Application Demonstration

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Classroom Tips and Techniques: Applying the Epsilon-Delta Definition of a Limit

Robert J. Lopez
Emeritus Professor of Mathematics and Maple Fellow
Maplesoft

Introduction

My experience in teaching calculus at two universities and an undergraduate engineering school was that students struggled to apply the epsilon-delta definition of a limit. This is understandable, given the history of the concept's development. Much of the objection to Newton's introduction of the derivative stemmed from complaints about the limiting process involved. Many of his contemporaries, steeped in a geometric mode of thought, had trouble with the notion of a ratio of vanishingly small quantities. In fact, it took nearly two centuries for the present definition of the limit to evolve, and before this could happen, a theory of the real number system had to be articulated.

I can remember the examples that were used even in my real analysis classes in grad school, examples where the tricks for manipulating the requisite inequalities all but obscured the underlying concept. Each different function to which the definition is applied evokes a new amalgamation of manipulations specific to the function. There seems to be no unifying principle that can be used for a range of examples, from which a student can extract the essence of what is really a pretty deep concept.

The following four examples demonstrate a unified approach to finding δ as a function of ε. Implemented in Maple, the necessary calculations are a straight-forward demonstration of how, given an $\varepsilon > 0$, one can find a $\delta > 0$ for which $|f(x \pm \delta) - L| < \varepsilon$.

Definition 1

The epsilon-delta definition of a limit is generally captured by a statement such as the following.

Definition 1
The limit of $f(x)$ as x approaches a is the number L, that is $$\lim_{x \to a} f(x) = L$$ if and only if for every $\varepsilon > 0$ there exists a number $\delta > 0$ with the property that if $0 < \lvert x - a \rvert < \delta$ (and x is in the domain of *f*), then $\lvert f(x) - L \rvert < \varepsilon$.

Explication of Definition 1 usually involves a discussion of "an ε -band" about the line $y = L$ and a corresponding "δ -band" around $x = a$. Describing these bands with inequalities, and then manipulating these inequalities so that the outcome is $\delta = \delta(\varepsilon)$ is difficult mathematically, and a challenge in Maple. However, describing the bands with equalities is an approach more suited to Maple's strengths, and more intuitive mathematically. Figure 1 is the basis for this approach for an increasing function. The appropriate modifications for a decreasing function are detailed at the end of the discussion.

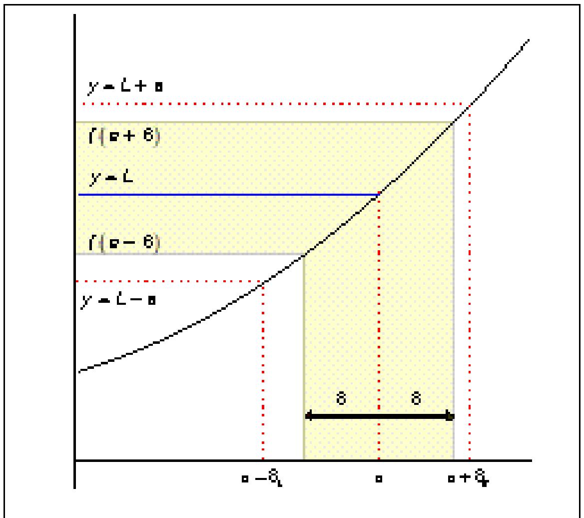


Figure 1 Schematic for Definition 1

- The essence of Definition 1 is finding a δ -band around $x = a$ inside of which function values $f(x)$ remain within an ε -band around L . See the region shaded in yellow in Figure 1.

- Solve the two equations

  $f\left(a + \delta_R\right) = L = \varepsilon$ and $f\left(a - \delta_L\right) = L - \varepsilon$

  where both δ_L and δ_R are positive, and generally different. See the dotted red lines in Figure 1.

- Choose $\delta(\varepsilon) \le \min\{\delta_L, \delta_R\}$. See the black arrows and the region shaded yellow in Figure 1.

- The final step is to show that $0 < |x - a| < \delta(\varepsilon) \Rightarrow |f(x) - L| < \varepsilon$, that is, that for all x -values in the δ -band around $x = a$, all the function values $f(x)$ remain inside the ε -band around $y = L$. (Having found a candidate for $\delta(\varepsilon)$, it's much easier to establish the validity of the appropriate inequalities than it is to use inequalities to find $\delta(\varepsilon)$ in the first place.)

- This demonstration is expedited by expressing the x -values in the δ -band $|x - a| < \delta$ as $x = a + t\,\delta(\varepsilon)$, where $0 < |t| < 1$. Since t can be both positive and negative, these x -values correspond to all those inside the δ -band. Then, to show that $|f(x) - L| < \varepsilon$ for these x -values, show instead that $|f(a + t\,\delta(\varepsilon)) - L| < \varepsilon$. The following four examples will verify that these steps form an algorithm that can be applied to a variety of functions without having to make radical modifications because of the peculiar properties of the function f .

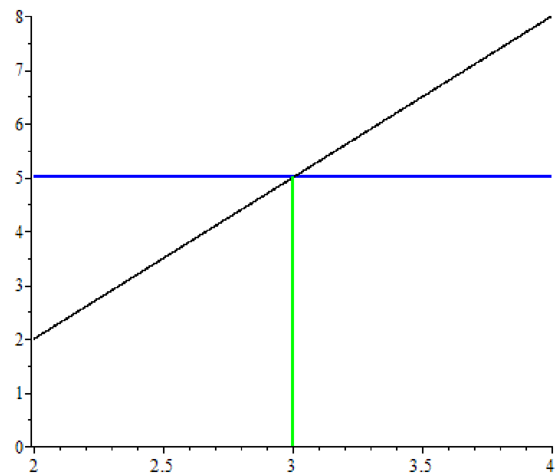
Note: For a function that decreases in the vicinity of $x = a$, the two equations for δ_L and δ_R would respectively be

$$f\left(a + \delta_L\right) = L + \varepsilon \quad \text{and} \quad f\left(a - \delta_R\right) = L - \varepsilon$$

Example 1

Use Definition 1 to verify $\lim_{x\to 3}(3x-4)=5$.

Solution

• Click the restart icon in the toolbar or execute the **restart** command to the right.	*restart*
• Type the equation $f(x)=\ldots$ • Context Menu: Assign Function	$f(x)=3x-4 \xrightarrow{\text{assign as function}} f$
Figure 2 is an animation in which $f(x)=3x-4$ is graphed in black, and the line $y=5$ is graphed in blue. • The red and green horizontal lines are drawn at $y=5\pm\varepsilon$, respectively, and the red and green vertical lines are drawn at the corresponding x-coordinates $x=3\pm\delta=f^{-1}(5\pm\varepsilon)$. (Because of the linearity of f, $\delta_L=\delta_R=\delta(\varepsilon)$). • The slider in the animation toolbar controls the value of ε . As the slider is moved, the red and green horizontal lines delineate the ε-band around $y=5$, and the red and green vertical lines delineate the corresponding δ-band around $x=3$.	 **Figure 2** Animation illustrating Definition 1
• Write the equation $f(a-\delta_L)=L-\varepsilon$ • Press the Enter key. • Context Menu: Solve>Isolate Expression for>δ_L	$f(3-\delta_L)=5-\varepsilon$ $5-3\,\delta_L=5-\varepsilon$ $\xrightarrow{\text{isolate for delta[L]}}$ $\delta_L=\frac{1}{3}\,\varepsilon$
• Write the equation $f(a+\delta_R)=L-\varepsilon$	$f(3+\delta_R)=5+\varepsilon$

• Press the Enter key. • Context Menu: Solve>Isolate Expression for>δ_R	$5+3\,\delta_R=5+\varepsilon$ $\xrightarrow{\text{isolate for delta[R]}}$ $\delta_R=\frac{1}{3}\,\varepsilon$

Consequently, $\delta_L=\delta_R=\delta(\varepsilon)=\varepsilon/3$. To complete the proof, show that $|x-3|<\varepsilon/3 \Rightarrow$ $|(3\,x-4)-5|<\varepsilon$. This is done by showing that $|f(3+t\,\varepsilon/3)-5|<\varepsilon$.

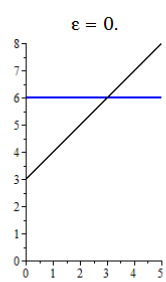
$$|f(3+t\,\varepsilon/3)-5|=|t\,\varepsilon| \;=\; |t|\,|\varepsilon| \;=\; |t|\,\varepsilon<\varepsilon$$

The first equality follows from straight-forward algebra, and the second from the recognition that the absolute value of a product is the product of the absolute values. The next equality follows because ε is positive, and the final inequality follows because $|t|<1$.

Example 2

Use Definition 1 to verify $\lim_{x\to 3}\frac{x^2-9}{x-3}=6$.

Solution

• Click the restart icon in the toolbar or execute the **restart** command to the right.	*restart*
• Type the equation $f(x)=\ldots$ • Context Menu: Assign Function	$f(x)=\frac{x^2-9}{x-3}\xrightarrow{\text{assign as function}} f$
• Figure 3 is an animation in which $f(x)=\frac{x^2-9}{x-3}$ is graphed in black, and the line $y=6$ is graphed in blue. • The red and green horizontal lines are drawn at $y=6\pm\varepsilon$, respectively, and the red and green vertical lines are drawn at the corresponding x -coordinates $x=3\pm\delta=f^{-1}(6\pm\varepsilon)$. (Because f reduces to the linear $x+3$ for $x\neq 3$, once again $\delta_L=\delta_R=\delta(\varepsilon)$).	 **Figure 3** Animation illustrating Definition 1

• The slider in the animation toolbar controls the value of ε. As the slider is moved, the red and green horizontal lines delineate the ε-band around $y = 6$ and the red and green vertical lines delineate the corresponding δ-band around $x = 3$.	
• Write the equation $f\left(a - \delta_L\right) = L - \varepsilon$ • Press the Enter key. • Context Menu: Simplify≻Simplify • Context Menu: Solve≻Isolate Expression for≻δ_L	$f\left(3 - \delta_L\right) = 6 - \varepsilon$ $-\frac{\left(3 - \delta_L\right)^2 - 9}{\delta_L} = 6 - \varepsilon$ $\overset{\text{simplify}}{=}$ $6 - \delta_L = 6 - \varepsilon$ $\xrightarrow{\text{isolate for delta[L]}}$ $\delta_L = \varepsilon$
• Write the equation $f\left(a + \delta_R\right) = L + \varepsilon$ • Press the Enter key. • Context Menu: Simplify≻Simplify • Context Menu: Solve≻Isolate Expression for≻δ_R	$f\left(3 + \delta_R\right) = 6 + \varepsilon$ $\frac{\left(3 + \delta_R\right)^2 - 9}{\delta_R} = 6 + \varepsilon$ $\overset{\text{simplify}}{=}$ $6 + \delta_R = 6 + \varepsilon$ $\xrightarrow{\text{isolate for delta[R]}}$ $\delta_R = \varepsilon$

Consequently, $\delta_L = \delta_R = \delta(\varepsilon) = \varepsilon$. To complete the proof, show that $|x - 3| < \varepsilon \Rightarrow |f(x) - 6| < \varepsilon$. This is done by showing that $|f(3 + t\,\varepsilon) - 6| < \varepsilon$.

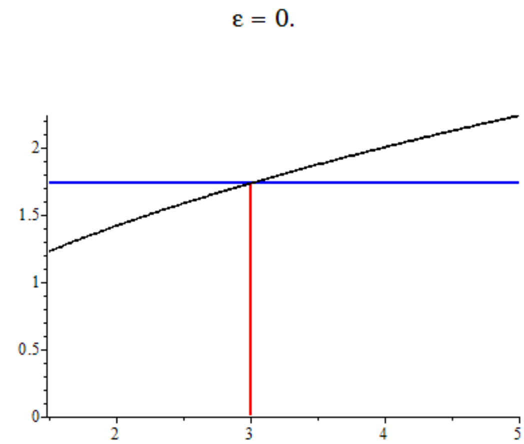
$$|f(3 + t\,\varepsilon) - 6| = \left|\frac{(3 + t\,\varepsilon)^2 - 9}{t\,\varepsilon} - 6\right| \overset{\text{simplify}}{=} |t\,\varepsilon| = |t|\,|\varepsilon| = |t|\,\varepsilon < \varepsilon$$

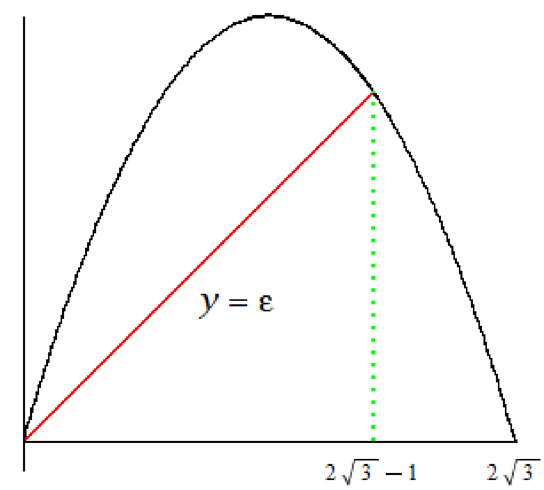
The first and second equalities follow from straight-forward algebra; the next, from the recognition that the absolute value of a product is the product of the absolute values. The next equality follows because ε is positive, and the final inequality follows because $|t| < 1$.

Example 3

Use Definition 1 to verify $\lim_{x \to 3} \sqrt{x} = \sqrt{3}$.

Solution

• Click the restart icon in the toolbar or execute the **restart** command to the right.	*restart*
• Type the equation $f(x) = \ldots$ • Context Menu: Assign Function	$f(x) = \sqrt{x} \xrightarrow{\text{assign as function}} f$
Figure 4 is an animation in which $f(x) = \sqrt{x}$ is graphed in black, and the line $y = \sqrt{3}$ is graphed in blue. • The red and green horizontal lines are drawn at $y = \sqrt{3} \pm \varepsilon$, respectively, and the red and green vertical lines are drawn at the corresponding x-coordinates $x = f^{-1}\left(\sqrt{3} - \varepsilon\right)$ and $x = f^{-1}\left(\sqrt{3} + \varepsilon\right)$. • The slider in the animation toolbar controls the value of ε. As the slider is moved, the red and green horizontal lines delineate the ε-band around $y = \sqrt{3}$ and the red and green vertical lines delineate the band $3 - \delta_L < x < 3 + \delta_R$.	 **Figure 4** Animation illustrating Definition 1
• Write the equation $f\left(a - \delta_L\right) = L - \varepsilon$ • Press the Enter key. • Context Menu: Solve>Isolate Expression for>δ_L • Context Menu: Simplify>Simplify	$f\left(3 - \delta_L\right) = \sqrt{3} - \varepsilon$ $\sqrt{3 - \delta_L} = \sqrt{3} - \varepsilon$ $\xrightarrow{\text{isolate for delta[L]}}$ $\delta_L = -\left(\sqrt{3} - \varepsilon\right)^2 + 3$ $\underset{=}{\text{simplify}}$ $\delta_L = 2\sqrt{3}\,\varepsilon - \varepsilon^2$
• Write the equation $f\left(a + \delta_R\right) = L + \varepsilon$ • Press the Enter key. • Context Menu: Solve>Isolate Expression for>δ_R	$f\left(3 + \delta_R\right) = \sqrt{3} + \varepsilon$ $\sqrt{3 + \delta_R} = \sqrt{3} + \varepsilon$ $\xrightarrow{\text{isolate for delta[R]}}$

• Context Menu: Simplify≻Simplify	$\delta_R = (\sqrt{3} + \varepsilon)^2 - 3$ $\overset{\text{simplify}}{=}$ $\delta_R = 2\sqrt{3}\,\varepsilon + \varepsilon^2$
• Clearly, $\delta_L < \delta_R$, but the choice $\delta(\varepsilon) = \varepsilon$ is simpler. • Figure 5 suggests that $\delta_L = 2\sqrt{3}\,\varepsilon - \varepsilon^2 > \varepsilon$ for $0 < \varepsilon < 2\sqrt{3} - 1 \doteq 2.46$. • To establish this inequality analytically, compare the left- and right-sides via the ratio $\dfrac{2\sqrt{3}\,\varepsilon - \varepsilon^2}{\varepsilon} = 2\sqrt{3} - \varepsilon$ which is greater than 1 for $0 < \varepsilon < 2\sqrt{3} - 1$.	 **Figure 5** Graph of $\delta(\varepsilon)$ and the line $y = \varepsilon$

To complete the proof, show that $|x - 3| < \delta = \varepsilon \Rightarrow \left|f(x) - \sqrt{3}\right| < \varepsilon$. This is done in Table 1 by showing that $\left|f(3 + t\varepsilon) - \sqrt{3}\right| < \varepsilon$.

$\left|f(3 + t\varepsilon) - \sqrt{3}\right| = \left|\sqrt{3 + t\varepsilon} - \sqrt{3}\right|$
$= \left|\left(\sqrt{3 + t\varepsilon} - \sqrt{3}\right)\dfrac{\sqrt{3 + t\varepsilon} + \sqrt{3}}{\sqrt{3 + t\varepsilon} + \sqrt{3}}\right|$
$= \left|\dfrac{t\varepsilon}{\sqrt{3 + t\varepsilon} + \sqrt{3}}\right|$
$< \left|\dfrac{t\varepsilon}{1}\right| = |t|\,\varepsilon < \varepsilon$

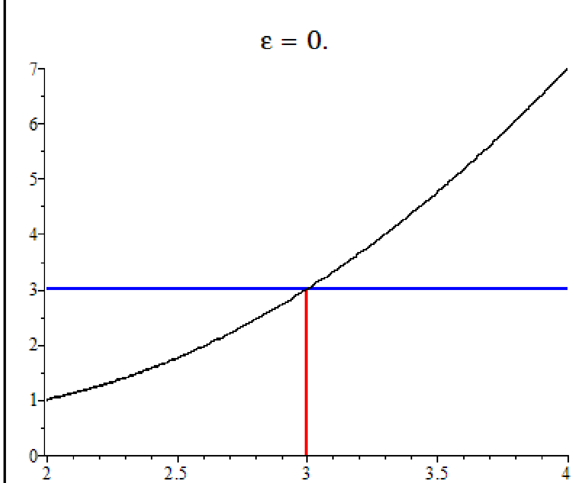
Table 1 Verification that $x = 3 + t\,\varepsilon \Rightarrow \left|f(3+t\,\varepsilon) - \sqrt{3}\right| < \varepsilon$	

The key step is in the second equality, where the "numerator" is rationalized, resulting in the third equality. The first inequality follows from the observation that the sum of the square roots in the preceding equality is greater than 1, so replacing this denominator with 1 makes the denominator smaller, and thus, the fraction larger. The remaining two steps are the same as in Examples 1 and 2.

Example 4

Use Definition 1 to verify $\lim_{x \to 3} \left(x^2 - 3\,x + 3\right) = 3$.

Solution

• Click the restart icon in the toolbar or execute the **restart** command to the right.	*restart*
• Type the equation $f(x) = \ldots$ • Context Menu: Assign Function	$f(x) = x^2 - 3\,x + 3 \xrightarrow{\text{assign as function}} f$
The animation in Figure 6 shows $f(x) = x^2 - 3\,x + 3$ in black, and the line $y = 3$ in blue. • The red and green horizontal lines are drawn at $y = 3 \pm \varepsilon$, respectively, and the red and green vertical lines are drawn at the corresponding x -coordinates $3 - \delta_L = f^{-1}(3 - \varepsilon)$ and $3 + \delta_R = f^{-1}(3 + \varepsilon)$. • • The slider in the animation toolbar controls the value of ε . As the slider is moved, the red and green horizontal lines delineate the ε -band around $y = 3$ and the red and green vertical lines delineate the band $3 - \delta_L < x < 3 + \delta_R$.	 **Figure 6** Animation illustrating Definition 1

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- Write the equation $f\left(a-\delta_L\right)=L-\varepsilon$
- Press the Enter key.
- Context Menu: Solve≻Obtain Solutions for≻δ_L

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$f\left(3-\delta_L\right)=3-\varepsilon$

$\left(3-\delta_L\right)^2-6+3\,\delta_L=3-\varepsilon$

$\xrightarrow{\text{solutions for delta[L]}}$

$\frac{3}{2}+\frac{1}{2}\sqrt{9-4\,\varepsilon},\ \frac{3}{2}-\frac{1}{2}\sqrt{9-4\,\varepsilon}$

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Select the solution that tends to 0 as $\varepsilon\rightarrow 0$. (The other solution tends to 3.) Hence,

$\delta_L := \frac{3}{2}-\frac{1}{2}\sqrt{9-4\,\varepsilon}$:

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- Write the equation $f\left(a+\delta_R\right)=L+\varepsilon$
- Press the Enter key.
- Context Menu: Solve≻Obtain Solutions for≻δ_R

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$f\left(3+\delta_R\right)=3+\varepsilon$

$\left(3+\delta_R\right)^2-6-3\,\delta_R=3+\varepsilon$

$\xrightarrow{\text{solutions for delta[R]}}$

$-\frac{3}{2}+\frac{1}{2}\sqrt{9+4\,\varepsilon},\ -\frac{3}{2}-\frac{1}{2}\sqrt{9+4\,\varepsilon}$

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Select the solution that tends to 0 as $\varepsilon\rightarrow 0$. (The other solution tends to -3 .) Hence,

$\delta_R := -\frac{3}{2}+\frac{1}{2}\sqrt{9+4\,\varepsilon}$:

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- To determine $\min\left\{\delta_L,\delta_R\right\}=\delta_R$, show $\delta_L/\delta_R>1$.
- Begin by simplifying the ratio δ_L/δ_R and finish via the calculations in Table 2.

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$\frac{\delta_L}{\delta_R} =$

$\frac{\frac{3}{2}-\frac{1}{2}\sqrt{9-4\,\varepsilon}}{-\frac{3}{2}+\frac{1}{2}\sqrt{9+4\,\varepsilon}} \overset{\text{simplify}}{=} -\frac{-3+\sqrt{9-4\,\varepsilon}}{-3+\sqrt{9+4\,\varepsilon}}$

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The first step in Table 2 is the rationalization of the denominator; the second is making the resulting numerator smaller by changing $9+4\,\varepsilon$ to $9-4\,\varepsilon$.

$\frac{\delta_L}{\delta_R} = \; = \frac{3-\sqrt{9-4\varepsilon}}{\sqrt{9+4\varepsilon}-3} \cdot \frac{\sqrt{9+4\varepsilon}+3}{\sqrt{9+4\varepsilon}+3}$	$= \; = \frac{1}{4\varepsilon}\left(\left(3-\sqrt{9-4\varepsilon}\right)\left(3+\sqrt{9+4\varepsilon}\right)\right)$
	$> \frac{1}{4\varepsilon}\left(\left(3-\sqrt{9-4\varepsilon}\right)\left(3+\sqrt{9-4\varepsilon}\right)\right)$
	$= \frac{4\varepsilon}{4\varepsilon} = 1$
Table 2 Show $\delta_L/\delta_R > 1$	

Consequently, $\delta(\varepsilon) = \delta_R = \left(\sqrt{9+4\varepsilon}-3\right)/2$. To complete the proof, show that $|x-3| < \delta = \delta_R \Rightarrow |f(x)-3| < \varepsilon$. This is done in Table 3 by showing that $\left|f\left(3+t\,\delta_R\right)-3\right| < \varepsilon$.

1	$\left|f\left(3+t\,\delta_R\right)-3\right| = \left|-\frac{9}{2}t+\frac{3}{2}t\sqrt{9+4\varepsilon}+\frac{9}{2}t^2-\frac{3}{2}t^2\sqrt{9+4\varepsilon}+t^2\varepsilon\right|$
2	$= \left|-\frac{9}{2}t+\frac{3}{2}t\sqrt{9+4\varepsilon}+\frac{9}{2}t^2-\frac{3}{2}t^2\sqrt{9+4\varepsilon}+t^2\varepsilon\right|$
3	$= \left|\left(\frac{9}{2}-\frac{3}{2}\sqrt{9+4\varepsilon}+\varepsilon\right)t^2+\left(-\frac{9}{2}+\frac{3}{2}\sqrt{9+4\varepsilon}\right)t\right|$
4	$< \left|\varepsilon t-3\,\delta_R\,t+3\,\delta_R\,t\right|$
5	$= \left|\varepsilon t\right|$
6	$= \varepsilon\,\left|t\right|$

7	$< \varepsilon$
Table 3 Verification that $x = 3 + t\,\delta_R \Rightarrow \left|f\left(3 + t\,\delta_R\right) - 3\right| < \varepsilon$	

Lines 1-3 in Table 3 are obtained with basic algebra. The result in (4) hinges on several observations. First, the coefficient of t^2 in (3) is positive, so replacing t^2 with t makes the expression larger. Then, this coefficient is $\varepsilon - 3\,\delta_R$ and the coefficient of t in (3) is $3\,\delta_R$. Once the result in (4) has been obtained, steps 5-7 are self evident.

The key, then, seems to be recognizing that, in (3), the coefficient of t^2 is positive. A graph of the coefficient would suggest this positivity, but applying the binomial expansion to the square root gives $(\varepsilon/2)^2$ as a first approximation to the coefficient would be more rigorous.