+,\lambda^-)}^{(n)}$, $\mathcal{Z}_{\mu}^{(n)}$, and $\mathcal{Z}_{2k,\ell}^{(n)}$, there are $\mathfrak{S}_n$-representation isomorphisms

$$\mathcal{Z}_{(\lambda^+,\lambda^-)}^{(n)} \cong \mathrm{Lie}_{(\lambda^+ \cup \lambda^-)}, \qquad \mathcal{Z}_{\mu}^{(n)} \cong \bigoplus_{\lambda:\ \mathrm{Odd}(\lambda)=\mu} \mathrm{Lie}_{\lambda}, \qquad \mathcal{Z}_{2k,\ell}^{(n)} \cong \bigoplus_{\substack{\lambda: \ell(\lambda)=n-\ell \\ \mathrm{odd}(\lambda)=n-2k}} \mathrm{Lie}_{\lambda}\,.$$

Example 4.5. When $n = 4$, the non-vanishing pieces $\mathcal{Z}_{\mu}^{(4)}$ are as follows:

$$\mathcal{Z}_{\varnothing}^{(4)} \cong \mathrm{Lie}_{(2,2)} \oplus \mathrm{Lie}_{(4)} \quad \mathcal{Z}_{(1,1)}^{(4)} \cong \mathrm{Lie}_{(2,1,1)} \quad \mathcal{Z}_{(3,1)}^{(4)} \cong \mathrm{Lie}_{(3,1)} \quad \mathcal{Z}_{(1,1,1,1)}^{(4)} \cong \mathrm{Lie}_{(1,1,1,1)}\,.$$

The non-vanishing bi-graded pieces $\mathcal{Z}^{(4)}_{2k,\ell}$ are

$$\mathcal{Z}^{(4)}_{0,0} \cong \mathrm{Lie}_{(1,1,1,1)} \quad \mathcal{Z}^{(4)}_{2,1} \cong \mathrm{Lie}_{(2,1,1)} \quad \mathcal{Z}^{(4)}_{2,2} \cong \mathrm{Lie}_{(3,1)} \quad \mathcal{Z}^{(4)}_{4,2} \cong \mathrm{Lie}_{(2,2)} \quad \mathcal{Z}^{(4)}_{4,3} \cong \mathrm{Lie}_{(4)}\,.$$

In fact, we now have all the tools necessary to provide a cohomological interpretation of the $\mathfrak{S}_n$-representations generated by the Peak idempotents, by analyzing the $\mathbb{Z}_2^n$ fixed spaces of Theorem 2.2 and applying Theorem 4.4.

Theorem 4.6. *The idempotent $E^{\mathcal{P}_n}_\mu$ does not vanish if and only if μ is an odd partition (including $\mu = \varnothing$) and $n - |\mu|$ is even. In this case, there are $\mathfrak{S}_n$-representation isomorphisms*

$$(\mathbf{k}\,\mathfrak{S}_n)E^{\mathcal{P}_n}_\mu \cong \mathcal{Z}^{(n)}_\mu \cong \bigoplus_{\lambda\colon \mathrm{Odd}(\lambda)=\mu} \mathrm{Lie}_\lambda\,.$$

Note that combining Proposition 3.5 with Theorems 4.4 and 4.6 implies Theorem 1.1.

5 Hilbert series and the free Jordan algebra

Having established the connection between the peak algebra and the ring $\mathcal{Z}_n$, we now develop enumerative and recursive properties of the latter.

Let Λ denote the *ring of symmetric functions* (of bounded degree, in infinitely many variables). It has a $\mathbb{Z}$-algebra isomorphism known as the *Frobenius characteristic map* $\mathrm{ch} : \oplus_{n\geq 0}\mathrm{Rep}(\mathfrak{S}_n) \to \Lambda$, where $\mathrm{Rep}(\mathfrak{S}_n)$ are the *virtual characters* of $\mathfrak{S}_n$. We will study the Frobenius characteristic of $\mathcal{Z}^{(n)}_{2k,\ell}$, using the fact that $\mathcal{Z}^{(n)}_{2k+1,\ell} = 0$ by Theorem 1.1.

Definition 5.1. Write $\Lambda_{\mathbb{Z}[t,q]}$ to be the ring Λ with coefficients in $\mathbb{Z}[t,q]$ and define

$$M_n(t,q) := \sum_{k,\ell} \dim\left(\mathcal{Z}^{(n)}_{2k,\ell}\right) t^k q^\ell \in \mathbb{Z}[t,q], \qquad \mathcal{M}^{(n)}(t,q) := \sum_{k,\ell} \mathrm{ch}\left(\mathcal{Z}^{(n)}_{2k,\ell}\right) t^k q^\ell \in \Lambda_{\mathbb{Z}[t,q]}.$$

For $w \in \mathfrak{S}_n$ let $\mathrm{even}(w), \mathrm{odd}(w)$ denote the number of even-sized and odd-sized cycles of w, and $\mathrm{cyc}(w)$ the number of cycles of w.

Theorem 5.2. *Write $L_\lambda := \mathrm{ch}(\mathrm{Lie}_\lambda)$. Then one can rewrite $M_n(t,q)$ and $\mathcal{M}^{(n)}(t,q)$ as follows:*

$$M_n(t,q) = \sum_{w\in\mathfrak{S}_n} t^{\frac{n-\mathrm{odd}(w)}{2}} q^{n-\mathrm{cyc}(w)}, \qquad \mathcal{M}^{(n)}(t,q) = \sum_{\lambda\vdash n} L_\lambda \cdot t^{\frac{|\lambda|-\mathrm{odd}(\lambda)}{2}} q^{|\lambda|-\ell(\lambda)}.$$

Using Theorem 5.2, we manipulate the symmetric functions in $\mathcal{M}^{(n)}(t,q)$ to give a branching rule recurrence for the bi-graded pieces $\mathcal{Z}^{(n)}_{2k,\ell}$. Let $\uparrow$ denote representation induction from $\mathfrak{S}_n$ to $\mathfrak{S}_{n+1}$ and $\downarrow$ denote representation restriction from $\mathfrak{S}_n$ to $\mathfrak{S}_{n-1}$.

Theorem 5.3. *The restriction of* $\mathcal{Z}_{2k,j}^{(n)}$ *from an* $\mathfrak{S}_n$ *to an* $\mathfrak{S}_{n-1}$*-module is given by*

$$\mathcal{Z}_{2k,\ell}^{(n)}\downarrow = \mathcal{Z}_{2k,\ell}^{(n-1)} + \mathcal{Z}_{2(k-1),\ell-1}^{(n-2)}\uparrow + \left(\mathcal{Z}_{2(k-1),\ell-2}^{(n-2)}\uparrow\right) * \chi^{(n-2,1)},$$

where $*$ *is the Kronecker product and* $\chi^{(n-2,1)}$ *is the irreducible reflection representation of* $\mathfrak{S}_{n-1}$.

Theorem 5.3 implies a recursive formula for $M_n(t,q)$ with interesting specializations:

$$M_n(1,q) = (1+q)(1+2q)\cdots(1+(n-1)q), \tag{5.1}$$
$$M_n(t,1) = (1+(n-1)q)\cdot M_{n-1}(1,q), \tag{5.2}$$

where (5.1) is the generating function for the *Stirling numbers of the first kind*, and (5.2) describes the *Sheffer polynomials* [15] counting permutations w according to $\mathrm{odd}(w)$.

5.1 The space of simple Jordan elements

Finally, we mention an interesting connection between $\mathcal{Z}_n$ and the multilinear part of the *space of simple Jordan elements* within the free associative algebra $\mathbf{k}\langle\mathbf{x}\rangle = \mathbf{k}\langle x_1,\ldots,x_n\rangle$.

Consider a deformation of the Lie bracket on $\mathbf{k}\langle\mathbf{x}\rangle$ by $\alpha\in\mathbb{C}$: $[x,y]_\alpha := xy-\alpha yx$. Let J_α be the smallest $\mathbf{k}$-subspace of $\mathbf{k}\langle\mathbf{x}\rangle$ containing the generators $\mathbf{x}$ and closed under $[\cdot,\cdot]_\alpha$.

For example, $J_1\subset\mathbf{k}\langle\mathbf{x}\rangle$ is the free Lie algebra. Define $V_n(\alpha)\subset J_\alpha$ to be the $\mathbf{k}$-subspace spanned by these multilinear bracketings of homogeneous degree n for $w\in\mathfrak{S}_n$:

$$[[\cdots[x_{w(1)},x_{w(2)}]_\alpha,x_{w(3)}]_\alpha,\cdots]_\alpha,x_{w(n)}]_\alpha$$

Then $V_n(1)\cong\mathrm{Lie}_n$ is the multilinear component of the free Lie algebra, while $V_n(-1)$ is the multilinear part of the *space of simple Jordan elements*. The following was proved by Robbins in [14, §6, Thm. 7] and later in [7, Thm 2.1] by Calderbank–Hanlon–Sundaram:

$$V_n(-1)\cong\bigoplus_{\substack{\lambda\vdash n\\ \mathrm{odd}(\lambda)=\ell(\lambda)}}\mathrm{Lie}_\lambda\,. \tag{5.3}$$

We combine Theorem 4.4 and (5.3), to give a cohomological interpretation for $V_n(-1)$.

Corollary 5.4. *The space* $V_n(-1)$ *is isomorphic as an* $\mathfrak{S}_n$*-representation to* $\bigoplus_k\mathcal{Z}_{2k,2k}^{(n)}$.

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