ON PARITY AND CHARACTERS OF SYMMETRIC GROUPS

ALEXANDER R. MILLER

ABSTRACT. We present a conjectural parity bias in the character values of the symmetric group. The main conjecture says that a character value chosen uniformly at random from the character table of S_n is congruent to 0 mod 2 with probability $\to 1$ as $n \to \infty$. A more general conjecture says that the same is true for all primes p, not only $p = 2$. We relate these conjectures to zeros, give generating functions for computing lower bounds, and present some computational data in support of the main conjecture.

1. Introduction

This note is a sequel to [11] where we studied zeros and random character values of the symmetric group S_n. By character we mean irreducible character. The purpose of this short note is to present two conjectures. The first and main conjecture (Conjecture 1) says that a character value chosen uniformly at random from the character table of S_n is congruent to 0 mod 2 with probability $\to 1$ as $n \to \infty$. The second conjecture (Conjecture 2) says that the same is true for all primes p, not only $p = 2$. It should be stated here at the start that we are unable to prove these conjectures.

1.1. Character values of $\mathbf{S_n}$. Given an integer partition λ of n, let χ_λ be the corresponding irreducible character of S_n, and let $\chi_\lambda(\mu)$ be short for the value of χ_λ at any g with cycle type μ, so that the matrix $[\chi_\lambda(\mu)]_{\lambda,\mu}$ is the character table of S_n [8]. These values can be computed for small n with the formulas of Frobenius and Murnaghan–Nakayama. Frobenius [3] says

$$\chi_\lambda(\mu) = \text{coeff. of } x_1^{n+\lambda_1-1} x_2^{n+\lambda_2-2} \dots x_n^{\lambda_n} \text{ in } \Delta(x) P_\mu(x) \tag{1}$$

where $\Delta(x) = \prod_{1 \le i < j \le n}(x_i - x_j)$ and $P_\mu(x) = P_{\mu_1}(x) P_{\mu_2}(x) \dots P_{\mu_n}(x)$, $P_0(x) = 1$, $P_k(x) = x_1^k + x_2^k + \dots + x_n^k$ for $k \ge 1$. Murnaghan–Nakayama says

$$\chi_\lambda(\mu) = \sum (-1)^{\mathrm{ht}(\rho)} \chi_{\lambda \setminus \rho}(\nu) \tag{2}$$

for any ν obtained by removing a nonzero part from μ, where the sum is over all rim hooks ρ in λ of size $|\mu| - |\nu|$, and $\mathrm{ht}(\rho)$ is one less than the number of rows occupied by ρ, see [8].

1.2. Background. The study of zeros of irreducible characters goes back to Burnside [2], who proved that each nonlinear irreducible character of a finite group is zero on some class. J. G. Thompson modified Burnside's argument to prove that for each irreducible character the values are roots of unity or zero on more than a third of the group [7, p. 46]. P. X. Gallagher proved similarly that on a larger than average class some irreducible character is zero [4], and that on a larger than average class the character values are roots of unity or zero for more than a third of the irreducible characters [5].

What can be said about a random character value of the symmetric group? Here there are two natural ways to choose a character value. The first is to choose χ uniformly at

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random from the irreducible characters and g uniformly at random from the group, and then evaluate $\chi(g)$. The main result of [11] is the remarkable fact that for the symmetric group S_n the probability that $\chi(g)$ equals 0 goes to 1 as $n \to \infty$ [11, Theorem 1].

The other natural way to choose a character value of S_n is to choose an entry $\chi_\lambda(\mu)$ uniformly at random from the character table. That is, choose λ and μ uniformly at random from the partitions of n, and then evaluate $\chi_\lambda(\mu)$. In this scheme the asymptotic behavior of $\mathrm{Prob}(\chi_\lambda(\mu) = 0)$ is not known, and experiments do not suggest a limit of 1. See Table 3 in Section 4 and cf. [11, Question 2 and p. 1015].

1.3. Working modulo 2. The present note considers zeros of S_n character values mod 2. In particular, we are interested in the asymptotic behavior of $\mathrm{Prob}(\chi_\lambda(\mu) \equiv 0 \pmod 2)$. What we find experimentally is a striking parity bias. See Figure 1.

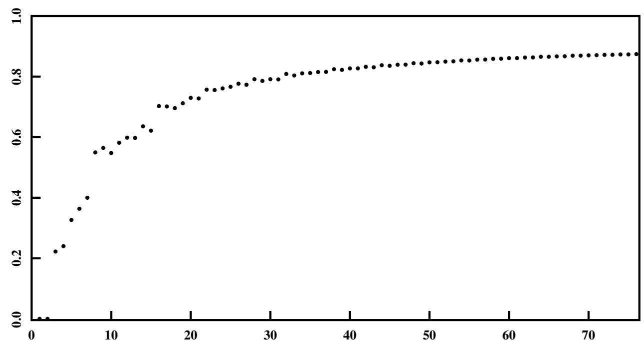

FIGURE 1. $\mathrm{Prob}(\chi_\lambda(\mu) \equiv 0 \pmod 2)$ for $1 \le n \le 76$.

Conjecture 1. $\mathrm{Prob}(\chi_\lambda(\mu) \equiv 0 \pmod 2) \to 1$ *as* $n \to \infty$.

Our second conjecture extends the first to other primes.

Conjecture 2. $\mathrm{Prob}(\chi_\lambda(\mu) \equiv 0 \pmod p) \to 1$ *as* $n \to \infty$ *for every prime* p.

Numerical evidence for Conjectures 1 and 2 is given in Section 4 in Tables 1 and 2. Table 1 lists the number of even entries in the character table of S_n for $1 \le n \le 76$. Table 2 lists $\mathrm{Prob}(\chi_\lambda(\mu) \equiv 0 \pmod q)$ for some small prime powers q and $1 \le n \le 38$. In Section 2 we relate Conjecture 1 and Conjecture 2 to zeros in the character table of S_n. We use a small set of well-understood zeros to obtain generating functions for computing lower bounds for the number of $\chi_\lambda(\mu)$'s congruent to 0 mod p. These crude bounds are surprisingly good. However, it seems that new tools will be needed in order to establish a parity bias, see Remark 1. In Section 3 we make two more remarks. The first remark reformulates Conjecture 1 as a problem in tableau enumeration, see Conjecture 1$'$. This gives another possible approach to Conjecture 1. The second remark explains why in Table 1 we find that the number of even entries in the character table of S_n is always even.

2. Zeros and a lower bound

2.1. The following is a well-known result for dealing with characters of S_n modulo a prime number p. See [12, §3] and [10, Proof of Theorem].

Proposition 1. *Let χ be a character of S_n, and let p be a prime number. Suppose that μ and ν are partitions of n such that ν is obtained from μ by replacing p parts of size k with a single part of size pk. Then* $\chi(\mu) \equiv \chi(\nu) \pmod p$.

Proof. This follows from Frobenius (1) and $(x+y)^p = x^p + y^p$ over $\mathbf{Z}/p\mathbf{Z}$. $\square$

2.2. Consider the graph $\Gamma_p(n)$ whose vertices are the partitions μ of n, and whose edges are the pairs $\{\mu, \nu\}$ where ν is obtained from μ by exchanging p parts of size k for a part of size pk. Let

$$\mathrm{Par}(n) = \{\text{partitions of } n\}, \tag{3}$$

$$\Omega_p(n) = \{\text{partitions of } n \text{ into parts not divisible by } p\}, \tag{4}$$

$$\Delta_p(n) = \{\text{partitions of } n \text{ with no part appearing } p \text{ or more times}\}. \tag{5}$$

Each connected component of $\Gamma(n)$ contains a unique representative $\mu \in \Delta_p(n)$ and a unique representative $\mu^* \in \Omega_p(n)$. The partition μ^* is obtained from μ by replacing each part $p^k d$ $(p \nmid d)$ by p^k parts of size d, and this mapping $\mu \mapsto \mu^*$ is a bijection from $\Delta_p(n)$ onto $\Omega_p(n)$. (The $p = 2$ case of this is the odd–distinct bijection of Euler [1].) Write $K_p(\mu)$ $(\mu \in \Delta_p(n))$ for the set of partitions in the same component as μ, so that

$$\mathrm{Par}(n) = \bigcup_{\mu \in \Delta_p(n)} K_p(\mu) \tag{6}$$

as a disjoint union, and $\mu, \mu^* \in K_p(\mu)$. Let $k_p(\mu) = |K_p(\mu)|$ and let

$$N_p(n) = \#\{(\lambda, \mu) \in \mathrm{Par}(n) \times \mathrm{Par}(n) : \chi_\lambda(\mu) \equiv 0 \pmod{p}\}, \tag{7}$$

$$z_p(\mu) = \#\{\lambda \in \mathrm{Par}(n) : \chi_\lambda(\mu) \equiv 0 \pmod{p}\}, \tag{8}$$

$$z(\mu) = \#\{\lambda \in \mathrm{Par}(n) : \chi_\lambda(\mu) = 0\}. \tag{9}$$

Proposition 2.

$$N_p(n) = \sum_{\mu \in \Delta_p(n)} k_p(\mu) z_p(\mu) \tag{10}$$

and

$$N_p(n) \geq \sum_{\mu \in \Delta_p(n)} k_p(\mu) z(\mu). \tag{11}$$

Proof. Proposition 1 implies (10), and (11) follows. $\square$

What makes Proposition 2 useful is: $\Delta_p(n)$ is much smaller than $\mathrm{Par}(n)$, the numbers $k_p(\mu)$ are straightforward to compute (Proposition 3), and even crude lower bounds on the number of zeros $z(\mu)$ in the μ column of the character table of S_n result in good lower bounds on the number $N_p(n)$ of p-divisible entries in the character table (Theorem 1).

2.3. Let $\mu = 1^{m_1} 2^{m_2} \ldots n^{m_n}$ be shorthand for the partition μ with m_1 many 1's, m_2 many 2's, and so on. Let $a_p(n)$ be the number of partitions of n into powers of p, so that

$$\sum_{n=0}^{\infty} a_p(n) q^n = \prod_{j=0}^{\infty} \frac{1}{1 - q^{p^j}} \tag{12}$$

and $a_p(n)$ is given by the recurrence relation

$$a_p(n) = \begin{cases} a_p(n-1) & \text{if } n \not\equiv 0 \pmod{p}, \\ a_p(n-1) + a_p(n/p) & \text{if } n \equiv 0 \pmod{p}, \end{cases} \tag{13}$$

where $a_p(0) = a_p(1) = \ldots = a_p(p-1) = 1$. See [13]. Then we have the following formula for $k_p(\mu)$.

Proposition 3. *Let $\mu \in \Delta_p(n)$ and write $\mu^* = 1^{m_1}2^{m_2}\dots n^{m_n}$. Then*

$$k_p(\mu) = a_p(m_1)a_p(m_2)\dots a_p(m_n). \tag{14}$$

Proof. Define

$$\pi : \mathrm{Par}(n) \to \Omega_p(n), \quad 1^{s_1}2^{s_2}\dots n^{s_n} \mapsto \prod_{p \nmid k} k^{s_k + ps_{kp} + p^2 s_{kp^2} + \cdots}. \tag{15}$$

Then

$$K_p(\mu) = \pi^{-1}(\mu^*) \tag{16}$$

and the elements of $\pi^{-1}(\mu^*)$ are in bijection with the tuples $(\nu_1, \nu_2, \dots, \nu_n)$ where ν_k is a partition of m_k into powers of p (s_k many 1's, s_{kp} many p's, and so on). □

2.4. By t-core we mean a partition λ with no rim hook of size t, which is the same as saying that λ has no hook of size t (cf. [8, pp. 76 and 56]), or equivalently λ has no hook of size divisible by t [8, p. 86]. Let $c_t(n)$ be the number of t-core partitions of n. Then [6, 9]

$$\sum_{n=0}^{\infty} c_t(n)q^n = \prod_{n=1}^{\infty} \frac{(1-q^{tn})^t}{(1-q^n)}. \tag{17}$$

Theorem 1.

$$N_p(n) \geq \sum_{t=1}^{n} c_t(n) \sum_{\substack{\mu \in \Delta_p(n) \\ \mu_1 = t}} k_p(\mu). \tag{18}$$

Proof. Let $\lambda, \mu \in \mathrm{Par}(n)$. If λ is a t-core and μ has a part of size t, then by the Murnaghan–Nakayama rule $\chi_\lambda(\mu) = 0$. In particular, the number of 0's in column μ of the S_n character table is bounded below by the number of μ_1-core partitions of n. In other words, $z(\mu) \geq c_{\mu_1}(n)$. This inequality and the one in (11) together give the result. □

Remark 1. We suspect that Theorem 1 can be used to obtain nontrivial asymptotic lower bounds for $\mathrm{Prob}(\chi_\lambda(\mu) \equiv 0 \pmod 2)$. For example, for $n = 30, 60, 90, 120$ the respective lower bounds for $\mathrm{Prob}(\chi_\lambda(\mu) \equiv 0 \pmod 2)$ given by Theorem 1 are approximately 0.45369, 0.47022, 0.47883, and 0.46521. Unfortunately, at least from these computations, Theorem 1 does not seem sufficient for a provable parity bias in the sense that there exists an $\epsilon > 0$ such that $\mathrm{Prob}(\chi_\lambda(\mu) \equiv 0 \pmod 2) \geq 1/2 + \epsilon$ for n sufficiently large.

3. Remarks

3.1. Conjecture 1 can be reformulated as a problem in tableau enumeration. Let $N_{\lambda\mu}$ be the number of rim hook tableaux of shape λ and content μ, see [14].

Proposition 4. $\chi_\lambda(\mu) \equiv N_{\lambda\mu} \pmod 2$.

Proof. By the Murnaghan–Nakayama rule (2). □

Hence the following enumerative reformulation of Conjecture 1.

Conjecture 1′. $\mathrm{Prob}(N_{\lambda\mu}$ is even$) \to 1$ *as* $n \to \infty$.

3.2. We are aware of one previous parity result for the character table of S_n. It is an enumerative result on the number of odd entries in the column of degrees $\chi_\lambda(1^n)$. J. McKay [10] proved that the number of odd entries in this column is a certain power of 2. It is natural to wonder about a similar result for the entire character table. Table 1 suggests that the number of even entries is always even. This is in fact true.

Theorem 2. *The number of even entries $\chi_\lambda(\mu)$ in the character table of S_n is even.*

Recall that the conjugate of λ is the partition λ' whose parts are $\lambda'_i = \#\{j : i \leq \lambda_j\}$ for $1 \leq i \leq \lambda_1$. Conjugation is the involution $\lambda \mapsto \lambda'$. The fixed points of this involution are self-conjugate partitions. Self-conjugate partitions λ of n are in one-to-one correspondence with partitions μ of n into odd distinct parts via $\lambda \mapsto \mu$ where $\mu_i = 2(\lambda_i - i) + 1$ for i such that $1 \leq i \leq \lambda_i$.

Proof of Theorem 2. Let O_n be the number of odd entries in the character table of S_n. Then

$$O_n \equiv \sum_\mu \sum_\lambda \chi_\lambda(\mu) \equiv \sum_\mu \sum_\lambda \chi_\lambda(\mu)^2 \pmod 2 \tag{19}$$

where the sums are over all partitions of n. By one of the orthogonality relations [3]

$$\sum_\lambda \chi_\lambda(\mu)^2 = 1^{m_1} m_1! 2^{m_2} m_2! \ldots n^{m_n} m_n! \tag{20}$$

where m_i is the number of i's in μ. Together (19) and (20) imply that

$$O_n \equiv OD_n \pmod 2 \tag{21}$$

where OD_n is the number of partitions of n into odd distinct parts. Let SC_n be the number of self-conjugate partitions of n so that $SC_n = OD_n$ and hence

$$O_n \equiv SC_n \pmod 2. \tag{22}$$

Let E_n be the number of even entries in the character table of S_n. Then

$$O_n + E_n = p_n^2 \equiv p_n \pmod 2 \tag{23}$$

where p_n is the number of partitions of n. Together (22) and (23) imply that

$$E_n \equiv p_n - SC_n \pmod 2. \tag{24}$$

But $p_n - SC_n \equiv 0 \pmod 2$ because conjugation restricts to a fixed-point-free involution on the set of non-self-conjugate partitions of n. □

4. Tables

The probabilities in Tables 2 and 3 are rounded to the number of digits shown.

TABLE 1. Number of even entries and number of odd entries in the character table of S_n for $1 \leq n \leq 76$.

n	no. of evens	no. of odds
1	0	1
2	0	4
3	2	7
4	6	19
5	16	33
6	44	77
7	90	135
8	266	218
9	508	392
10	966	798
11	1824	1312
12	3548	2381
13	6094	4107
14	11586	6639
15	19254	11722
16	37492	15869
17	61876	26333
18	103110	45115
19	170932	69168
20	286916	106213
21	456554	170710
22	759962	244042
23	1190034	384991
24	1887766	592859
25	2937820	895944
26	4608084	1326012
27	7004646	2055454
28	10938762	2884762
29	16372732	4466493
30	24851432	6553384
31	37014368	9798596
32	56368810	13336991
33	82688102	20192347
34	122855526	28680574
35	179808396	41695293
36	263406424	59766105
37	381814902	86344867
38	557951490	118828735
39	799580980	172923245
40	1152977342	241148902
41	1644080076	343563813
42	2352923494	474550782
43	3324344208	677609913
44	4732761850	918518775
45	6639049122	1305820834
46	9351080036	1791411328
47	13067332410	2496228106
48	18309958344	3379378185
49	25390864566	4720061059
50	35331180090	6377078986
51	48786461562	8786181687
52	67367826002	11924538919
53	92571070272	16283394489
54	127268025536	21847658489
55	173744388742	29905639434
56	237567368138	39975105191
57	323002974632	54182161084
58	439208932802	72330715598
59	594363393060	97561119340
60	804101537262	129956924827
61	1082902860136	174870604889
62	1458789177232	231616447104
63	1956705210484	309822028517
64	2625259647972	408015408928
65	3505898738012	544490965352
66	4679753246976	718991943424
67	6226771093726	953962042995
68	8285512851154	1248594579071
69	10979998587386	1653369791639
70	14541318538948	2170163830076
71	19209876952108	2853857859917
72	25351409083192	3730699401897
73	33363529811282	4899218593439
74	43886589872232	6374420377768
75	57554118617836	8352091755860
76	75434276878574	10852934727707

TABLE 2. Prob($\chi_\lambda(\mu) \equiv 0 \pmod q$) for some q's and $1 \le n \le 38$.

n	$q=2$	$q=2^2$	$q=2^3$	$q=3$	$q=3^2$	$q=3^3$	$q=5$	$q=5^2$	$q=5^3$
1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3	0.222	0.111	0.111	0.111	0.111	0.111	0.111	0.111	0.111
4	0.240	0.160	0.160	0.240	0.160	0.160	0.160	0.160	0.160
5	0.327	0.245	0.204	0.224	0.204	0.204	0.245	0.204	0.204
6	0.364	0.248	0.248	0.322	0.256	0.240	0.289	0.240	0.240
7	0.400	0.271	0.244	0.324	0.244	0.244	0.284	0.244	0.244
8	0.550	0.386	0.333	0.374	0.322	0.316	0.368	0.316	0.316
9	0.564	0.409	0.359	0.473	0.359	0.350	0.373	0.341	0.341
10	0.548	0.386	0.350	0.455	0.353	0.334	0.412	0.340	0.333
11	0.582	0.406	0.348	0.464	0.348	0.329	0.389	0.328	0.325
12	0.598	0.459	0.414	0.529	0.405	0.387	0.450	0.382	0.376
13	0.597	0.444	0.388	0.518	0.382	0.355	0.427	0.351	0.348
14	0.636	0.463	0.409	0.548	0.409	0.386	0.457	0.386	0.378
15	0.622	0.454	0.395	0.547	0.407	0.378	0.458	0.371	0.363
16	0.703	0.520	0.446	0.556	0.422	0.395	0.475	0.395	0.387
17	0.701	0.512	0.433	0.564	0.420	0.391	0.478	0.389	0.379
18	0.696	0.521	0.448	0.598	0.445	0.409	0.490	0.402	0.392
19	0.712	0.527	0.443	0.583	0.428	0.390	0.483	0.389	0.377
20	0.730	0.547	0.468	0.599	0.449	0.411	0.504	0.407	0.396
21	0.728	0.544	0.460	0.607	0.441	0.401	0.494	0.396	0.383
22	0.757	0.567	0.477	0.615	0.454	0.412	0.509	0.409	0.396
23	0.756	0.562	0.470	0.615	0.447	0.404	0.505	0.400	0.386
24	0.761	0.577	0.485	0.627	0.460	0.413	0.513	0.409	0.394
25	0.766	0.577	0.479	0.623	0.449	0.401	0.509	0.397	0.382
26	0.777	0.591	0.493	0.635	0.465	0.415	0.521	0.411	0.394
27	0.773	0.583	0.483	0.636	0.456	0.403	0.515	0.399	0.381
28	0.791	0.603	0.501	0.641	0.465	0.412	0.524	0.408	0.390
29	0.786	0.597	0.493	0.644	0.463	0.407	0.523	0.402	0.383
30	0.791	0.605	0.501	0.648	0.465	0.409	0.524	0.403	0.385
31	0.791	0.602	0.495	0.649	0.463	0.404	0.524	0.399	0.378
32	0.809	0.619	0.509	0.654	0.470	0.410	0.530	0.405	0.384
33	0.804	0.611	0.500	0.655	0.464	0.401	0.526	0.396	0.374
34	0.811	0.621	0.509	0.658	0.469	0.407	0.531	0.401	0.379
35	0.812	0.619	0.504	0.662	0.469	0.403	0.530	0.397	0.374
36	0.815	0.626	0.511	0.662	0.468	0.402	0.531	0.397	0.373
37	0.816	0.625	0.508	0.666	0.469	0.400	0.532	0.394	0.370
38	0.824	0.635	0.516	0.669	0.472	0.403	0.533	0.396	0.371

TABLE 3. Number of zeros in the character table of S_n for $1 \leq n \leq 38$.

n	no. of zeros	Prob$(\chi_\lambda(\mu) = 0)$
1	0	0.0000
2	0	0.0000
3	1	0.1111
4	4	0.1600
5	10	0.2041
6	29	0.2397
7	55	0.2444
8	153	0.3161
9	307	0.3411
10	588	0.3333
11	1018	0.3246
12	2230	0.3761
13	3543	0.3473
14	6878	0.3774
15	11216	0.3621
16	20615	0.3863
17	33355	0.3781
18	57980	0.3912
19	90194	0.3757
20	155176	0.3947
21	239327	0.3815
22	395473	0.3939
23	604113	0.3836
24	970294	0.3911
25	1453749	0.3792
26	2323476	0.3915
27	3425849	0.3781
28	5349414	0.3870
29	7905133	0.3793
30	11963861	0.3810
31	17521274	0.3743
32	26472001	0.3798
33	38054619	0.3699
34	56756488	0.3745
35	81683457	0.3688
36	119005220	0.3682
37	170498286	0.3642
38	247619748	0.3659

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FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT WIEN, WIEN, AUSTRIA