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Learning Optimal Feedback Operators and their Sparse Polynomial Approximations

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Abstract

A learning based method for obtaining feedback laws for nonlinear optimal control problems is proposed. The learning problem is posed such that the open loop value function is its optimal solution. This infinite dimensional, function space, problem, is approximated by a polynomial ansatz and its convergence is analyzed. An ℓ_1 penalty term is employed, which combined with the proximal point method, allows to find sparse solutions for the learning problem. The approach requires multiple evaluations of the elements of the polynomial basis and of their derivatives. In order to do this efficiently a graph-theoretic algorithm is devised. Several examples underline that the proposed methodology provides a promising approach for mitigating the curse of dimensionality which would be involved in case the optimal feedback law was obtained by solving the Hamilton Jacobi Bellman equation.

Keywords: Optimal feedback control, nonlinear systems, learning theory, Hamilton Jacobi Bellman equation, polynomial based approximation.

1. Introduction

Designing optimal feedback laws for non-linear control problems is a challenging problem from both the theoretical and applied points of view. The main approach for obtaining an optimal feedback law is based on dynamic programming. Its solution involves the control theoretic Hamilton-Jacobi-Bellman (HJB) equation. For high dimensional problems the computational cost of directly solving the HJB equation makes this approach non-viable. In the last years many efforts have been put forward to overcome this difficulty and to partially alleviate the curse of dimensionality. Here we can only mention a very small sample of the large number of contributions: representation formulas (Chow et al., 2017, 2019a,b; Darbon and Osher, 2016), approximating the HJB equation by neural networks (Han et al., 2018; Darbon et al., 2020; Nüsken and Richter, 2020; Onken et al., 2021; Ito et al., 2021; Kunisch and Walter, 2021; Ruthotto et al., 2020), data driven approaches (Nakamura-Zimmerer et al., 2021a,b; Azmi et al., 2021; Kang et al., 2021; Albi et al., 2022), max-plus methods (Akian et al., 2008; Gaubert et al., 2011; Dower et al., 2015), polynomial approximation (Kalise and Kunisch, 2018; Kalise et al., 2020), tensor decomposition methods (Horowitz et al., 2014; Stefansson and Leong, 2016; Gorodetsky et al., 2018; Dolgov et al., 2021; Oster et al., 2019, 2022), POD methods (Alla et al., 2017; Kunisch et al., 2004), tree structure algorithms (Alla et al., 2019), and sparse grids techniques(Bokanowski et al., 2013; Garcke and Kröner, 2017; Kang and Wilcox, 2017), see also the proceedings volume (Kalise et al., 2018). Among the classical methods for solving the HJB equation we mention finite difference schemes (Bonnans et al., 2003), semi-Lagrangian schemes (Falcone and Ferretti, 2013), and policy iteration (Alla et al., 2015; Beard et al., 1997; Puterman and Brumelle, 1979; Santos and Rust, 2004). Learning techniques have also been investigated in the context of model predictive control, see for instance (Drgoňa et al., 2022).

In the present work we propose, analyze, and numerically test a learning approach to obtain optimal feedback laws. The problem is formulated in a way that all its solutions are optimal feedback laws. The learning problem is based on finding a feedback-law which minimizes the average of the objective functions with respect to a set of the initial conditions. The feedback-law is obtained in terms of the gradient of a scalar valued function. Due to the infinite dimensional nature of the problem, a finite dimensional approximation is required. We propose polynomials as ansatz functions and add a ℓ_1 penalty term, in order to promote sparsity in the solutions. This work is an extension of the developments commenced in (Kunisch and Walter, 2021), where the feedback was parametrized by a neural network. Appropriate hypotheses on the value function are provided which ensure the existence of a solution to this problem. Furthermore, convergence is established as the dimension of the ansatz space tends to infinity. In order to efficiently evaluate elements of the chosen polynomial basis and their first and second order derivatives, a tree-based procedure is devised. The choice of monomials as ansatz functions turned out to be computationally very promising. Certainly it would also be of interest to investigate other non-grid based, approximation schemes in the future.

The structure of this work is as follows. In Section 2 we introduce the learning problem and in Section 3 we present the finite dimensional approximation by polynomials. In Section 4 the existence of solutions and convergence of the finite dimensional problems are established. The optimality conditions for the finite dimensional problem are studied in Section 5, together with a basis reduction procedure. The learning algorithm is developed in Section 6. An efficient polynomial basis evaluation method is developed in Section 7. In Section 8 we present a result concerning the generalization capability of our approach. Finally, in Section 9 we present four numerical experiments which show that our algorithm is able to solve non-linear and high (here the dimension is 40) dimensional control problems in a standard laptop environment.

We end the section by introducing some notation which is needed in the following. For k, m, d all integers greater than or equal 1, and a domain $A \subset \mathbb{R}^m$, the spaces $H^k(A; \mathbb{R}^d)$ and $H^k_{loc}(A; \mathbb{R}^d)$ denote the Sobolev spaces and local Sobolev spaces of order k from A to $\mathbb{R}^d$. Analogously, for $p \geqslant 1$ the spaces $L^p(A; \mathbb{R}^d)$ and $L^p_{loc}(A; \mathbb{R}^d)$ are the spaces of p integrable and locally p integrable functions from A to $\mathbb{R}^d$. In addition, for a compact set $K \subset \mathbb{R}^m$ and $\alpha \in (0, 1]$, we define $C^{k,\alpha}(K; \mathbb{R}^d)$ to be the class of k times differentiable functions with $\alpha-$Hölder continuous derivatives up to order k from K to $\mathbb{R}^d$. For a Lipschitz continuous function $v : K \to \mathbb{R}^m$ we define

$$|v|_{Lip(K)} = \max_{x \neq y \in K} \frac{|v(x) - v(y)|}{|x - y|},$$

where $|\cdot|$ is the usual euclidean norm. For $y \in \mathbb{R}^m$ we denote the $p-$norm with $p \in (1, \infty) \setminus \{2\}$ by $|y|_p$, and the maximum norm by $|y|_\infty$.

2. Statement of the Problem

In this work we study the infinite horizon optimal control problem:

$$\min_{u \in L^2((0,\infty);\mathbb{R}^m)} J(u, y_0) := \int_0^\infty \ell(y(t))dt + \frac{\beta}{2} \int_0^\infty |u(t)|^2 dt \tag{1}$$

where $y \in H^1_{loc}((0, \infty); \mathbb{R}^d)$ is the unique solution of

$$y'(t) = f(y(t)) + Bu(t), \quad t \in (0, \infty), \quad y(0) = y_0. \tag{2}$$

Here $f : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz on bounded sets, $\ell : \mathbb{R}^d \to \mathbb{R}$ is of class C^1, bounded from below by 0, $\beta > 0$ is the penalization for the control, and $B \in \mathbb{R}^{d \times m}$ is the control matrix, with $d \geqslant m \in \mathbb{N}$. We also assume that $\ell(0) = 0$ and $f(0) = 0$. This implies that 0 is an equilibrium for system (2). In (1) the cost J is considered as extended real-valued function.

A solution of Problem (1) in feedback form can be obtained by means of dynamic programming. Namely, defining the value function of (1) by

$$V(y_0) = \min_{u \in L^2((0,\infty);\mathbb{R}^m)} J(u, y_0), \tag{3}$$

and assuming that V is differentiable in an open neighborhood $U \subset \mathbb{R}^d$ of y_0, then V solves the Hamilton Jacobi Bellman equation

$$\min_{u \in \mathbb{R}^m} \left\{ \nabla V(y)^\top (f(y) + Bu) + \frac{\beta}{2}|u|^2 + \ell(y) \right\} = 0 \tag{4}$$

in U. By the verification's Theorem (see Theorem 5.1 in (Fleming and Soner, 2006)), the optimal control in (1) is given by the feedback law:

$$u^*(t) = -\frac{1}{\beta} B^\top \nabla V(y^*(t)), \tag{5}$$

provided $y^*(t) \in U$. Here y^* is the solution of (2) corresponding to u^*. Replacing u by u^* in (2) we get the closed loop system

$$y'(t) = f(y(t)) - \frac{1}{\beta} BB^\top \nabla V(y(t)), \quad t \in (0, \infty), \quad y(0) = y_0. \tag{6}$$

This approach involves solving the Hamilton Jacobi Bellman equation, which is computationally expensive or even unfeasible for problems of high dimension. Therefore, in this work we propose to find a feedback law by solving a learning problem. For this purpose we define a computational domain $\Omega = (-l, l)^d$ with $l > 0$, and a set of initial conditions $\{y_0^1, \ldots, y_0^I\} \subset \mathbb{R}^d$, which will be called the training set. We assume throughout that the value function V is of class $C^{1,1}(\overline{\Omega})$ and that the image of the solutions of the closed loop problems (6) are strictly contained in Ω, that is, there exists $\delta \in (0, l)$ such that

$$|y_i(t)|_\infty \leqslant l - \delta \text{ for all } t \in (0, \infty), \tag{7}$$

where y_i for $i \in \{1, \ldots, I\}$ are the solutions to the closed loop problem (6) for $y_0 = y_0^i$. For $T \in (0, \infty)$ we define $\mathcal{J}_T : C^1(\overline{\Omega}) \to [0, \infty]$ by

$$v \mapsto \mathcal{J}_T(v) = \frac{1}{I} \sum_{i=1}^{I} \int_0^T \left(\ell(y_i(t)) + \frac{1}{2\beta} |B^\top \nabla v(y_i(t))|^2 \right) dt, \tag{8}$$

where $y_i \in C^1([0, T]; \mathbb{R}^d)$ are the solutions of the closed loop problems

$$\begin{cases} y_i'(t) = f(y_i(t)) - \frac{1}{\beta} B B^\top \nabla v(y_i(t)), \ y_i(0) = y_0^i, \\ y_i(t) \in \overline{\Omega}, \ t \in (0, T). \end{cases} \tag{9}$$

If there exists an index $i \in \{1, \ldots, I\}$ such that problem (9) has no solution for a given $v \in C^{1,1}(\overline{\Omega})$, then we set $\mathcal{J}_T(v) = \infty$. Further, we define $\mathcal{J}_\infty : C^1(\overline{\Omega}) \to [0, \infty]$ as the pointwise limit of $\mathcal{J}_T$ when T goes to infinity, i.e.

$$v \mapsto \mathcal{J}_\infty(v) := \lim_{T \to \infty} J_T(v). \tag{10}$$

Under these definitions, we formulate the learning problem

$$\min_{v \in C^{1,1}(\overline{\Omega}), \ \nabla v(0) = 0, \ v(0) = 0} \quad \mathcal{J}_\infty(v). \tag{11}$$

We notice that assuming $V \in C^{1,1}(\overline{\Omega})$ and (7), the value function is a solution of (11). Moreover, if there exits an optimal solution v^* of (11), then the controls defined by $u_i^*(t) = -\frac{1}{\beta} \nabla v^*(y_i^*(t))$ with y_i^* the solutions of (9) are optimal solutions of (1).

In order to solve this problem numerically we replace the infinite dimensional function space $C^{1,1}(\overline{\Omega})$ by a finite dimensional space. In this case we add a penalty term in order to ensure the existence of at least one solution. Moreover for numerical purposes we introduce a finite horizon formulation. This problem will be formulated in Section 3, where we also state results regarding the existence of solutions and their convergence to a solution of (1).

3. Polynomial Learning Problem

In this section we formulate the finite dimensional learning problem. For this purpose we first introduce some notation. Let $n \in \mathbb{N}$ and $d \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. We denote the space of polynomials with total degree less than or equal to n in $\mathbb{R}^d$ by $\mathcal{P}_n$ and its dimension by m_n. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ we define a monomial ϕ_α by

$$\phi_\alpha(y) = \prod_{j=1}^{d} y_j^{\alpha_j}, \quad y \in \mathbb{R}^d. \tag{12}$$

We denote by Λ_n the set of multi-indexes such the sum of all its elements is lower or equal than n, that is

$$\Lambda_n = \left\{ \alpha \in \mathbb{N}^d : \sum_{j=1}^{d} \alpha_j \leqslant n \right\}. \tag{13}$$

We assume that the set of all the multi-index $\mathbb{N}^d$ is ordered such that

$$\mathbb{N}^d = \left\{ \alpha^i \right\}_{i=1}^{\infty} \text{ and } \Lambda_{n+1} = \Lambda_n \bigcup \left\{ \alpha^i \right\}_{i=m_n+1}^{m_{(n+1)}}, \tag{14}$$

for example, for $d = 2$, we have $\Lambda_1 = \{(0,0);(0,1);(1,0)\}$. We denote the set of all the monomials with total degree lower or equal to n by $\mathcal{B}_n$. Therefore, by (13) and (14) we have

$$\mathcal{B}_n = \{\phi_\alpha : \alpha \in \Lambda_n\} \text{ and } \mathcal{B}_{n+1} = \mathcal{B}_n \bigcup \left\{ \phi_{\alpha^i} \right\}_{i=m_n+1}^{m_{(n+1)}}, \tag{15}$$

We denote the *hyperbolic cross* multi-index set by Γ_n, i.e.

$$\Gamma_n = \Big\{ \alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d : \quad \prod_{j=1}^{d} (\alpha_j + 1) \leqslant n + 1 \Big\}. \tag{16}$$

We also introduce the subset $\mathcal{S}_n$ of $\mathcal{B}_n$ composed by the elements of $\mathcal{B}_n$ associated to the multi-indexes in Γ_n, i.e.

$$\mathcal{S}_n = \{\phi_\alpha : \ \alpha \in \Gamma_n\} \tag{17}$$

and the subspace $\mathcal{A}_n$ of $\mathcal{P}_n$ generated by $\mathcal{S}_n$.

It is important to observe that the cardinality of Λ_n is $\sum_{j=1}^{n} \binom{d+j+1}{j}$, on the other hand the cardinality of Γ_n is bounded by $\min\{2n^3 4^d, e^2 n^{2+\log_2(n)}\}$ (Adcock et al., 2017). For an intuitive description of the hyperbolic cross basis see the figures in e.g. (Azmi et al., 2021). Hence, for high d the cardinality of the hyperbolic cross is smaller than cardinality of Λ_n. For this reason, Γ_n is more suitable for high dimensional problems.

To introduce a family of approximating computationally tractable problems, we consider a finite set of monomials $X = \{\phi_i\}_{i=1}^{M}$, which can be $\mathcal{B}_n \setminus \mathcal{B}_1$ or $S_n \setminus \mathcal{B}_1$. Here we subtract $\mathcal{B}_1$ to ensure $v(0) = 0$ and $\nabla v(0) = 0$. Then, for $\theta \in \mathbb{R}^M$, setting

$$v = \sum_{i=1}^{M} \theta_i \phi_i,$$

we define

$$\tilde{\mathcal{J}}_T(\theta) = \mathcal{J}_T(v). \tag{18}$$

Further we define a penalty function $P_{\gamma,r}$, with $\gamma > 0$ and $r \in [0,1]$ by

$$\theta \mapsto P_{\gamma,r}(\theta) = \gamma\left(\frac{(1-r)}{2}|\theta|_2^2 + r|\theta|_1\right). \tag{19}$$

Now we are in a position to introduce the finite dimensional version of the learning problem. That is, we replace $C^{1,1}(\overline{\Omega})$ with the space of polynomial spanned by X, add the penalty $P_{\gamma,r}$ to the objective function and consider a finite time horizon $T \in (0,\infty)$, namely

$$\min_{\theta\in\mathbb{R}^M} \tilde{\mathcal{J}}_T(\theta) + P_{\gamma,r}(\theta). \tag{20}$$

The penalty term $P_{\gamma,r}$ ensures the coercivity of the objective function. Moreover the non-smooth term in (19) promotes the sparsity of the solution of (20). However, unless we assume some further hypotheses on the structure of ℓ, f, B and/or the value function V, we do not yet know if there exist a solution of (20).

4. Existence and Convergence

In this section we are concerned with the existence of solutions for (20). For $n \geqslant 2$, $T \in (0,\infty]$, $X = \mathcal{B}_n \setminus \mathcal{B}_1$ or $X = \mathcal{S}_n \setminus \mathcal{B}_1$, and M the cardinality of X, we say that $\theta \in \mathbb{R}^M$ is a feasible solution for problem (20) if $\tilde{\mathcal{J}}_T(\theta) < \infty$. If there exits a feasible solution for problem (20), we say that the problem is feasible.

Theorem 1 *Consider $\gamma > 0$, $r \in [0,1]$, $T \in (0,\infty]$ and $X = \{\phi_i\}_{i=1}^M \subset C^{1,1}(\overline{\Omega})$. If problem* (20) *is feasible, then it has at least one optimal solution.*

The proof of this theorem as well as of the remaining results of this section are given in Appendix A.

In general, we do not know if there exists any feasible solution for the learning problem. Nevertheless, given that V is in $C^{1,1}(\overline{\Omega})$ and the density of the polynomials in $C^{1,1}(\overline{\Omega})$, we prove that for every finite time horizon $T > 0$ there exists a degree high enough, such that (20) has at least one feasible solution. Moreover, assuming the exponential stability of the closed loop problem (6) and that $V \in C^2(\overline{\Omega})$, we obtain that there exists a feasible solution for $T = \infty$.

Proposition 1 *For every $T \in (0,\infty)$, there exits a positive integer n and $V_{n,T} \in \mathcal{P}_n$ such that $\mathcal{J}_T(V_{n,T}) < \infty$. Moreover, assuming that the value function V is $C^2(\overline{\Omega})$, that*

$$\lim_{t\to\infty} y_i(t) = 0, \text{ for all } i \in \{1,\dots,I\}, \tag{21}$$

where $\{y_i\}_{i=1}^I$ are the solutions of the closed loop problems (6)*, and that the linearized system*

$$z' = \left(Df(0) - \frac{1}{\beta}BB^\top\nabla\tilde{V}(0)\right)z, \quad z(0) = z_0 \tag{22}$$

is exponentially stable, we have that there exists a positive integer $\tilde{n}$ and $V_{\tilde{n}} \in \mathcal{P}_{\tilde{n}}$ such that $\mathcal{J}_\infty(V_{\tilde{n}}) < \infty$.

Above we call system (22) exponentially stable if there exist $C > 0$ and $\mu > 0$ such that $|z| \leqslant Ce^{-\mu t}|z_0|$ for all $t \in (0, \infty)$ and $z_0 \in \mathbb{R}^d$. Exponential stability is guaranteed if the real parts of all the eigenvalue of the system matrix in (22) are strictly negative. Proposition 1 is a direct consequence of Theorem 9 in (Hájek and Johanis, 2014, Section 7.2), Proposition 3 and Lemma 3, which can be found in Appendix A. We now address the convergence of problem (20) to (11).

Theorem 2 *There exist sequences $T_k \in (0, \infty)$, $\gamma_k \in (0, \infty)$, $r_k \in [0, 1]$, $X_k = \mathcal{B}_{n_k} \setminus \mathcal{B}_1$ respectively $X_k = \mathcal{S}_{n_k} \setminus \mathcal{B}_1$ with $n_k \in \mathbb{N}$ and M_k the cardinality of X_k, and $\theta^k \in \mathbb{R}^{M_k}$ solution of (20), such that $n_k \to \infty$, $T_k \to \infty$, $\gamma_k \to 0$ and $\tilde{\mathcal{J}}_{T_k}(\theta_k)$ converges to the value of (11) when $k \to \infty$. Moreover, setting $v_k = \sum_{j=1}^{M_k} \theta^k_j \phi_j$, where $X_k = \{\phi_i\}_{i=1}^{M_k}$, and defining*

$$u_i^k = -\frac{1}{\beta} B^T \nabla v_k(y_i^k), \tag{23}$$

where y_i^k is the solution of the closed loop problem (9) for $v = v_k$, $T = T_k$ and $y_0 = y_0^i$, we have that

$$y_i^k \rightharpoonup y_i^* \text{ in } H^1_{loc}((0;\infty);\mathbb{R}^d) \text{ and } u_i^k \rightharpoonup u_i^* \text{ in } L^2_{loc}((0;\infty);\mathbb{R}^m), \text{ when } k \to \infty \tag{24}$$

where u_i^* is a solution of the open loop problem (1) and y_i^* the solution of (2).

The proof of this theorem can be found in Appendix A.

Remark 1 Assuming that (11) admits a solution $v^* \in C^{1,1}(\overline{\Omega})$ and that there exist $v_n \in \mathcal{P}_n$ such that

$$\lim_{n\to\infty} v_n = v^* \text{ in } C^{1,1},\ \mathcal{J}_\infty(v_n^*) < \infty \text{ and } \lim_{n\to\infty} \mathcal{J}_\infty(v_n) = \mathcal{J}_\infty(v^*), \tag{25}$$

it is possible to take $T_k = \infty$ for all $k \in \mathbb{N}$. In this case one can formulate (20) as infinite horizon problem.

5. Optimality Conditions

Throughout this section we consider a basis $X = \{\phi_i\}_{i=1}^M$ for $M \in \mathbb{N}$, where for each $i \in \{1, \dots, M\}$ the function ϕ_i is a monomial given by (12) for a multi-index $\alpha^i \in \mathbb{N}^d$. We recall that we have defined the function $\tilde{\mathcal{J}}_T$ such that $\theta \mapsto \tilde{\mathcal{J}}_T(\theta) = \mathcal{J}_T(v)$, where $v = \sum_{k=1}^M \theta_k \phi_k$.

Consider $\theta \in \mathbb{R}^M$ and $T > 0$ finite. Assume that for each $i \in \{1, \dots, I\}$, there exists a unique solution of (9) with $v = \sum_{k=1}^M \theta_k \phi_k$, denoted by $y_i \in C^1([0, T], \overline{\Omega})$. Further, assume that $y_i(t) \in \Omega$ for all $t \in [0, T]$. Then, $\tilde{\mathcal{J}}_T$ is differentiable in θ and its partial derivatives are given by

$$\frac{\partial}{\partial \theta_k} \tilde{\mathcal{J}}_T(\theta) = \frac{1}{I\beta} \int_0^T \sum_{i=1}^I \nabla \phi_k^\top(y_i) BB^\top \left(\nabla v(y_i) + p_i\right) dt \text{ for } k \in \{1, \dots, M\}, \tag{26}$$

where p_i is the solution of

$$-p_i' - Df(y_i)^\top p + \frac{1}{\beta} \nabla^2 v(y_i) BB^\top (\nabla v(y_i) + p_i) = -\nabla \ell(y_i), \quad p_i(T) = 0 \tag{27}$$

for $i \in \{1, \dots, I\}$, where Df is the Jacobian matrix of f.

For the non differentiable term in $P_{\gamma,r}$ we recall that the subgradient of $|\cdot|_1$ is given by

$$\partial|\cdot|_1(\theta)_k = \begin{cases} \{1\} & \text{if } \theta_k > 0, \\ \{-1\} & \text{if } \theta_k < 0, \\ [-1,1] & \text{if } \theta_k = 0, \end{cases} \quad \text{for } k \in \{1, \dots, M\}.$$

Hence, if $\theta^* \in \mathbb{R}^M$ is a solution of (20) it satisfies the following optimality condition

$$\nabla \tilde{\mathcal{J}}_T(\theta^*) + \gamma(1-r)\theta^* \in -\gamma r \cdot \partial|\cdot|_1(\theta^*). \tag{28}$$

For each $k \in \{1, \dots, M\}$, we deduce from (28) that

$$\left| \frac{\partial}{\partial \theta_k} \tilde{\mathcal{J}}(\theta^*) + \gamma(1-r)\theta_k^* \right| < \gamma r \implies \theta_k^* = 0$$

and

$$\text{if } \theta_k^* \neq 0, \text{ then } \frac{\partial}{\partial \theta_k} \tilde{\mathcal{J}}(\theta^*) + \gamma(1-r)\theta_k^* = \begin{cases} \gamma r & \text{if } \theta_k^* < 0, \\ -\gamma r & \text{if } \theta_k^* > 0. \end{cases}$$

In the remainder of this section we shall verify the following property which is enjoyed by any optimal solution θ^*:

$$\text{for each } k \in \{1, \dots, M\},\ \theta_k^* = 0 \text{ if and only if } B^\top \nabla \phi_k(y) = 0 \text{ for all } y \in \mathbb{R}^d. \tag{29}$$

We define the subset $\mathcal{O}(X)$ of X by

$$\mathcal{O}(X) := \{\phi \in X : \ B^\top \nabla \phi(y) = 0 \ \forall \ y \in \mathbb{R}^d\} \tag{30}$$

It is possible to further characterize the elements of $\mathcal{O}(X)$.

Lemma 1 *Let $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ be a multi-index and consider ϕ_α the monomial given by* (12). *Then,*

$$B^\top \nabla \phi_\alpha(y) = 0 \text{ for all } y \in \mathbb{R}^d \text{ if and only if } B^\top \cdot e_i = 0 \text{ for all } i \in \mathcal{I}(\alpha), \tag{31}$$

where e_i is the $i-$th canonical vector of $\mathbb{R}^d$ and

$$\mathcal{I}(\alpha) = \{i \in \{1, \dots, d\} : \alpha_i > 0\}$$

Proof Let assume first that

$$B^\top \nabla \phi_\alpha(y) = 0 \text{ for all } y \in \mathbb{R}^d. \tag{32}$$

We prove now

$$B^\top e_i = 0 \text{ for all } i \in \mathcal{I}(\alpha). \tag{33}$$

If $\mathcal{I}(\alpha) = \{i\}$ for some $i \in \mathbb{N}$, then we have $\nabla \phi_\alpha(y) = \alpha_i y_i^{\alpha_i - 1} e_i$ if $\alpha_i \geqslant 1$ and therefore (33) is evident. On the other hand, if $\mathcal{I}(\alpha)$ contains more than one element, we take any $i \in \mathcal{I}(\alpha)$. Then, for $\varepsilon > 0$, we define $y^\varepsilon \in \mathbb{R}^d$ by

$$y_k^\varepsilon = \begin{cases} \varepsilon^{-a} & \text{if } k = i, \\ \varepsilon & \text{if } k \neq i, \end{cases} \quad \text{for all } k \in \{1, \dots, M\},$$

where in the case $\alpha_i \neq 1$

$$a = \sum_{j \neq i, j=1}^{N} \alpha_j/(\alpha_i - 1) > 0.$$

Evaluating $\nabla \phi_\alpha$ in y^ε we obtain

$$\frac{\partial \phi_\alpha}{\partial y_k}(y^\varepsilon) = \begin{cases} \alpha_i & \text{if } k = i, \\ \alpha_k \varepsilon^{-a-1} & \text{if } k \neq i \end{cases}.$$

Taking $\varepsilon \to \infty$ we obtain

$$\nabla \phi_\alpha(y^\varepsilon) \to \alpha_i e_i$$

and therefore

$$B^\top e_i = \lim_{\varepsilon \to \infty} B^\top \nabla \phi_\alpha(y^\varepsilon) = 0.$$

If $\alpha_i = 1$, then we have

$$\frac{\partial \phi_\alpha}{\partial y_k}(y) = \begin{cases} \displaystyle\prod_{j \neq i} y^{\alpha_j} & \text{if } i = k, \\ \alpha_k y^{\alpha_k - 1} \displaystyle\prod_{j \neq k} y^{\alpha_j} & \text{if } i \neq k, \end{cases} \quad \text{for all } y \in \mathbb{R}^d.$$

We choose $\bar{y} \in \mathbb{R}^d$ such that $\bar{y}_i = 0$ and $\bar{y}_j = 1$ for all $j \in \{1, \ldots, d\} \setminus \{i\}$. Evaluating $\nabla \phi_\alpha$ in $\bar{y}$ we again obtain

$$B^\top e_i = B^\top \nabla \phi_\alpha(\bar{y}) = 0.$$

Since the $i \in \mathcal{I}(\alpha)$ is arbitrary, we have proved (33).

Now we assume that (33) holds and we prove (32). For every $j \in \{1, \ldots, d\} \setminus \mathcal{I}(\alpha)$, it is clear that

$$\frac{\partial \phi_\alpha}{\partial y_j}(y) = 0 \text{ for all } y \in \mathbb{R}^d.$$

From this, we have that

$$B^\top \nabla \phi_\alpha(y) = \sum_{i \in \mathcal{I}(\alpha)} \frac{\partial \phi_\alpha}{\partial y_i}(y) B^\top e_i = 0,$$

which concludes the proof. ■

Lemma 2 *Consider $\theta \in \mathbb{R}^M$ and $T \in (0, \infty)$. Assume that $y_i(t) \in \Omega$ for all $i \in \{1, \ldots, I\}$ and $t \in [0, T]$, where y_i is the solution of the closed loop problem (9) with $v = \sum_{k=1}^M \theta_k \phi_k$ and $y_0 = y_0^i$. Then,*

$$\frac{\partial}{\partial \theta_k} \tilde{\mathcal{J}}_T(\theta) = 0 \text{ for every } k \in \{1, \ldots, M\}, \text{ such that } \phi_k \in \mathcal{O}(X). \tag{34}$$

Moreover, if $\theta^* \in \mathbb{R}^M$ is an optimal solution of (20), then

$$\theta_k^* = 0 \text{ for every } k \in \{1, \ldots, M\}, \text{ such that } \phi_k \in \mathcal{O}(X). \tag{35}$$

Proof For every $\theta \in \mathbb{R}^M$, (34) is a direct consequence of (26). To prove (35) we proceed by contradiction. Let $\theta^* \in \mathbb{R}^M$ be an optimal solution for (20) and assume that there exists $\bar{k} \in \{1, \ldots, M\}$ such that

$$B^\top \nabla \phi_{\bar{k}} = 0 \text{ in } \overline{\Omega}, \text{ but } \theta^*_{\bar{k}} \neq 0. \tag{36}$$

Then, we define $\tilde{\theta} \in \mathbb{R}^M$ by

$$\tilde{\theta}_j = \begin{cases} \theta^*_j & \text{if } j \neq \bar{k} \\ 0 & \text{if } j = \bar{k} \end{cases} \quad \text{for all } j \in \{1, \ldots, M\}.$$

By (36) we have

$$\sum_{k=1}^{M} B^\top \nabla \phi_k(y) \tilde{\theta}_k = \sum_{k=1}^{M} B^\top \nabla \phi_k(y) \theta^*_k \text{ for all } y \text{ in } \Omega.$$

Consequently, for each initial condition y_0^i, the solution of (9) for $\tilde{v} = \sum_{k=1}^{M} \phi_k \tilde{\theta}_k$ is the same as for $v^* = \sum_{k=1}^{M} \phi_k \theta^*_k$. Therefore we get

$$\tilde{\mathcal{J}}_T(\tilde{\theta}) = \tilde{\mathcal{J}}_T(\theta^*).$$

Further, it is clear that $P_{\gamma,r}(\theta^*) > P_{\gamma,r}(\tilde{\theta})$, because $\theta^*_{\bar{k}} \neq 0$. Thus

$$\tilde{\mathcal{J}}_T(\tilde{\theta}) + P_{\gamma,r}(\tilde{\theta}) < \tilde{\mathcal{J}}_T(\theta^*) + P_{\gamma,r}(\theta^*),$$

which is a contradiction. ∎

Remark 2 From Lemma 2 we conclude that basis functions $\phi_k \in \mathcal{O}(X)$ do not contribute to the optimal solution of (20). Therefore they should be dismissed before computing the minimizers (20). This can be done utilizing Lemma 1. In this way we replace X by $X \backslash \mathcal{O}(X)$.

6. Optimization Algorithm

In this section we consider $T \in (0, \infty)$ and $X = \{\phi_i\}_{i=1}^M$ with $M \in \mathbb{N}$, where for each $i \in \{1, \ldots, M\}$ the function ϕ_i is a monomial of the form (12) for a multi-index $\alpha_i \in \mathbb{N}^d$. To solve (20) we use a linear proximal point method with an adaption Barzilai-Borwein method for choosing the step length, which proved to be efficient for high dimensional problems (see (Azmi and Kunisch, 2020), (Barzilai and Borwein, 1988), and (Raydan, 1997) for a convergence analysis in the smooth case). In contrast to the smooth setting, we are not aware of a thorough convergence analysis of this particular step size choice in the nonsmooth case. However, from a practical point of view, the method performs reliably for our purposes.

We now describe the algorithm that we use to solve (20). We denote the $k-$th element of the sequence produced by the algorithm by θ^k, the step size by s^k, and we define at each iteration

$$d^k := \nabla \tilde{\mathcal{J}}_T(\theta^k) + \gamma(1 - r)\theta^k. \tag{37}$$

We use the proximal point update rule as is described in section 10.2 in (Beck, 2017), namely we take θ^{k+1} such that

$$\theta^{k+1} = \underset{\vartheta\in\mathbb{R}^M}{\operatorname{argmin}} \left\{ d^k \cdot \left(\vartheta - \theta^k\right) + \frac{1}{2s^k}|\theta^k - \vartheta|_2^2 + \gamma r|\vartheta|_1 \right\}. \tag{38}$$

Defining

$$shrink(a,b) = \begin{cases} a-b & \text{if } a-b>0 \\ a+b & \text{if } a+b<0 \\ 0 & \text{if } |a| \leqslant |b|. \end{cases}$$

the update rule (38) can be expressed as

$$\theta_j^{k+1} = shrink\left(\theta_j^k - s^k d_j^k, s^k\gamma r\right). \tag{39}$$

for each $j = 1, \ldots, M$.

If the cardinality of X is large and θ^k has many non-zero entries, the evaluation of $\mathcal{J}_T$ and $\nabla\mathcal{J}_T$ can be very expensive. Consequently it is useful to initialize sparsely and to monitor the sparsity level during the iterations of the algorithm. The ℓ^1 term will enhance sparsity in the limit. During the iterations we only update one coordinate j^k chosen by a greedy rule proposed in (Wu and Lange, 2008) (see also (Shi et al., 2017)), in order to keep θ^k as sparse as possible. Namely, instead of updating all the coordinates of θ^{k+1} by (39), we determine the coordinates to be updated by (39) by means of

$$j^{k+1} \in \underset{j\in\{1,\ldots,M\}}{\operatorname{argmax}} \min_{z\in\partial|\cdot|(\theta_j)} \left| d_j^k + \gamma r z \right|. \tag{40}$$

Concerning initialization of θ it is not always possible to do this by 0 since the solution of (9) with $v = 0$ could have a large norm causing numerical difficulties due to the evaluation of the polynomials or it may not exist for all $t \in [0,T]$. For this reason an initial guess for θ has to be chosen that at least ensures the boundedness of the solutions of the closed loop problems (9). This depends on the nature of f and the choice of T in (20). We shall return to this point in the course of discussing the numerical examples.

To choose the step size s^k we use the backtracking line search described in section 10.3.3 in (Beck, 2017), starting from an initial guess s_0^j. That is, for $\kappa \in (0,1)$ and $\beta \in (0,1)$, we take $s^k = s_0^k\beta^i$ such that i is the smallest natural number which satisfies

$$\tilde{\mathcal{J}}_T(\theta^+) \leqslant \tilde{\mathcal{J}}_T(\theta^k) - \frac{\kappa}{s_0^k\beta^i}|\theta^k - \theta^+|^2, \tag{41}$$

where θ^+ is obtained by (39) for either all the coordinates or only the coordinate determined by (40). We use the Barzilai-Borwein step size as initial guess, namely we take s_0^k as

$$s_0^k = \begin{cases} \left[(\theta_k - \theta_{k-1})\cdot(d_k - d_{k-1})\right]/|d_k - d_{k-1}|^2 & \text{if } k \text{ is } odd, \\ |\theta_k - \theta_{k-1}|^2/\left[(\theta_k - \theta_{k-1})\cdot(d_k - d_{k-1})\right] & \text{if } k \text{ is } even. \end{cases} \tag{42}$$

We iterate until the following stopping criterion is fulfilled

$$\max_{j\in\{1,\ldots,M\}} \min_{z\in\partial|\cdot|(\theta_j^k)} \left| d_j^k + \gamma r z \right| \leqslant gtol \text{ or } |J_k - J_{k-1}| \leqslant tol. \tag{43}$$

We summarize the algorithm as follows:

Algorithm 1 Sparse polynomial learning algorithm.

Require: An initial guess $\theta^0 \in \mathbb{R}^M$,$\gamma > 0$, $\kappa > 0$, $\beta \in (0,1)$, $T > 0$, $r \in [0,1]$, $s_0 \in (0,\infty)$.
Ensure: An approximated stationary point θ^* of (20).

```
k = 1
For θ^0, set d^0 (37) and J_0 = J̃_T(θ^0) + P_{γ,r}(θ^0).
Use (41) to obtain s_1.
Obtain θ^1 by (39) in all the coordinates or only with j = j^1 from (40).
Obtain d^1 by (37) and set J_1 = J̃_T(θ^1) + P_{γ,r}(θ^1).
while condition (43) is not satisfied do
    Obtain s_0^k by using (42) and choose s^k using (41).
    Obtain θ^{k+1} by (39) in all the coordinates or only with j = j^{k+1} from (40).
    Obtain d^{k+1} by (37) and set J_{k+1} = J̃_T(θ^{k+1}) + P_{γ,r}(θ^{k+1}).
    Set k = k + 1.
return θ^* := θ_k.
```

7. Polynomial Basis Evaluation

Concerning the implementation, we address the problem of an efficient evaluation of $\tilde{\mathcal{J}}(\theta)$ and $\nabla\tilde{\mathcal{J}}(\theta)$. Indeed, to evaluate $\tilde{\mathcal{J}}$ and $\nabla\tilde{\mathcal{J}}$, we need to solve (9) and (27). We solve these systems numerically, which involves multiple evaluations of the elements of the basis $\mathcal{B}_n$ or $\mathcal{S}_n$ and their derivatives. Therefore it is essential to do this efficiently. For simplicity, we only describe how to evaluate the elements of $\mathcal{B}_n$, but the case of $\mathcal{S}_n$ is analogous. Our approach for polynomial evaluation is related to (Carnicer and Gasca, 1990) and (Lodha and Goldman, 1997).

Before describing how we evaluate the elements of $\mathcal{B}_n$, we need to recall some concepts from graph theory. We only give some basic definitions following (Rosen, 2019), and refer to (Korte and Vygen, 2010) for further description.

A directed graph $G = (V, E)$ is a pair, where V is the set of nodes or vertices of the graph and $E \subset V \times V$ is the set of edges of G.

For a graph $G = (V, E)$ a directed path that connects $a \in V$ and $b \in V$ is a sequence of vertices $\{v_i\}_{i=1}^k \subset V$ such that $(v_i, v_{i+1}) \in E$ for all $i \in \{1, \dots, k-1\}$, $a = v_1$ and $b = v_k$, furthermore we say that a directed path is a directed circuit or cycle if $a = b$. Similarly, an undirected path that connects a and b is a sequence of vertices $\{v_i\}_{i=1}^k \subset V$ such that $(v_i, v_{i+1}) \in E$ or $(v_{i+1}, v_i) \in E$ for all $i \in \{1, \dots, k-1\}$, $a = v_1$ and $b = v_k$, furthermore we say that an undirected path is a circuit or cycle if $a = b$.

A directed rooted tree is a graph $G = (V, E)$ such that there is no undirected circuit in G and it has a node $v_r \in V$ which is connected to every $v \in V \setminus \{v_r\}$.

For two graphs $G = (V, E)$ and $G' = (V', E')$, we say that G is a subgraph of G if $V' \subset V$ and $E' \subset E$.

For a graph $G = (V, E)$, a minimum spanning rooted tree is a subgraph $T = (V, E')$ of G such that T is a rooted tree.

We also need to recall a fundamental algorithm to traverse a graph, which is called the breadth-first search (BFS), (Korte and Vygen, 2010). For G a directed graph and r a node in G connected to every other node in G, the BFS algorithm returns a minimum spanning rooted tree with r as its root.

Algorithm 2 Breadth-first search (BFS)

Require: A graph $G = (V, E)$ and $v_r \in V$.
Ensure: A subgraph $G' = (V, E')$

Set $E' = \emptyset$
For every $v \in V$ set $color(v) = 0$.
Choose $v_r \in V$, set $I = 1$, and $q = \{(1, v_r)\}$.
while $q \neq \emptyset$. **do**
  Set v to be such $(1, v) \in q$.
  for $\tilde{v} \in V$ such that $(v, \tilde{v}) \in E$ **do**
    if $color(\tilde{v}) = 0$ **then**
      Set $color(\tilde{v}) := 1$.
      Set $I := I + 1$, $q := q \cup \{(I, \tilde{v})\}$.
      Set $E' := E' \cup \{(v, \tilde{v})\}$
  Set $q := q \setminus \{(1, v)\}$, $I := I - 1$ and $q := \{(i-1, u) : \forall (i, u) \in q\}$.
return V'

We are now prepared to describe the evaluation of all the elements of X in a given point $y \in \mathbb{R}^d$. We recall that for simplicity we only consider the case $X = \mathcal{B}_n$, later we explain how to do it in other cases.

Let us consider the directed graph $G = (\Lambda_n, E_n)$ where Λ_n is given (13) and $E_n \subset \Lambda_n \times \Lambda_n$ is defined by

$$\forall\ \tilde{\alpha}, \alpha \in \Lambda_n :\ (\tilde{\alpha}, \alpha) \in E_n \text{ if and only if } \alpha = \tilde{\alpha} + e_j \text{ for an unique } j \in \{1, \ldots, d\}, \qquad (44)$$

where e_j is the j-th canonical vector of $\mathbb{R}^d$. Let T be a minimum spanning rooted tree of G, where $\alpha^0 = (0, \ldots, 0) \in \mathbb{N}^d$ is the root of T. Then, for every $\alpha \in \Lambda_n \setminus \{\alpha^0\}$ there exists a unique $\tilde{\alpha} \in \Lambda_n$ such that $(\tilde{\alpha}, \alpha)$ is a vertex of T, which in turn implies that there exists $j \in \{1, \ldots, d\}$ such $\alpha = \tilde{\alpha} + e_j$. Therefore we have

$$\phi_\alpha(y) = y_j^{\alpha_j} \prod_{i=1, i\neq j}^{d} y_i^{\alpha_i} = y_j \cdot y_j^{\alpha_j - 1} \prod_{i=1, i\neq j}^{d} y_i^{\alpha_i} = y_j \phi_{\tilde{\alpha}}(y),\ \forall\ y \in \mathbb{R}^d. \qquad (45)$$

For a given $y \in \mathbb{R}^d$ we evaluate all the elements of $\mathcal{B}_n$ by performing a BFS in T starting from the root and using (45). More precisely, we denote by $c(\alpha)$ the value of $\phi_\alpha(y)$. It is clear that $c(\alpha^0) = 1$. Now, let $\tilde{\alpha}^i$ be the node visited in the $i-$th iteration of BFS. Then for each element in $\alpha \in \Lambda_n$ such that $(\tilde{\alpha}^i, \alpha) \in T$ we obtain $c(\alpha)$ using (45), i.e.

$$c(\alpha) = y_j c(\tilde{\alpha}^i), \qquad (46)$$

where j is the unique index such that $\alpha = \tilde{\alpha}^i + e_j$.

Similarly to (45), the partial derivatives of ϕ_α satisfy

$$\frac{\partial \phi_\alpha}{\partial y_i}(y) = \begin{cases} \alpha_i \phi_{\alpha - e_i}(y) & \text{if } \alpha_i > 0, \\ 0 & \text{if } \alpha_i = 0 \end{cases} \quad \text{for all } i \in \{1, \dots, d\} \tag{47}$$

and

$$\frac{\partial^2 \phi_\alpha}{\partial y_i \partial y_j}(y) = \begin{cases} \alpha_i \alpha_j \phi_{\alpha - e_i - e_j}(y) & \text{if } i \neq j, \ i \geqslant 1 \text{ and } j \geqslant 1, \\ \alpha_i(\alpha_i - 1)\phi_{\alpha - 2e_i}(y) & \text{if } i = j \text{ and } \geqslant \alpha_i = 2, \\ 0 & \text{if } i = j \text{ and } \alpha_i = 1, \\ 0 & \text{if } i \neq j \text{ and } \alpha_i = 0 \text{ or } \alpha_j = 0, \end{cases} \tag{48}$$

for all $i, j \in \{1, \dots, m\}$, where $e_i \in \mathbb{R}^d$ is the $i - th$ canonical vector in $\mathbb{R}^d$. Therefore, by the definition of c, we have

$$\frac{\partial \phi_\alpha}{\partial y_i}(y) = \begin{cases} \alpha_i c(\alpha - e_i) & \text{if } \alpha_i > 0, \\ 0 & \text{if } \alpha_i = 0 \end{cases} \quad \forall i \in \{1, \dots, d\} \tag{49}$$

and

$$\frac{\partial^2 \phi_\alpha}{\partial y_i \partial y_j}(y) = \begin{cases} \alpha_i \alpha_j c(\alpha - e_i - e_j) & \text{if } i \neq j, \ i \geqslant 1 \text{ and } j \geqslant 1, \\ \alpha_i(\alpha_i - 1)c(\alpha - 2e_i) & \text{if } i = j \text{ and } \geqslant \alpha_i = 2, \\ 0 & \text{if } i = j \text{ and } \alpha_i = 1, \\ 0 & \text{if } i \neq j \text{ and } \alpha_i = 0 \text{ or } \alpha_j = 0, \end{cases} \tag{50}$$

for all $i, j \in \{1, \dots, d\}$.

Now we address the evaluation of the elements of $\mathcal{S}_n$ and their derivatives. We consider a graph $\bar{G} = (\Gamma_n, \bar{E}_n)$, where Γ_n is given by (16) and $\bar{E}_n \subset \Gamma_n \times \Gamma_n$ is defined by

$$\forall\ \tilde{\alpha}, \alpha \in \Gamma_n : \ (\tilde{\alpha}, \alpha) \in \bar{E}_n \text{ if and only if } \alpha = \tilde{\alpha} + e_j \text{ for an unique } j \in \{1, \dots, d\}. \tag{51}$$

By the definition of Γ_n, it is clear that for all $\alpha \in \Lambda_n \setminus \alpha^0$, there exists at least one $\alpha \tilde{\in} \Lambda$ which satisfies $\alpha = \tilde{\alpha} + e_j$ for some $j \in \{1, \dots, m\}$, therefore α^0 is connected to every $\alpha \in \Lambda_n$. Then, the evaluation of the elements of $\mathcal{S}_n$ is analogous to the evaluation of $\mathcal{B}_n$.

Finally, in virtue of Remark 2, we know that not all the elements of either $X = \mathcal{S}_n$ or $X = \mathcal{B}_n$ are contributing to the optimal solution. In this case we should consider a reduced basis given by $\mathcal{O}(X)$ in (30). Nevertheless, we can not use our approach directly, because it is not possible to ensure that we can construct an spanning tree with only the multi-indexes that correspond to $X \setminus \mathcal{O}(X)$. More generally, for a given $\theta \in \mathbb{R}^M$ we only need to evaluate the intersection between $X \setminus \mathcal{O}(X)$ and $\{\phi_i \in X : \ \theta_i \neq 0\}$, i.e. the intersection of the support of θ and $X \setminus \mathcal{O}(X)$. To address this problem, consider a subset $\tilde{X}$ of X and a spanning tree T rooted at α^0 for either Λ_n or Γ_n, as appropriate. Then we extract a sub-tree $\tilde{T}$ from T by traversing T starting from each element of $\tilde{X}$ by the BFS algorithm.

8. Generalization

When learning approximation schemes on a finite training set, it is of special interest whether the design objective can also be accomplished for configurations which are not contained in the training set. In our case this amounts to achieving stable trajectories for initial data outside of the training set.

To demonstrate a scenario of what can be expected, we focus on the case when we train with only one initial condition $\tilde{y}_0 \in \Omega$. We consider $v^* \in C^2(\overline{\Omega})$, which may represent an optimal solution of the learning problem (11). For $y_0 \in \mathbb{R}^d$, we denote the solution of (9) on $(0, \infty)$ with $v = v^*$ by $y(\cdot, y_0)$.

We assume that there exist neighbourhoods around $\tilde{y}_0$ and 0 where system (9) is asymptotically stable, in the sense that there exists a monotonically decreasing function $\mu : [0, \infty) \to [0, \infty)$ with $\lim_{t\to\infty} \mu(t) = 0$ and such that

$$|y(t, y_0)| \leqslant \mu(t) \text{ for all } y_0 \in B(\tilde{y}_0, \rho) \cup B(0, \rho) \text{ and } t \in [0, \infty). \tag{52}$$

We next show that this provides a sufficient condition for the existence of an open neighbourhood around the trajectory $\mathcal{T} = \{y(\cdot, \tilde{y}_0)(t) : \ t > 0\} \subset \mathbb{R}^d$ such that for all initial conditions in this neighbourhood the solution is asymptotically stable.

Below we shall utilize the linearised system

$$x' = A(t)x, \quad x(0) = \delta x, \tag{53}$$

where $A(t) = Df(y(t, \tilde{y}_0)) - \frac{1}{\beta} BB^\top \nabla^2 v^*(y(t, \tilde{y}_0))$, and define the associated solution mapping $S(t)\delta x = x(t, \delta x)$, where $t \geqslant 0$ and $\delta x \in \mathbb{R}^n$.

Proposition 2 *Assume that* (52) *holds for* $\tilde{y}_0$. *Then, there exits* $\tilde{\rho} > 0$ *such that*

$$|y(t, y_0)| \leqslant \mu(t), \text{ for all } y_0 \in N(\overline{\mathcal{T}}) \text{ and } t \in [0, \infty), \tag{54}$$

where $N(\overline{\mathcal{T}}) = \{y_0 \in \mathbb{R}^d : \ dist(y_0, \overline{\mathcal{T}}) < \tilde{\rho}\}$.

Proof For arbitrary $\tau > 0$ define the set

$$B(\tau) = \{y(\tau, y_0) : \ y_0 \in B(\tilde{y}_0, \rho)\}.$$

Note that $y(\tau, \tilde{y}_0) \in B(\tau)$ and that by the Bellman's principle and monotonicity of μ

$$|y(t, y_0)| \leqslant \mu(t), \ t > 0, \text{ for all } y_0 \in B(\tau).$$

We need to argue that $\inf_{\tau>0} diam(B(\tau)) > 0$. For this purpose we apply the implicit function theorem to the mapping

$$G : B(\tilde{y}_0, \rho) \times \mathbb{R}^d \subset \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$$

$$G(y_0, z) = y(\tau, y_0) - z,$$

to argue that $y(\tau, \tilde{y}_0) \in int(B(\tau))$. Indeed, $G(\tilde{y}_0, y(\tau, \tilde{y}_0)) = 0$ and DG_{y_0} is characterized by

$$DG_{y_0} \delta x = S(\tau)\delta x.$$

By Liouville's theorem $S(\tau)$ is an isomorphism. Hence for each $\tau > 0$ there exits $\rho_\tau > 0$ such that $B(y(\tau, \tilde{y}_0), \rho_\tau) \subset B_\tau$. By construction,

$$|y(t, y_0)| \leqslant \mu(t), \ \forall \ t > 0, \forall \ y_0 \in B(y(\tau, \tilde{y}_0), \rho_\tau).$$

We consider the covering

$$\bigcup_{\tau>0} B(y(\tau,\tilde{y_0}),\rho_\tau) \cup B(\tilde{y_0},\rho)) \cup B(0,\rho) \supset \bar{\mathcal{T}}.$$

Since $\lim_{\tau\to\infty} y(\tau,\tilde{y}_0) = 0$, the set $\bar{\mathcal{T}}$ is compact. Therefore there exits a finite subcover, and consequently some $\tilde{\rho} > 0$ such that (54) holds. ■

Remark 3 In the case that $\{\tilde{y}_0^i\}_{i=1}^I$ initial conditions are used for the learning step, the construction of Theorem 2 can be repeated for each one of them leading to I tubes containing the trajectories $\{y(\cdot,\tilde{y}_0^i)\}_{i=1}^I$. For initial conditions in these tubes we have guaranteed exponential stabilization. Moreover, since these tubes all intersect at the origin it can be expected that, as I increases, the neighorhood of the origin for which stabilization is guaranteed increases as well.

9. Numerical Experiments

We implement Algorithm 1 to solve problem (20) for 4 different problems. These problems are the stabilization of an LC-circuit, stabilization of a modified Van der Pol oscillator, stabilization of the Allen-Cahn equation, and optimal consensus for the Cucker-Smale model. For every experiment we shall specify the computational time horizon, and the sets of initial conditions for training and testing. For all experiments, we utilized a Monte Carlo based uniform sampling for the training sets. The ordinary differential equations are solved by the Crank-Nicolson algorithm with step size of 10^{-2}. The arguments of the monomials are normalized by l, i.e. we redefine $\phi_\alpha(y)$ by $\phi_\alpha(y) = \prod_{i=1}^d \left(\frac{y}{l}\right)^{\alpha_i}$.

We measure the performance of our approach by comparing the control u^* obtained by solving the open loop problem for every initial condition in the test set with $\hat{u} = -\frac{1}{\beta}B^\top\nabla\hat{v}(y)$, where $\hat{v}$ is the solution of (20) and y is the corresponding solution of (9). We then compute the mean normalized squared error in $L^2((0,T);\mathbb{R}^m)$ for the controls by

$$MNSE_u(\{\hat{u}_i\}_{i=1}^N, \{u_i^*\}_{i=1}^N) = \sum_{i=1}^N \int_0^T |\hat{u}_i - u_i^*|^2 dt \Big/ \sum_{i=1}^N \int_0^T |u_i^*|^2 dt,$$

and analogously $MNSE_y(\{\hat{y}_i\}_{i=1}^N, \{y_i^*\}_{i=1}^N)$ for the states. We also compare the optimal value of the open loop problem with the objective function of (1) evaluated in $\hat{u}$ by computing the mean normalized squared error

$$MNSE_J(\{\hat{u}_i\}_{i=1}^N, \{u_i^*\}_{i=1}^N) = \sum_{i=1}^N |J(u_i^*, y_0^i) - J(\hat{u}_i, y_0^i)|^2 \Big/ \sum_{i=1}^N J(u_i^*, y_0^i)^2.$$

In order to compute an optimal control for the non-linear problems, we solve the open loop problem by a gradient descent algorithm with a backtracking line-search. For the linear-quadratic problem (55) we use the algebraic Riccati equation to obtain the optimal feedback controls.

The hyper-parameters selection was carried out by the following heuristic guidelines. Since r expresses the weight of the desired sparsity it was chosen close to 0 in small size problems and large for the remaining ones. The time horizon T was chosen after experiments such that the running cost was close to 0 for that choice of T. The parameter l was chosen according to experiments observing the controlled trajectories. The usage of the total degree and of the hyperbolic cross bases was based on the dimension of the state space. All learning problems were solved by Algorithm 1 with the choice (40) in steps 4 and 8, $\kappa = 0.5$, and $\beta = 0.9$, except for the Optimal Consensus for Cucker Smale problem where (38) was used. In all cases the stopping criterion (43) was satisfied with $gtol = 10^{-3}$ and $tol = 10^{-5}$.

9.1 LC-circuit

We consider the linear-quadratic problem

$$\begin{aligned} &\min_{u \in L^2((0,T),\mathbb{R})} \frac{1}{2}\int_0^T |y|^2 dt + \frac{\beta}{2}\int_0^T |u|^2 dt \\ &s.t.\ y' = Ay + Bu,\ y(0) = y_0, \end{aligned} \tag{55}$$

with

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{56}$$

We set $T = 10$, $l = 10$, $\gamma = 10^{-10}$, and $r = 0.1$, and randomly choose 5 sets $\mathcal{Y}^j_{train}$ with $j \in \{1, \dots, 5\}$ each of cardinality 10 from Ω. A test set of cardinality 200 was randomly and uniformly sampled from Ω. From each $\mathcal{Y}^j_{train}$ we take a sequence of increasing subsets $\{\mathcal{Y}^j_i\}_{i=1}^{10}$, such that for each $i \in \{1, \dots, 10\}$ the cardinality of $\mathcal{Y}^j_i$ is i. For each $i \in \{1, \dots, 10\}$ we solve (20) with $X = \mathcal{B}_2 \setminus (\mathcal{B}_1 \cup \mathcal{O}(\mathcal{B}_2))$, where $\mathcal{O}(\cdot)$ is given by (30). Subsequently for every $y_0 \in \mathcal{Y}_{test}$ the control obtained through the solution of (20) for each $\mathcal{Y}^j_i$ is compared with the optimal one and denote this solutions by $\hat{v}^j_i$ respectively. The initialisation is chosen as $v_0 = 0$, and $N_{train} \in \{1, \dots, 10\}$ refers to the number of initial conditions which is chosen from each $\mathcal{Y}^j_{train}$.

In Figure 1 we present the mean normalized squared errors for the objective functions, the controls, and the states, for increasing training sizes. The endpoints of the error-bars along the y-axis correspond to the maximum and minimum error obtained for each N_{train}. The curved line in each of the plots connects the average errors with respect to j for each of the $(MNSE_*)^j_i$ where $*$ stand for J, u, respectively y. We notice that in all the cases the mean percentage error is smaller than 1 %. Now that we have established that the results are robust with respect to different choices of training sets, we are curious to look at test sets.

In Figure 2 we show the errors for the test set again for increasing number of training initial conditions. As before we can observe robustness with respect to the choice and the number of initial conditions.

To further illustrate the performance of our approach, in Figure 3, we provide the scatter plot between the true value of the open loop problem and the objective function evaluated in the learned control for every point in the test set. On the x-axis, the label *learned objective*

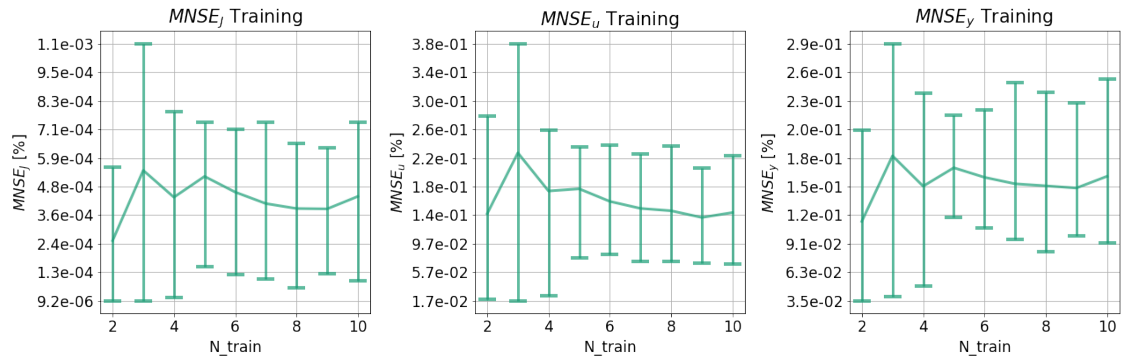

Figure 1: SSE_u, SSE_y, SSE_J training for LC-circuit example.

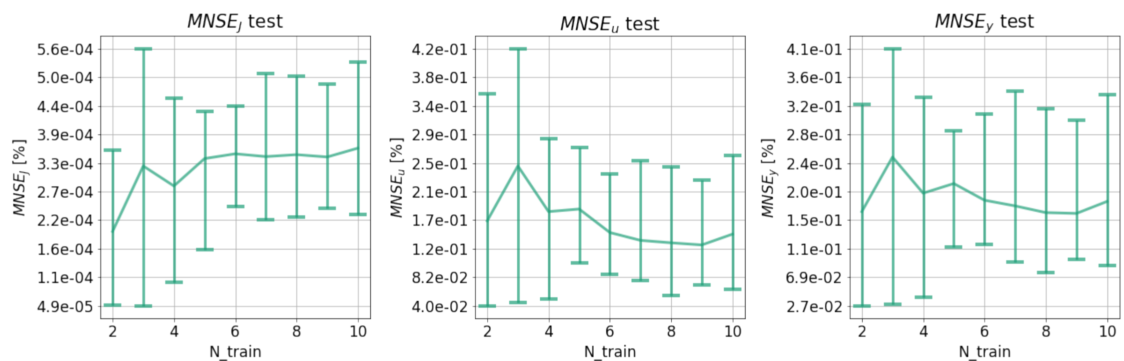

Figure 2: SSE_u, SSE_y, SSE_J test for LC-circuit example.

value is given by $\frac{1}{5}\sum_{j=1}^{5} J(y_0^k, u_{i,k}^j)$, where $u_{i,k}^j = -\frac{1}{\beta}B^T\nabla\hat{v}_i^j(y_{i,k}^j)$, and $y_{i,k}^j$ is the solution to the closed loop problem (9) with $v = \hat{v}_i^j$, for $k = 1, \ldots, 200$ corresponding to the test set. This is carried out for $N_{train} = 1, 2$ and 10. Additionally, for each N_{train}, the endpoints of the error-bars along the x-axis correspond to the maximum and minimum objective value obtained for the corresponding training subsets. For $N_{train} = 2$, and $N_{train} = 10$ the regression line of the scatter points in the test set are close to the identity line, which is already suggested by the results in Figure 2. With only one initial condition in each $\mathcal{Y}_{train}^j$ the error-bars are distinctly larger than in the other cases.

9.2 Modified Van der Pol Oscillator

We investigate the problem

$$\begin{aligned} &\min_{u\in L^2((0,T);\mathbb{R})} \frac{1}{2}\int_0^T |y|^2 dt + \frac{\beta}{2}\int_0^T |u|^2 dt \\ &s.t. \quad y'' = \nu(1-y^2)y' - y + \mu y^3 + u, \ (y(0), y'(0)) = (y_0, v_0). \end{aligned} \tag{57}$$

In the previous linear-quadratic example we addressed the convergence when the cardinality of the training set increases. Given the structure of the problem we only used

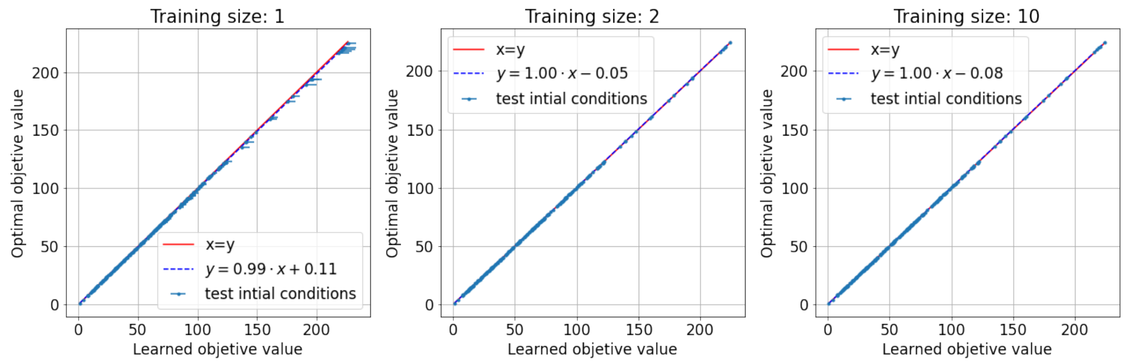

Figure 3: Validation scatter for LC-circuit example.

polynomials of degree 2. In the example we investigate the effect of the degree of the polynomials.

The parameters are set to be $T = 3$, $\beta = 10^{-3}$, $\nu = \frac{3}{2}$, $l = 10$, $\mu = \frac{4}{5}$, r=0.5, $\gamma = 10^{-5}$ and $X_n = \mathcal{B}_n \setminus (\mathcal{B}_1 \cup \mathcal{O}(\mathcal{B}_n))$ for $n \in \{4, 5, 6, 7, 8\}$. The training and test set were created as in the previous example. It is important to mention that if we choose $v = 0$ as initial guess, the norm of the solutions of the closed loop problem (9) may increase exponentially with time. Therefore, in order to ensure the boundedness of the state of the closed loop problem we proceed as follows: for X_4 we choose $v_0(y_1, y_2) = \mu\beta y_1^3 y_2 + \frac{\beta\nu}{2} y_2^2$ as initial guess and for $n > 4$ we use the solution of the previous degree as initial guess. Note that this requires the polynomial degree to be at least 4.

In Figure 4 the training and test errors are depicted as in the LC-circuit example. It is observed that the training errors $MNSE_J$ and $MNSE_u$ are decreasing. The training $MNSE_y$ stays small throughout. Additionally, the widths of the error-bars decrease with the cardinality as well.

In the first row of Figure 5 the cardinalities of the supports $\{i : \theta_i \neq 0\}$ of the coefficients of the learned feedback laws are presented in the first graph, whereas in the second graph the same cardinality is presented as a proportion with respect of the cardinality of the basis. It is observed that the cardinality of the support increases with the degree, but it decreases as percentage of the cardinality of X_n. (The colored curves appear in reversed order in these two graphs.) Moreover, for each degree, the cardinality decreases as the training size increases. In first graph of the second row of Figure 5 the training times are shown. The training time increases with the number of training points, but in all the cases it is lower than 14 minutes. Further, in the second graph of the second row of Figure 5 the maximum time that take to solve a closed loop problem from the test set is depicted. It is noted that the solving time for the closed loop problem decreases with the cardinality of the training set and it increases with the degree of the approximation, which is consistent with the fact that the size of the support increases with the degree. It is noteworthy that the solving time is less than 0.42 seconds in all the experiments.

We turn our attention to the phase planes in Figure 6. In the first phase we see that trajectories originating from a 'dot' representing an initial condition moving towards a one dimensional manifold with two branches meeting at the origin. These results are obtained

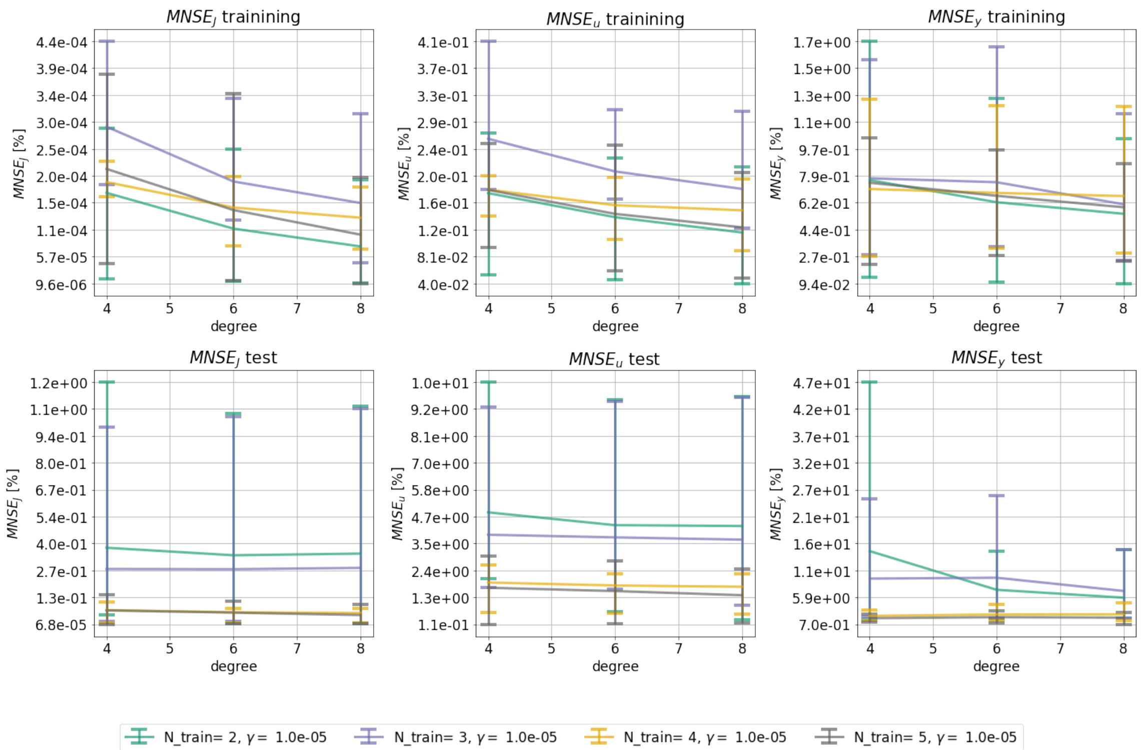

Figure 4: Training and test errors $MNSE_u$, $MNSE_y$ and $MNSE_J$ for modified Van der Pol oscillator example.

by our feedback strategy with polynomials of degree four and training cardinality two. If, on the other hand we train with only one initial condition the the learned feedback law is not capable of stabilizing the test initial conditions near the right-hand branch of the manifold, as depicted in the second graph of Figure 6. We conclude that when training with two or more initial conditions we are able to steer all the initial conditions in the test set towards the origin.

9.3 Allen-Cahn Equation.

We turn to the control of the Allen-Cahn equation with the Neumann boundary conditions and consider

$$\begin{gathered}\min_{u_i\in L^2([0,T),\mathbb{R})}\int_0^T\int_{-1}^1|y(x,t)|^2dxdt+\beta\int_0^T|u(t)|^2dt\\ y'(t,x)=\nu\frac{\partial^2 y}{\partial x^2}(t,x)+y(t,x)(1-y^2(t,x))+\sum_{i=1}^3\chi_{\omega_i}(x)u_i(t)\qquad(58)\\ \frac{\partial y}{\partial x}(t,-1)=\frac{\partial y}{\partial x}(t,1)=0,\quad y(0,x)=y_0(x)\end{gathered}$$

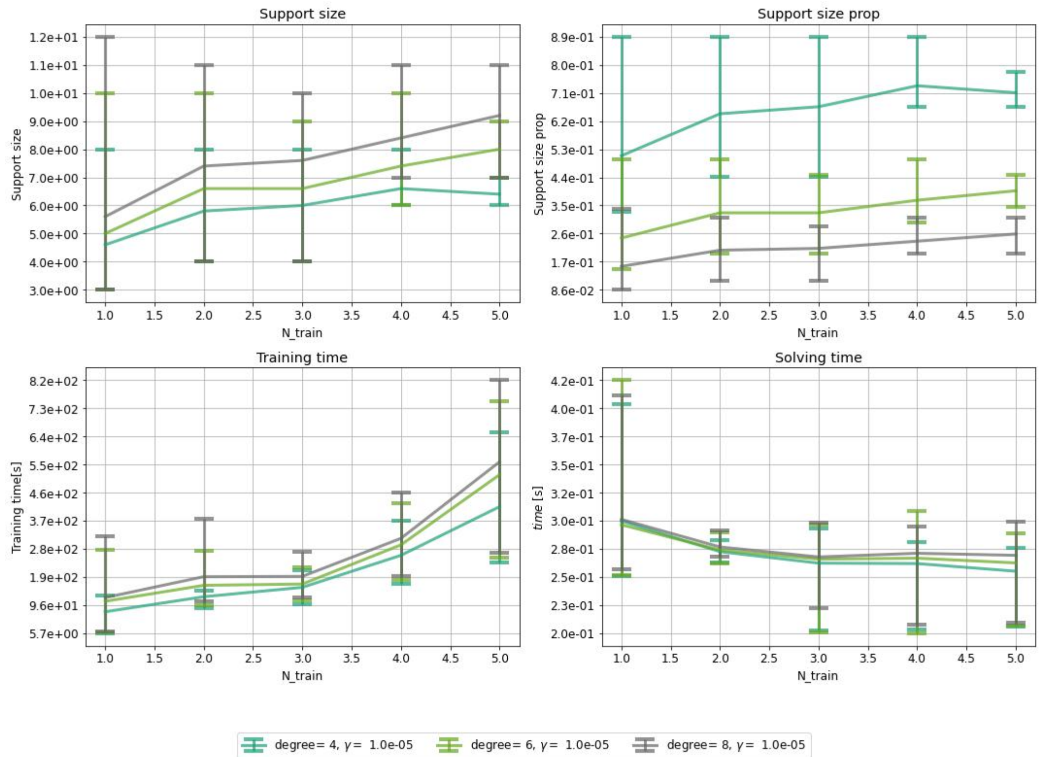

Figure 5: Support cardinality, training time and solving time of the solutions to the learning problems for modified Van der Pol oscillator example.

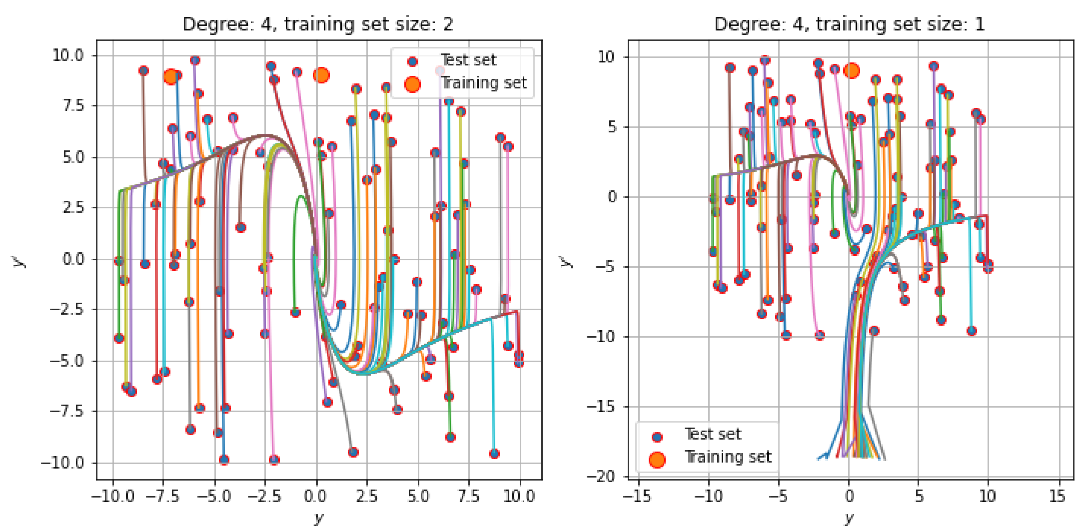

Figure 6: Phase plane for test initial conditions for modified Van der Pol oscillator example.

for $x \in (-1,1)$ and $t > 0$, where $\nu = 0.1$, $T = 3$ and χ_{ω_i} are the indicators functions of the sets $\omega_1 = (-0.7, -0.4)$, $\omega_2 = (-0.2, 0.2)$, and $\omega_3 = (0.4, 0.7)$. This problem admits the 3 steady states given by the constant functions with values -1, 0 and 1, with 0 being unstable.

Since problem (58) is infinite-dimensional, we discretize it by using a Chebyshev spectral collocation method with 19 degrees of freedom. The first integral in (58) is approximated by means of the Clenshaw-Curtis quadrature. For further details on the Chebyshev spectral collocation method and the Clenshaw-Curtis quadrature we refer to (Boyd, 2000, Chapters 6, 19), and (Trefethen, 2020, Chapters 7, 12, 13).

Due to the high dimensionality of this problem, the evaluation of the feedback law is computationally expensive. In order to mitigate this difficulty the hyperbolic cross technique is used for the construction of the basis. Further, sparsity of solution, can be influenced by the penalty coefficient γ. With this in mind, we pay attention to the influence of γ on the sparsity of the solution and performance of the obtained feedback laws.

For the results presented below, we choose $r = 0.9$, $l = 10$, $X = \mathcal{S}_4 \setminus (\mathcal{B}_1 \cup \mathcal{O}(\mathcal{B}_4))$, $v_0 = 0$, and $\gamma \in \{10^{-i} : \ i = 1, \ldots, 6\}$ and sampling as before. We train progressively starting with $\gamma = 10^{-1}$, for which we use $v = 0$ as initial guess for the value function. For the remaining γ values initialization is done with the solution of the previous γ value.

In Figure 7 we present the normalized errors calculated for test sets for each chosen γ. We observe that the overall error given by $MNSE_J$ achieves the lowest value at about $\gamma = 10^{-3}$. Moreover, all errors $MNSE_J$, $MNSE_u$, and $MNSE_y$ decrease while training cardinality increases more distinctly than in the other examples.

Concerning the efficiency of the method, in Figure 8 the support sizes of the solutions of the learning problem (20) and the training times are depicted. We observed that the support size decreases with γ. Of course, this also depends on the cardinality of the training set. With respect to the training time, in the rightmost subplot in Figure 8 we observe that the training time increases linearly with the cardinality as expected and decreases with γ, which can be explained by the fact that the support cardinality decreases with γ.

In Figure 9 we present the scatter plot between the value of the objective of the closed loop problem and the value obtained by our approach when $\gamma = 10^{-1}$ with training cardinality 5, and $\gamma = 10^{-3}$ for training cardinalities 5 and 20. In the first scatter we see that the slope and the intercept of the regression line are around 0.6 and 0.13, respectively. Moreover we observe a high dispersion of the point around the regression line. On the other hand, in the second scatter the regression line is closer to the identity line and the dispersion around it is clearly lower than in the first scatter. Subsequently, for training cardinality of 20 we see that the regression is sligthly closer to the identity line with respect to the previous case, but the dispersion around it is smaller.

9.4 Optimal Consensus for Cucker-Smale Model

We consider a set of N agents with states $(x_i(t), y_i(t)) \in \mathbb{R}^2 \times \mathbb{R}^2$ for $i \in \{1, \ldots, N\}$ governed by the Cucker-Smale (see (Cucker and Smale, 2007)) dynamics. The system is controlled in such a way that the velocity of every agent asymptotically approaches the mean velocity. In order to achieve this in an optimal way, we solve (see (Bailo et al., 2018; Caponigro et al.,

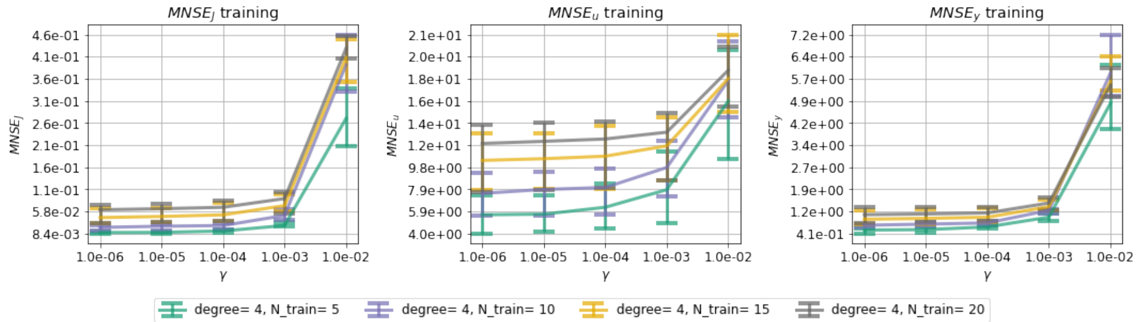

Figure 7: Test mean normalized errors for Allen-Cahn equation example. The x-axis is in logarithmic scale in all the subplots.

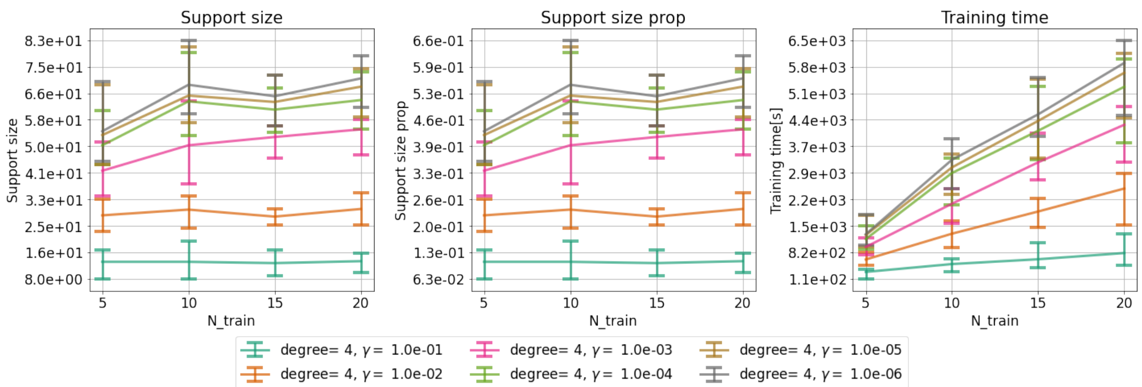

Figure 8: Support cardinality and training time of the solution obtained for each γ for Allen-Cahn equation example.

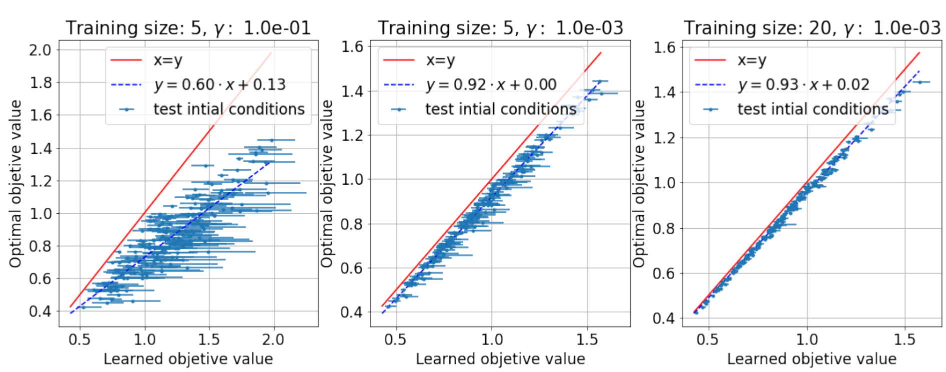

Figure 9: Validation scatter for Allen-Cahn equation example.

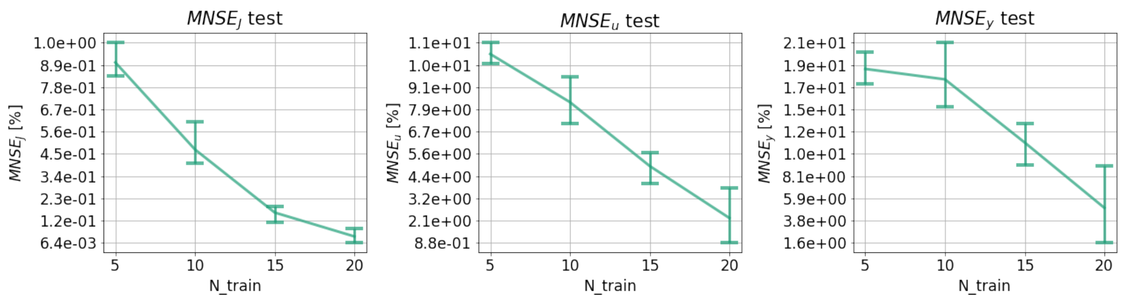

Figure 10: Training and test SSE_u, SSE_y, SSE_J in percent for optimal consensus example.

2015))

$$\begin{aligned} &\min_{u_i \in L^2((0,\infty);\mathbb{R}^2)} \frac{1}{N}\sum_{i=1}^{N}\int_0^T |y_i - \bar{y}|^2 dt + \beta \sum_{i=1}^{N}\int_0^T |u_i|^2 dt \\ &s.t. \quad x_i' = y_i, \quad y_i' = \frac{1}{N}\sum_{j=1}^{N} a(|x_i - x_j|)(y_j - y_i) + u_i, \\ &\qquad x_i(0) = x_0^i, \quad y_i(0) = y_0^i, \end{aligned} \tag{59}$$

where $a : [0,\infty) \to \mathbb{R}$ is a communication kernel given by $a(r) = \frac{K}{(1+r^2)}$ and $\bar{y}(t)$ is the mean velocity, that is $\bar{y}(t) = \frac{1}{N}\sum_{j=1}^{N} y_j(t)$.

We set $N = 10$, $T = 3$, $K = 10^{-1}$, and $\beta = 10^{-2}$. For the learning problem we take $X = \mathcal{S}_4 \setminus (\mathcal{B}_1 \cup O(\mathcal{S}_4))$, $\gamma = 10^{-5}$, $r = 0.9$, and $l = 5$. The training and test sets are sampled as before and we take $v_0(x_1,\ldots,x_N,y_1,\ldots,y_N) = NK\beta\sum_{i=1}^{N}|y_i|^2$ as initial guess, which ensures the boundedness of the solutions for the closed loop problem (9).

The test errors for different training sets of cardinalities 5, 10, 15 and 20 are presented in Figure 10. As expected, the errors decrease with the training cardinality together with the length of the error-bars. Moreover, in all the cases errors are smaller than 2.1 %.

We also provide the scatter plot between the value of the open loop problem and the value obtained by our approach in Figure 11 for cardinalities 5,10 and 20. In the scatter plots the slopes of the regression lines progressively increase towards 1, as the training cardinality increases. Further, the dispersion around the regression line and the width of the error-bars decrease with the training.

Finally, in Figure 12 the support size of the solutions of the learning problem and the training times are shown. We notice that the cardinality of the support tends to decrease with the number of training initial conditions. We point out that in this case the (38) was used instead of (40) in Algorithm 1. This choice was made because in previous experiments for this problem using (40), the errors and the training times were larger than in the case presented. However, the solutions found by (40) tend to be sparser.

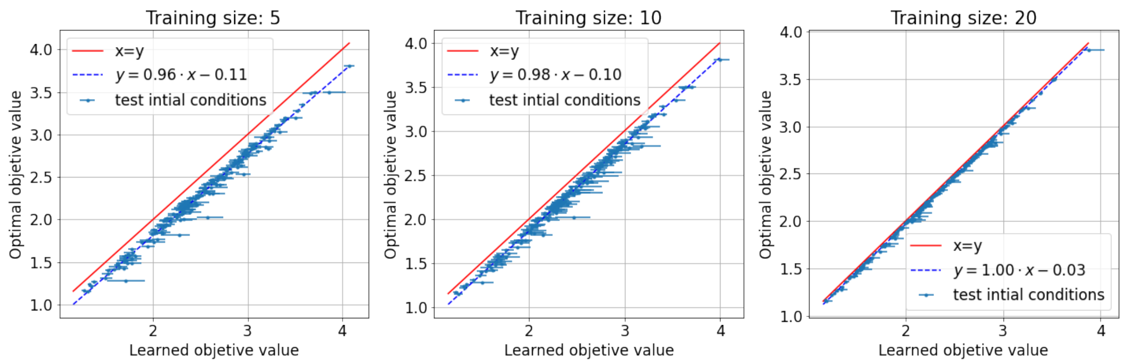

Figure 11: Learned value vs optimal value on the test set for optimal consensus example.

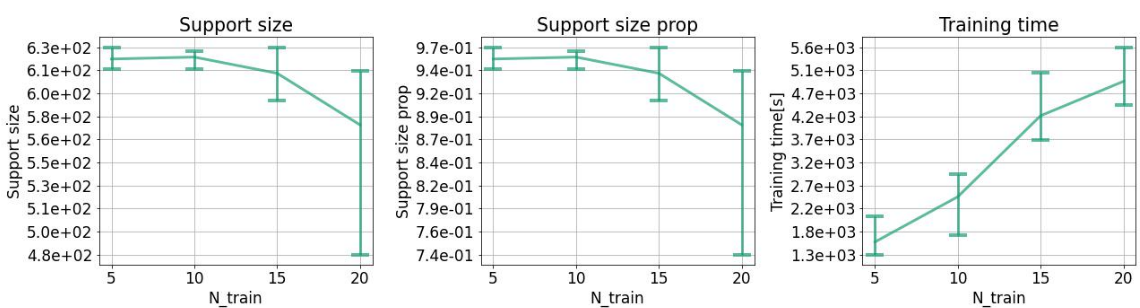

Figure 12: Support cardinality and training time of the solution obtained for the optimal consensus example.

10. Conclusion

A learning based method to obtain feedback-laws for nonlinear optimal control problems and their approximation by polynomials was presented. The proposed methodology was implemented in python and tested on 4 problems. These experiments demonstrate the robustness and efficiency of the approach for obtaining approximative feedback-laws for non-linear and high dimensional problems. For the linear problem that we tested, our approach was capable of finding feedback-laws close to those provided by the Riccati synthesis. Of course, this relates to the fact that for linear-quadratic problems the value function is a quadratic polynomial, and it is thus contained in our ansatz space. The nonlinear examples also suggest possible further research directions. In the case of the Van der Pol problem we saw the proposed method is capable of solving highly nonlinear problems. It also became evident that the initialization is important. This leads to the question for guidelines for efficient initializations. In the case of the Allen-Cahn problem it would be of interest to investigate the convergence of the feedback-laws for the finite dimensional approximations to the infinite dimensional one. For the Cucker-Smale problem we used the full gradient method rather than the greedy version. This raises the question of the interplay between performance, sparsity, and the choice of the gradient method.

Further extensions of our approach to more general problems are possible. A key ingredient for such extensions will be the possibility of a representation formula of the optimal feedback-law in terms of the value function as in (5).

Appendix A. Appendix

Lemma 3 *Let $\varepsilon \in (0,1)$, $\sigma \in (0,l)$, and v and v_ε be functions in $C^{1,1}(\overline{\Omega})$ satisfying*

$$\left|B^\top(\nabla v - \nabla v_\varepsilon)\right|_{C(\overline{\Omega})} < \varepsilon, \ \text{and } \mathcal{J}_\infty(v) < \infty. \tag{60}$$

Further assume that

$$|y_i(t)| \leqslant l - \sigma, \ \text{for } i = 1, \dots, I, \ \forall t \geqslant 0, \tag{61}$$

where y_i is the solution of (9), and define

$$C = 2\left|f - \frac{1}{\beta}BB^\top \nabla v\right|_{Lip(\overline{\Omega})} + \frac{|B|^2}{\beta^2}. \tag{62}$$

Then, there exists $T_\varepsilon \in (0,\infty]$ satisfying

$$T_\varepsilon \geqslant \frac{1}{C}\ln\left(1 + \frac{\sigma^2 C}{4\varepsilon^2}\right), \tag{63}$$

such that for each $i \in \{1,\dots,I\}$ problem (9) admits a solution $y_i^\varepsilon \in C^1([0,T_\varepsilon];\Omega)$, and

$$|y_i^\varepsilon(t) - y_i(t)|^2 \leqslant \frac{\varepsilon^2}{C}(e^{Ct}-1) \ \text{and } |y_i^\varepsilon(t)| \leqslant l - \frac{\sigma}{2} \ \text{for all } t \in [0,T_\varepsilon]. \tag{64}$$

Moreover, defining $\tilde{T}_\varepsilon > 0$ by

$$\tilde{T}_\varepsilon := \frac{1}{C}\ln\left(1 + \frac{C\sigma^2}{4\varepsilon^{1/2}}\right) \tag{65}$$

we have

$$\left|\mathcal{J}_{\tilde{T}_\varepsilon}(v_\varepsilon) - \mathcal{J}_{\tilde{T}_\varepsilon}(v)\right| \leqslant K\varepsilon^{1/4}, \tag{66}$$

where K is a constant independent of ε.

Proof [Proof of Lemma 3] Since f is Lipschitz on bounded sets, there exists a time $T_\varepsilon > 0$ such that for each $i \in \{1,\dots,I\}$ the closed loop problem (9) admits a solution $y_i^\varepsilon \in C([0,T_\varepsilon];\mathbb{R}^d)$ where $T_\varepsilon \in (0,\infty]$ is defined as the largest time such that

$$\sup_{\substack{i \in \{1,\dots,I\},\\ t\in[0,T_\varepsilon].}} |y_i^\varepsilon(t)| = l - \frac{\sigma}{2}. \tag{67}$$

We next verify that T_ε satisfies (63) where C is given by (62) and $\sigma \in (0,l)$ satisfies (61). We assume that T_ε is finite, otherwise this claim is trivially satisfied. Let y_i be the solution of (9) for v. Then, subtracting the equations (9) for y_i^ε and y_i we obtain

$$(y_i^\varepsilon - y_i)' = (f(y_i^\varepsilon) - f(y_i)) - \frac{1}{\beta}BB^\top\left(\nabla v_\varepsilon(y_i^\varepsilon) - \nabla v(y_i)\right) \text{ in } (0,T_\varepsilon), \tag{68}$$

and $y_i^\varepsilon(0) - y_i(0) = 0$. We have

$$\nabla v_\varepsilon(y_i^\varepsilon) - \nabla v(y_i) = (\nabla v_\varepsilon(y_i^\varepsilon) - \nabla v(y_i^\varepsilon)) + (\nabla v(y_i^\varepsilon) - \nabla v(y_i)) \tag{69}$$

in $(0,T_\varepsilon)$. Multiplying (68) by $y_i^\varepsilon - y_i$ and using (60), (62) and (69) we get

$$\frac{d}{dt}\left(\frac{1}{2}|y_i^\varepsilon - y_i|^2\right) \leqslant \frac{C}{2}|y_i^\varepsilon - y_i|^2 + \frac{\varepsilon^2}{2} \text{ in } (0,T_\varepsilon). \tag{70}$$

Multiplying both sides of (70) by e^{-Ct} and integrating between 0 and t we obtain for each $i \in \{1,\dots,I\}$

$$|y_i^\varepsilon - y_i|^2 \leqslant \frac{\varepsilon^2}{C}\left(e^{Ct}-1\right) \text{ in } [0,T_\varepsilon], \tag{71}$$

and (64) holds. By (71) and (61) we have

$$|y_i^\varepsilon|_\infty \leqslant |y_i^\varepsilon - y_i|_\infty + |y_i|_\infty \leqslant \frac{\varepsilon}{C^{1/2}}\left(e^{Ct}-1\right)^{1/2} + (l-\sigma) \text{ in } [0,T_\varepsilon]. \tag{72}$$

Combining (67) and the previous inequality we obtain

$$\frac{\sigma}{2} \leqslant \frac{\varepsilon}{C^{1/2}}\left(e^{CT_\varepsilon}-1\right)^{1/2}$$

which clearly implies (63) .

Now we turn to the proof of (66). Since $\varepsilon \in (0,1)$ and recalling the definition of $\tilde{T}_\varepsilon$ we have

$$\tilde{T}_\varepsilon = \frac{1}{C}\ln\left(1 + \frac{C\sigma^2}{4\varepsilon^{1/2}}\right) < \frac{1}{C}\ln\left(1+\frac{C\sigma^2}{4\varepsilon^2}\right) \leqslant T_\varepsilon.$$

Therefore, we have $\tilde{T}_\varepsilon < T_\varepsilon$, and using that $\ln(x+1) \leqslant x$ for all $x \geqslant 0$ we also get $\tilde{T}_\varepsilon \leqslant \frac{\sigma^2}{4\varepsilon^{1/2}}$. Since ℓ is C^1, and $\ell(0) = 0$ we can estimate

$$\left| \int_0^{\tilde{T}_\varepsilon} \ell(y_i^\varepsilon) dt - \int_0^{\tilde{T}_\varepsilon} \ell(y_i) dt \right| \leqslant |\ell|_{Lip(\overline{\Omega})} \int_0^{\tilde{T}_\varepsilon} |y_i^\varepsilon - y_i| dt. \tag{73}$$

Together with (71) this implies that

$$\left| \int_0^{\tilde{T}_\varepsilon} \ell(y_i^\varepsilon) dt - \int_0^{\tilde{T}_\varepsilon} \ell(y_i) dt \right| \leqslant \frac{\varepsilon}{C^{1/2}} \tilde{T}_\varepsilon \left(e^{C\tilde{T}_\varepsilon} - 1 \right)^{1/2} |\ell|_{Lip(\overline{\Omega})}. \tag{74}$$

By the definition of $\tilde{T}_\varepsilon$ and since $\tilde{T}_\varepsilon \leqslant \frac{\sigma^2}{4\varepsilon^{1/2}}$, we obtain

$$\left| \int_0^{\tilde{T}_\varepsilon} \ell(y_i^\varepsilon) dt - \int_0^{\tilde{T}_\varepsilon} \ell(y_i) dt \right| \leqslant \varepsilon^{1/4} \frac{\sigma^3}{8} |\ell|_{Lip(\overline{\Omega})}. \tag{75}$$

To estimate the second summand in $\mathcal{J}_{\tilde{T}_\varepsilon}(v_\varepsilon) - \mathcal{J}_{\tilde{T}_\varepsilon}(v)$ we first observe that due to (60), (61), and (64) there exists a constant K independent of $\varepsilon \in (0,1)$ such that for all $i \in \{1, \ldots, I\}$

$$\max_{t \in [0, \tilde{T}_\varepsilon]} \left| B^\top \left(\nabla v_\varepsilon(y_i^\varepsilon(t)) + \nabla v(y_i(t)) \right) \right| \leqslant K.$$

Consequently we find by (60) and (64)

$$\begin{aligned}
&| \int_0^{\tilde{T}_\varepsilon} |B^\top \nabla v_\varepsilon(y_i^\varepsilon)|^2 dt - \int_0^{\tilde{T}_\varepsilon} |B^\top \nabla v(y_i)|^2 dt \,| \\
&\leqslant \int_0^{\tilde{T}_\varepsilon} |B^\top \nabla (v_\varepsilon(y_i^\varepsilon) + v(y_i))| \, |B^\top \nabla (v_\varepsilon(y_i^\varepsilon) - v(y_i))| \, dt \\
&\leqslant K \int_0^{\tilde{T}_\varepsilon} |B^\top \nabla (v_\varepsilon(y_i^\varepsilon) - v(y_i^\varepsilon))| + |B^\top \nabla (v(y_i^\varepsilon) - v(y_i))| \, dt \\
&\leqslant K(\varepsilon \tilde{T}_\varepsilon + |B^\top \nabla v|_{Lip(\bar{\Omega})} \int_0^{\tilde{T}_\varepsilon} |y_i^\varepsilon - y_i| \, dt) \\
&\leqslant K(\varepsilon \tilde{T}_\varepsilon + |B^\top \nabla v|_{Lip(\bar{\Omega})} \tilde{T}_\varepsilon \frac{\varepsilon}{\sqrt{C}} \sqrt{e^{C\tilde{T}_\varepsilon} - 1}) \leqslant K[\frac{\varepsilon^{\frac{1}{2}} \sigma^2}{4} + |B^\top \nabla v|_{Lip(\bar{\Omega})} \frac{\varepsilon^{\frac{1}{4}} \sigma^3}{8}].
\end{aligned} \tag{76}$$

Inequality (66) is obtained from (75) and (76). ■

Lemma 4 *Consider $T \in (0, \infty]$ and a sequence $v_k \in C^{1,1}(\overline{\Omega})$ converging in $C^{1,1}(\overline{\Omega})$ to v, such that $\mathcal{J}_T(v_k) < \infty$ and $\mathcal{J}_T(v) < \infty$. Then we have*

$$\lim_{k \to \infty} \mathcal{J}_T(v_k) = \mathcal{J}_T(v), \tag{77}$$

for $T \in (0, \infty)$ and otherwise

$$\mathcal{J}_\infty(v) \leqslant \liminf_{k \to \infty} \mathcal{J}_\infty(v_k). \tag{78}$$

Proof [Proof of Lemma 4] Consider first $T \in (0, \infty)$, y_i^k and y_i the solutions of the closed loop problems (9) for v_k and v^* respectively. Recalling the definition of $\mathcal{J}_T$ in (8) and the assumption that $\mathcal{J}_T(v_k) < \infty$, we know that $|y_i^k(t)| \leqslant l$ for all $t \in [0,T]$, $i \in \{1, \dots, I\}$, and $k \in \{1, 2, \dots\}$. By the Lipschitz continuity of f on $\overline{\Omega}$ and (9), we get that the set $\{y_i^k \;:\; i = 1, \dots, I;\; k = 1, 2, \dots, \}$ is bounded in $H^1((0,T); \mathbb{R}^d)$. Therefore, for every $i \in \{1, \dots, I\}$ there exists a function $\bar{y}_i \in H^1((0,T); \mathbb{R}^d) \cap L^\infty((0,T); \mathbb{R}^d)$ such that, passing to a sub-sequence,

$$y_i^k \rightharpoonup \bar{y}_i \text{ in } H^1((0,T); \mathbb{R}^d) \text{ and } y_i^k \to \bar{y}_i \text{ in } C([0,T]; \mathbb{R}^d). \tag{79}$$

Further, as v_k converges to v in $C^{1,1}(\overline{\Omega})$ and y_i^k converges to $\bar{y}$ in $C([0,T])$, we get

$$\lim_{k \to \infty} \nabla v_k(y_i^k) = \nabla v(\bar{y}_i) \text{ in } C([0,T]) \text{ for all } i \in \{1, \dots, I\} \tag{80}$$

and

$$\lim_{k \to \infty} \left\{ f(y_i^k) - \frac{1}{\beta} B B^\top \nabla v_k(y_i^k) \right\} = f(\bar{y}_i) - \frac{1}{\beta} B B^\top \nabla v(\bar{y}_i) \text{ in } C([0,T]), \tag{81}$$

for all $i \in \{1, \dots, I\}$. This implies that the functions $\{\bar{y}_i\}_{i=1}^I$ are solutions of (9) and by uniqueness of the solutions of this problem, we have

$$\bar{y}_i = y_i, \quad \text{for all } i \in \{1, \dots, I\}.$$

Hence we obtain

$$\lim_{k \to \infty} y_i^k = y_i \text{ in } C^1([0,T]; \mathbb{R}^d) \text{ for all } i \in \{1, \dots, I\}. \tag{82}$$

By the continuity of ℓ, (80) and (82), we get that (77) is verified for $T \in (0, \infty)$. For $T = \infty$, we find that (82) holds for all $\bar{T} \in (0, \infty)$. Since ℓ is bounded from below by 0, we have

$$\mathcal{J}_{\bar{T}}(v_k) \leqslant \mathcal{J}_\infty(v_k), \text{ for every } \bar{T} \in (0, \infty).$$

Then, taking the limit inf when $k \to \infty$ on both sides of the previous inequality we get

$$\mathcal{J}_{\bar{T}}(v) \leqslant \liminf_{k \to \infty} \mathcal{J}_\infty(v_k), \text{ for every } \bar{T} \in (0, \infty),$$

where we use that $\liminf = \lim$ on the left hand side. Finally, taking $\bar{T} \to \infty$ we obtain (78). ■

Proof [Proof of Theorem 1] Assume that there exists a feasible solution of problem (20). Since the objective function is bounded from below, there exists an infimizing sequence $\theta^k \in \mathbb{R}^M$. We denote the infimum of (20) by $\mathcal{J}_T^*$ and set $v_k = \sum_{i=1}^M \theta_i^k \phi_i$, where θ_i^k is the i-th component of θ^k. Since θ^k is an infimizing sequence, the sequence $\{P_{\gamma,r}(\theta^k)\}_{k \in \mathbb{N}}$ is bounded, that is

$$P_{\gamma,r}(\theta^k) = \gamma \left(\frac{(1-r)}{2} |\theta^k|_2^2 + r |\theta^k|_1 \right) \leqslant C,$$

for some $C > 0$ independent of k. This implies that there exists $\theta^* \in \mathbb{R}^M$ such that, passing to a sub-sequence

$$\theta^k \to \theta^* \text{ and } v_k \to v^* = \sum_{i=1}^{n} \theta_i^* \phi_i \text{ in } C^{1,1}(\overline{\Omega}).$$

Hence, by Lemma 4 we have that $\tilde{\mathcal{J}}_T(\theta^*) + P_{\gamma,r}(\theta^*) \leqslant \mathcal{J}_T^*$, and we conclude that θ^* is a solution of (20). ■

Proof [Proof of Theorem 2] We provide the proof in several steps.

Step 1. Let V be the value function of (1). Since V is assumed to be $C^{1,1}(\overline{\Omega})$, by Theorem 9 in (Hájek and Johanis, 2014, Section 7.2), for every $\varepsilon \in (0,1)$ there exists a natural number $k(\varepsilon)$ and a polynomial $V_\varepsilon \in \mathcal{P}_{k(\varepsilon)}$ such that

$$\left\| B^\top(\nabla V_\varepsilon - \nabla V) \right\|_{C(\overline{\Omega})} < \varepsilon. \tag{83}$$

Since $\nabla V(0) = 0$ and $V(0) = 0$, we assume that $V_\varepsilon(0) = 0$ and $\nabla V_\varepsilon(0) = 0$, otherwise we can redefine it by subtracting $V_\varepsilon(0) + \nabla V_\varepsilon(0) \cdot x$ from it. We denote the coefficients of V_ε with respect to the basis $X_{k(\varepsilon)} = \mathcal{B}_{k(\varepsilon)} \setminus \mathcal{B}_1$ by $\theta^\varepsilon \in \mathbb{R}^{M_{k(\varepsilon)}}$. By Lemma 3, for every $i \in \{1, \dots, I\}$ problem (9) has a solution $y_i^\varepsilon \in C^1([0, \tilde{T}_\varepsilon], \overline{\Omega})$, with $T = \tilde{T}_\varepsilon$ and $v = V_\varepsilon$, where $\tilde{T}_\varepsilon$ is given by (65). Moreover, by (66) we know that there exists $K > 0$ independent of ε, such that

$$\mathcal{J}_{\tilde{T}_\varepsilon}(V_\varepsilon) \leqslant \mathcal{J}_{\tilde{T}_\varepsilon}(V) + K\varepsilon^{1/4}. \tag{84}$$

Step 2. We consider problem (20) with $X = \mathcal{B}_{k(\varepsilon)} \setminus \mathcal{B}_1$, $T = \tilde{T}_\varepsilon$, $\gamma > 0$ and $r \in [0,1]$, and denote its solution by $\theta^{\gamma,r,k(\varepsilon),\tilde{T}_\varepsilon}$, which we know to exist by Lemma 3 and Theorem 1. We point out that it is possible to use $X = \mathcal{S}_{\tilde{k}(\varepsilon)} \setminus \mathcal{B}_1$ instead of $B_{k(\varepsilon)}$, provided that $\tilde{k}(\varepsilon)$ is sufficiently large such that $\mathcal{B}_{k(\varepsilon)} \subset \mathcal{S}_{\tilde{k}(\varepsilon)}$. We set

$$v_{\gamma,r,k(\varepsilon),\tilde{T}_\varepsilon} = \sum_{i=1}^{M_{k(\varepsilon)}} \theta_i^{\gamma,r,k(\varepsilon),\tilde{T}_\varepsilon} \phi_i.$$

Since $\theta^{\gamma,r,k(\varepsilon),\tilde{T}_\varepsilon}$ is optimal, we have

$$\mathcal{J}_{\tilde{T}_\varepsilon}(v_{\gamma,k(\varepsilon),\tilde{T}_\varepsilon}) + P_{\gamma,r}(\theta^{\gamma,r,k(\varepsilon),\tilde{T}_\varepsilon}) \leqslant \mathcal{J}_{\tilde{T}_\varepsilon}(V_\varepsilon) + P_{\gamma,r}(\theta^\varepsilon) \tag{85}$$

and by (84) we obtain

$$\mathcal{J}_{\tilde{T}_\varepsilon}(v_{\gamma,k(\varepsilon),\tilde{T}_\varepsilon}) + P_{\gamma,r}(\theta^{\gamma,r,k(\varepsilon),\tilde{T}_\varepsilon}) \leqslant \mathcal{J}_\infty(V) + K\varepsilon^{1/4} + P_{\gamma,r}(\theta^\varepsilon). \tag{86}$$

We now choose $\gamma = \gamma_\varepsilon$ such that $P_{\gamma_\varepsilon,r}(\theta^\varepsilon) = K\varepsilon^{1/4}$. Then, we obtain

$$\mathcal{J}_{\tilde{T}_\varepsilon}(v_{\gamma_\varepsilon,r,n(\varepsilon),\tilde{T}_\varepsilon}) + P_{\gamma_\varepsilon,r}(\theta^{\gamma_\varepsilon,r,k(\varepsilon),T_\varepsilon}) \leqslant \mathcal{J}_\infty(V) + 2K\varepsilon^{1/4} \tag{87}$$

and taking $\varepsilon \to 0$, we get for every $r \in [0,1]$

$$\limsup_{\varepsilon \to 0} \mathcal{J}_{\tilde{T}_\varepsilon}(v_{\gamma_\varepsilon,r,n(\varepsilon),\tilde{T}_\varepsilon}) \leqslant \mathcal{J}_\infty(V). \tag{88}$$

Step 3. For $i \in \{1, \ldots, I\}$, we denote the solutions of (9), for $y_0 = y_0^i$, $T = \tilde{T}_\varepsilon$, and $v = v^{\gamma_\varepsilon, r, k(\varepsilon), \tilde{T}_\varepsilon}$, by $y_i^{\gamma_\varepsilon, r, k(\varepsilon), \tilde{T}_\varepsilon}$, and we define the controls $u_i^{\gamma_\varepsilon, r, k(\varepsilon), \tilde{T}_\varepsilon} \in L^2((0, \tilde{T}_\varepsilon); \mathbb{R}^m)$ by

$$u_i^{\gamma_\varepsilon, r, k(\varepsilon), \tilde{T}_\varepsilon}(t) = -\frac{1}{\beta} B^\top \nabla v^{\gamma_\varepsilon, r, k(\varepsilon), \tilde{T}_\varepsilon}(y_i^{\gamma_\varepsilon, r, k(\varepsilon), \tilde{T}_\varepsilon}(t)) \text{ in } (0, \tilde{T}_\varepsilon).$$

Now, by the definition of the controls we have for $i \in \{1, \ldots, I\}$ and all $\bar{T} \in (0, \tilde{T}_\varepsilon)$

$$\int_0^{\bar{T}} |u_i^{\gamma_\varepsilon, r, k(\varepsilon), \tilde{T}_\varepsilon}|^2 dt = \int_0^{\bar{T}} \left| B^\top \nabla v^{\gamma_\varepsilon, r, k(\varepsilon), \tilde{T}_\varepsilon}(y_i^{\gamma_\varepsilon, r, k(\varepsilon), \tilde{T}_\varepsilon}) \right|^2 dt \leqslant 2\beta I |B|^2 \tilde{\mathcal{J}}_{\tilde{T}_\varepsilon}(v^{\gamma_\varepsilon, r, k(\varepsilon), \tilde{T}_\varepsilon}).$$

Using (87) and $\varepsilon \leqslant 1$ in the previous inequality we get

$$\int_0^{\bar{T}} |u_i^{\gamma_\varepsilon, r, k(\varepsilon), \tilde{T}_\varepsilon}|^2 dt \leqslant 2\beta I |B|^2 (2K + \mathcal{J}_\infty(V)). \tag{89}$$

In virtue of (89), (9), and the fact that $|y_i^{\gamma_\varepsilon, r, k(\varepsilon), \tilde{T}_\varepsilon}(t)| \leqslant l$ for all $t \in [0, \tilde{T}_\varepsilon]$, we get

$$\int_0^{\bar{T}} \left| \frac{d}{dt} y_i^{\gamma_\varepsilon, r, k(\varepsilon), \tilde{T}_\varepsilon}(t) \right|^2 dt \leqslant \bar{T} \sup_{x \in \overline{\Omega}} |f(x)|^2 + 2\beta I |B|^2 (2K + \mathcal{J}_\infty(V)). \tag{90}$$

Thus, for every $\bar{T} \in (0, \infty)$, $i = 1, \ldots, I$ and taking $\varepsilon \to 0$, there exist $y_i^* \in H^1_{loc}((0, \infty); \mathbb{R}^d)$ and $u_i^* \in L^2_{loc}((0, \infty); \mathbb{R}^m)$ such that, passing to a sub-sequence

$$y_i^{\gamma_\varepsilon, r, k(\varepsilon), \tilde{T}_\varepsilon} \rightharpoonup y_i^* \text{ in } H^1((0, \bar{T}); \mathbb{R}^d) \text{ and } u_i^{\gamma_\varepsilon, r, k(\varepsilon), \tilde{T}_\varepsilon} \rightharpoonup u_i^* \text{ in } L^2((0, \bar{T}); \mathbb{R}^m). \tag{91}$$

Further, by the compact inclusion of $C([0, \bar{T}]; \mathbb{R}^d)$ into $H^1((0, \bar{T}), \mathbb{R}^d)$, we have

$$y_i^{\gamma_\varepsilon, r, k(\varepsilon), \tilde{T}_\varepsilon} \to y_i^* \text{ in } C([0, \bar{T}]; \mathbb{R}^d) \tag{92}$$

when $\varepsilon \to 0$, for every $\bar{T} \in (0, \infty)$.

For $i \in \{1, \ldots, d\}$, we use (91), (92) and take $\varepsilon \to 0$ in (9) to obtain

$$(y_i^*)'(t) = f(y_i^*(t)) + B u_i^*(t), \ \forall\, t \in (0, \infty), \quad y_i^*(0) = y_0^i. \tag{93}$$

Additionally, using the definitions of $y_i^{\gamma_\varepsilon, r, k(\varepsilon), T_\varepsilon}$ and $u_i^{\gamma_\varepsilon, r, k(\varepsilon), T_\varepsilon}$ for $i \in \{1, \ldots, I\}$ together with (91), (92), and the lower semi-continuity of $|\cdot|^2$ we have

$$\frac{1}{I} \sum_{i=1}^{I} \int_0^{\bar{T}} \ell(y_i^*) dt + \frac{\beta}{2} \int_0^{\bar{T}} |u_i^*|^2 dt \leqslant \liminf_{\varepsilon \to 0} \mathcal{J}_{\tilde{T}_\varepsilon}(v_{\gamma_\varepsilon, r, n(\varepsilon), \tilde{T}_\varepsilon}) \ \forall \bar{T} \in (0, \infty). \tag{94}$$

In particular, since $\bar{T} \in (0, \infty)$ in (94) is arbitrary, we get

$$\frac{1}{I} \sum_{i=1}^{I} \int_0^{\infty} \ell(y_i^*) dt + \frac{\beta}{2} \int_0^{\infty} |u_i^*|^2 dt \leqslant \liminf_{\varepsilon \to 0} \mathcal{J}_{\tilde{T}_\varepsilon}(v_{\gamma_\varepsilon, r, n(\varepsilon), \tilde{T}_\varepsilon}). \tag{95}$$

By (93), (95) and the definition of the value function we have

$$\begin{aligned}\mathcal{J}_\infty(V) &= \frac{1}{I}\sum_{i=1}^{I}\int_0^\infty \ell(y_i)dt + \frac{\beta}{2}\int_0^\infty |B^\top \nabla V(y_i)|^2 dt \\ &\leqslant \frac{1}{I}\sum_{i=1}^{I}\int_0^\infty \ell(y_i^*)dt + \frac{\beta}{2}\int_0^\infty |u_i^*|^2 dt \leqslant \liminf_{\varepsilon\to 0} \mathcal{J}_{\tilde{T}_\varepsilon}(v_{\gamma_\varepsilon,r,n(\varepsilon),\tilde{T}_\varepsilon}).\end{aligned} \tag{96}$$

Finally, (88) and (96) imply

$$\lim_{\varepsilon\to 0} \mathcal{J}_{\tilde{T}_\varepsilon}(v_{\gamma_\varepsilon,r,n(\varepsilon),\tilde{T}_\varepsilon}) = \mathcal{J}_\infty(V). \tag{97}$$

which concludes the proof. ■

Proposition 3 *Assume that $\nu \in C^2(\overline{\Omega})$ with $\nabla\nu(0) = 0$ and $\sigma \in (0, l)$ are such that (9) with $v = \nu$, $T = \infty$ and $i \in \{1, \dots, I\}$ admits a solution y_i, satisfying*

$$\lim_{t\to\infty} y_i(t) = 0,\ |y_i(t)|_\infty \leqslant l - \sigma,\ \forall t \in [0,\infty),\ \textit{for all } i \in \{1,\dots,I\}. \tag{98}$$

Suppose further that the linearized system

$$z' = \left(Df(0) - \frac{1}{\beta}BB^\top \nabla^2\nu(0)\right) z, \quad z(0) = z_0 \tag{99}$$

is exponentially stable, i.e. there exist $C > 0$ and $\mu > 0$ such that

$$|z| \leqslant Ce^{-\mu t}|z_0| \ \textit{for all } t \in (0,\infty) \ \textit{and } z_0 \in \mathbb{R}^d.$$

Then, there exist $\varepsilon_0 \in (0, 1)$, $\rho > 0$, $K > 0$, and $\kappa > 0$, such that for every $\tilde{\nu} \in C^2(\overline{\Omega})$ which satisfies

$$\|\nu - \tilde{\nu}\|_{C^2(\overline{\Omega})} \leqslant \varepsilon_0 \ \textit{and } \nabla\tilde{\nu}(0) = 0, \tag{100}$$

we have that the closed loop system (9) with $v = \tilde{\nu}$ is exponentially stable for every $y_0 \in B(0, \rho)$, and for every $i \in \{1, \dots, I\}$

$$|\tilde{y}_i(t)| \leqslant Ke^{-\kappa t}|y_0^i| \ \textit{for all } t \in (0,\infty), \tag{101}$$

and $\mathcal{J}_\infty(\tilde{\nu}) < \infty$, where $\{\tilde{y}_i\}_{i=1}^I$ are the solution of (9) with $v = \tilde{\nu}$.

Proof [Proof of Proposition 3] Consider $\varepsilon \in (0, 1)$ and a function $\nu_\varepsilon \in C^2(\overline{\Omega})$ such that

$$\|\tilde{\nu} - \nu\|_{C^2(\overline{\Omega})} \leqslant \varepsilon \text{ and } \nabla\tilde{\nu}(0) = 0. \tag{102}$$

By Lemma 3 there exists a time $T_\varepsilon > 0$, such that for each $i \in \{1, \dots, I\}$, problem (9) with $v = \nu^\varepsilon$ and $T = T_\varepsilon$ admits a solution $y_i^\varepsilon \in C^1([0, T_\varepsilon]; \overline{\Omega})$, which satisfies (64). Moreover, we know that T_ε fulfills (63).

Due to the exponential stability of (99), there exists a symmetric and positive definite matrix $M \in \mathbb{R}^{d\times d}$ (see Theorem 4.6 in (Khali, 2002, p. 136)), such that

$$y^\top (A^\top M + MA)y = -|y|^2, \text{ for all } y \in \mathbb{R}^d,$$

where $A = Df(0) - \frac{1}{\beta}BB^\top \nabla^2 \nu(0)$. This equality and the fact that ν is $C^2(\overline{\Omega})$, implies that there exists $\rho > 0$, such that

$$2(f(y) - \frac{1}{\beta}BB^\top \nabla \nu(y))^\top My < -\frac{3}{4}|y|^2 \text{ for all } y \in B(0,\rho) \subset \Omega. \tag{103}$$

By (102) and the integral mean value theorem we have

$$|\nabla \tilde{\nu}(y) - \nabla \nu(y)| \leqslant \varepsilon |y| \text{ for all } y \in \overline{\Omega}. \tag{104}$$

For $\varepsilon < \frac{\beta}{8|B|^2|M|}$ and combining (102), (104), and (103) we have

$$2(f(y) - \frac{1}{\beta}BB^\top \nabla \tilde{\nu}(y))^\top My < -\frac{1}{2}|y|^2 \text{ for all } y \in B(0,\rho). \tag{105}$$

Thus, $\psi(y) = y^\top My$ is a Lyapunov function for (9) with $v = \nu^\varepsilon$ in $B(0,\rho)$, (Khali, 2002, Section 4.4, Theorem 4.10).

By (98), we know that there exists $T > 0$ such that

$$|y_i(t)| < \frac{\rho}{4} \text{ for all } t \geqslant T \text{ and } i \in \{1,\dots,I\}.$$

Further, by (63), we know that $T_\varepsilon > T$ if $\varepsilon < C^{1/2}\sigma/(e^{CT}-1)^{1/2}$. Hence, by (64) and choosing ε satisfying $\varepsilon < C^{1/2}\sigma/(e^{CT}-1)^{1/2}$ we have

$$|y_i^\varepsilon(T)| \leqslant |y_i^\varepsilon(T) - y_i(T)| + |y_i(T)| \leqslant \frac{\varepsilon}{C^{1/2}}(e^{CT}-1)^{1/2} + |y_i(T)|. \tag{106}$$

Choosing

$$\varepsilon \leqslant \varepsilon_0 := \min\left\{ \frac{\rho C^{1/2}}{4(e^{CT}-1)^{1/2}}, \frac{C^{1/2}\sigma}{(e^{CT}-1)^{1/2}}, \frac{\beta}{8|B|^2|M|} \right\},$$

we have that $\frac{\varepsilon}{C^{1/2}}(e^{CT}-1)^{1/2} \leqslant \frac{\rho}{4}$ and by (106) we obtain

$$|y_i^\varepsilon(T)| \leqslant \frac{\rho}{2} \text{ for all } i \in \{1,\dots,I\}.$$

Given that ψ is a Lyapunov function in $B(0,\rho)$, we have that there exist $K > 0$ and $\kappa > 0$ such that (101) holds for all $\tilde{\nu}$ satisfying (100). Further, using that $\tilde{\nu}$ is $C^{1,1}(\overline{\Omega})$, ℓ is C^1, and (101), we get that $\mathcal{J}_\infty(\tilde{\nu}) < \infty$. Therefore, Proposition 3 holds with ε_0. ■

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