A Hotelling-Downs Framework for Party Nominees

Paul Harrenstein
University of Oxford
United Kingdom
Paul.Harrenstein@cs.ox.ac.uk

Grzegorz Lisowski
University of Warwick
United Kingdom
Grzegorz.Lisowski@warwick.ac.uk

Ramanujan Sridharan
University of Warwick
United Kingdom
R.Maadapuzhi-Sridharan@warwick.ac.uk

Paolo Turrini
University of Warwick
United Kingdom
P.Turrini@warwick.ac.uk

ABSTRACT

We present a model for the strategic selection of party nominees, where competing groups choose their representatives based on the expected electoral returns. Technically, we look at a generalisation of the Hotelling-Downs model, where each nominee has a predefined position on the political spectrum and attracts the closest voters compared to all other representatives. Within this framework we explore the algorithmic properties of Nash equilibria, which are not guaranteed to exist even in two party competitions. We show that finding a Nash equilibrium is NP-complete for the general case. However, if there are only two competing parties, this can be achieved in linear time. The results readily extend to games with restricted positioning options for the players involved, such as facility location and Voronoi games.

KEYWORDS

Hotelling-Downs Games; Strategic Candidacy; Primaries

ACM Reference Format:
Paul Harrenstein, Grzegorz Lisowski, Ramanujan Sridharan, and Paolo Turrini. 2021. A Hotelling-Downs Framework for Party Nominees. In *Proc. of the 20th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2021), Online, May 3–7, 2021*, IFAAMAS, 9 pages.

1 INTRODUCTION

Strategic voting [23] has become a central topic of algorithmic game theory and computational social choice. The analysis of collective decision-making among self-interested individuals poses deep computational problems with neat applications to economic interaction. This is particularly true for the analysis of representative democracy, where parties compete for electoral returns: deciding which candidate to select as the party's representative at an upcoming election warrants a careful examination of the behaviour of voters and candidates alike.

In US elections, for example, there is an explicit two-stage process, where parties undergo an internal nominee selection process, referred to as *primaries.* The candidates selected by the parties in the primaries then face one another in the general elections. This mechanism has been widely studied in the political science literature (see, e.g. [8, 32]) and has recently also attracted the attention of researchers in artificial intelligence (see, e.g. [5]). The latter focused on the parties' internal selection mechanisms rather than the strategic aspects thereof. As also emphasised by Borodin et al. [5], an important challenge has remained open, namely, that of understanding how parties can best select their nominees, as a function of the voters' opinions and the other competing parties' nominees.

The Hotelling-Downs model [21] is perhaps the most impactful and well-established framework to study strategic positioning of self-interested players on a spacial dimension, e.g., candidates on a political spectrum. In Hotelling and Downs's original setup, two self-interested ice cream vendors strategically place themselves on a beach so as to attract as many customers as possible, in the knowledge that the relaxed beachgoers will always opt for the one closer to them. This game has a unique Nash equilibrium, in which both agents choose the most central location. The simplicity and depth of this observation has led to applications in corporate strategy, strategic candidacy, and spatial design [13]. In [9], Downs himself mentions the potential of this framework to predict how parties will set their agendas and how they will position themselves in the political spectrum, suggesting that party politics will tend to more moderate choices.

It is fair to say that Downs's observation relies on severely restrictive assumptions. First and foremost, his model involves only two agents. Indeed, it has been shown that, if the number of agents is increased, there are cases without (pure) Nash equilibria [12]. Second, and perhaps more importantly, agents are allowed unrestricted movement. While this might be a reasonable assumption for ice cream vendors on a beach, this is certainly not the case for political parties, which can only count on a few potential nominees, typically tied up to relatively fixed political stances. Similarly, from an economics perspective, producers might be limited to a fixed number of products which can potentially be released.

Surprisingly, variations of the Hotelling-Downs model involving multiple participants with restricted options—and which could thus capture the mathematics behind real-world situations like the US primaries—have been largely overlooked. A notable exception is the work by Sabato et al. [28] on *real candidacy games*, where competing candidates select intervals on a line and are then chosen based on a given social choice rule. Notwithstanding the similarities of this work with our framework, it also displays some important technical differences. In particular, their restricted action spaces, in our view not suitable to model nominee selection, can force equilibrium existence, while they do not consider computation.

Proc. of the 20th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2021), U. Endriss, A. Nowé, F. Dignum, A. Lomuscio (eds.), May 3–7, 2021, Online.

Our Contribution. We provide a game-theoretic analysis of nominee selection, where parties choose independently and simultaneously from their respective pools of potential candidates. We assume that both voters and party candidates occupy a fixed position on a line, with voters always voting for the (nominated) candidate that is closest to them. We carry out an algorithmic analysis of verifying the existence of Nash equilibria (NEs), while focussing on the differences between two-party and n-party systems. Specifically, we show that if there are only two parties, the problem of establishing whether an NE exists can be achieved in linear time (Theorem 3.7). By contrast, finding an NE is NP-complete in the multi-party case (Theorem 3.14). We also look at some natural restrictions, such as having parties with non-overlapping political spectra, and provide equilibrium existence results for these, as well.

Related Literature. Our work connects to both to the literature on the Hotelling-Downs model in economics and the research on Voronoi games and facility location in computer science.

The Hotelling-Downs model has been widely studied and applied to various contexts (see, e.g. the highly impactful [30] and [14], as well as [13] for a survey). In a similar vein to ours, much research has been devoted to lifting the assumptions made in the original model, including scenarios with multiple players and voting rules (see, e.g. [12], [3],[29]) or dimensions in the metric space (e.g. [31], see also [13]). In the field of algorithmic game theory, Feldman et al. [19] have analysed the case in which candidates attract voters only in a limited range. In the context of voting, Brusco et al. [7] have looked at an application of the model with employment of the plurality with the run-off rule. As noted previously, the work of Sabato et al. [28] is certainly the closest to ours, but it presents important differences in terms of equilibrium existence and algorithmic analysis. In particular their setup restricts agents' positioning to intervals, which, forces equilibrium existence under simple voting rules (also see our Proposition 3.2 and Example 4.2). Moreover, it is not focussed on computing equilibria.

In algorithmic game theory, *Voronoi games* feature players selecting points in a given space, with their utility being equal to the sum of points in the space for which their selection is the closest. Voronoi games have been studied as sequential decision problems (see, e.g. Ahn et al. [1], Bandyapadhyay et al. [2]), where two players select their (potentially multiple) locations in rounds. It the simultaneous variant, which is the closer to our setup, Dürr and Thang [10] show that checking if a Nash equilibrium exists is NP-complete, although studying games played on arbitrary graphs and using more complex computational gadgets. Furthermore, Mavronicolas et al. [22] provide a characterisation of Nash equilibria in games played on cycle graphs. Also, Fournier [20] considers a setting in which consumers (our voters) are distributed non-uniformly, but players are allowed to position themselves anywhere on a graph. In a related contribution Boppana et al. [4] consider Voronoi games with restricted positioning, but on a different spacial domain, namely a k-dimensional unit torus. Similarly, Núñez and Scarsini, in a series of works [25, 26], study players with limited available positions from a finite set of locations. In contrast to our model, the action spaces are the same for all of the players. They then show that Nash equilibria exist provided a large number of players.

Facility location is an important related problem, where a planner selects the location of various facilities to satisfy as many agents as possible, given their positioning. This setup, originating from Moulin [24], has been extensively studied in the social choice literature. In particular, Feldman et al. [18] have considered how to locate facilities when participants can strategically misrepresent their position in order to benefit from the planner's decisions.

Our framework is closely linked to strategic voting, (see e.g. [23]), the emerging area of computational social choice where participants may misrepresent their preferences to potentially manipulate the result of an election and, in particular, strategic candidacy.

In the typical settings of strategic candidacy (see, e.g. [6, 11, 16]), candidates are equipped with preferences over their opponents, and are allowed to step down to let their favourite rival win. In some models (e.g. [15, 27]) participation in the elections incurs a cost, in which case it might be beneficial for candidates to simply abstain, if they cannot themselves win. An approach that is closer in spirit to ours is that of Faliszewski et al. [17], who study the behaviour of coalitions of potential candidates, which can be thought of as political parties. Then, given a profile of voters and a voting rule, parties select a single candidate maximizing the party's chances of winning. The authors analyse the algorithmic properties of checking if a party has a potential or a necessary winner.

Finally, our approach contributes to the study of primaries, along the lines of the work by Borodin et al. [5], who focus on the protocols for nominee selection within a party.

Paper Structure. In Section 2 we define our basic setup, including examples and key observations used in the remainder of the paper. In Section 3 we study the structural and algorithmic properties of Nash equilibria. Section 4 concludes with a discussion of our main findings and of some interesting directions for future research.

2 COALITIONAL STRATEGIC CANDIDACY

Parties, Voters, and Games. The games we are concerned with are played on a discrete *line* $\{0, 1, 2, \ldots, k\}$. For $x, y \in \mathbb{R}_0$ with $x \leq y$, we denote by $[x, y]$ the *segment* $\{\lfloor x \rfloor, \lfloor x \rfloor + 1, \ldots, \lceil y \rceil\}$. On this line, a positive number of *voters* are placed according to a distribution function $f\colon [0, k] \to \mathbb{N}_0$. We have $V(f) = \sum_{i \in [0,k]} f(i)$ denote the total number of voters on a line. At times we will restrict attention to distribution functions f that are *uniform*, i.e., for which $f(i) = 1$, for each $i \in [0, k]$.

The players of the game are given by a set $P = \{P_1, P_2, \ldots, P_n\}$ of *parties*. Each party P_i is fully described by a set of points on the line, i.e., $P_i \subseteq [0, k]$. Intuitively, these points correspond to the positions of the candidates the party has to choose its nominee from. Formally, they make up the party's *strategies*. We allow different parties to have candidates that occupy the same position, i.e., P_i and P_j need not be disjoint for distinct i and j.

Parties strategise over which candidate to select as their *nominee*. Thus, we define a *strategy profile* as a tuple $\mathbf{c} = (c_1, c_2, \ldots, c_n)$, where $c_i \in P_i$. With each party P_i and strategy profile $\mathbf{c} = (c_1, \ldots, c_n)$ we associate a utility, which is intuitively given by the number of voters that are closer to c_i than any other party P_j's chosen nominee c_j. A voter that is just as far removed from nominees on the left as from nominees to the right will contribute a half to the utility of the former and half to the utility of the latter. Nominees that are in

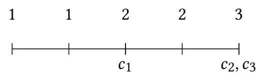


Figure 1: Party choices on a line: candidates are at the bottom, number of voters per position at the top.

the same position share the number voters they attract evenly. The following example illustrates the setup and introduces the intuition behind the concept of utility.

Example 2.1. Figure 1 depicts the line $[0, 4]$ and three parties $P_1 = \{0, 2\}$, $P_2 = \{4\}$, and $P_3 = \{1, 2, 4\}$. The distribution f of the voters, indicated at the top of the line, is given by $f(0) = f(1) = 1$, $f(2) = f(3) = 2$ and $f(4) = 3$. Below the line the figure displays the strategy profile $\mathbf{c} = (c_1, c_2, c_3)$, where $c_1 = 2$, $c_2 = 4$, and $c_3 = 4$. Given this strategy profile, P_1 attracts $1 + 1 + 2 + \frac{1}{2} \cdot 2 = 5$ voters, while P_2 and P_3 attract $\frac{1}{2}(\frac{1}{2} \cdot 2 + 3) = 2$ voters each.

Let now P_i be a party and $\mathbf{c} = (c_1, \ldots, c_n)$ a strategy profile. We set $L(c_i)$ to return the immediate predecessor of c_i in $\mathbf{c}$ on the line, whenever one exists. Likewise, we use $R(c_i)$ for c_i's immediate successor in $\mathbf{c}$ on the line, whenever one exists. That is, $L(c_i) = \sup\{c_j \in \mathbf{c} \colon c_j < c_i\}$ and $R(c_i) = \inf\{c_j \in \mathbf{c} \colon c_j > c_i\}$, on the understanding that $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$. We associate with each candidate $c_i \in \mathbf{c}$ an indicator function $\sigma_{c_i} \colon [0, k] \to \{0, \frac{1}{2}, 1\}$ that assumes value 1 if $m \in [0, k]$ is strictly closer to c_i than to any other $c_j \in \mathbf{c}$, value $\frac{1}{2}$ if m is equally close to c_i as some other $c_j \in \mathbf{c}$, with $c_i \neq c_j$ and not farther removed from m than any, and value 0 if m is strictly closer to some $c_j \in \mathbf{c}$ other than c_i, i.e.,

$$\sigma_{c_i}(m) = \begin{cases} 1 & \text{if } |c_i - m| < \min\{|L_{c_i} - m|, |R_{c_i} - m|\}, \\ \frac{1}{2} & \text{if } |c_i - m| = \min\{|L_{c_i} - m|, |R_{c_i} - m|\}, \\ 0 & \text{otherwise.} \end{cases}$$

The *range* of candidate c_i on line $[0, k]$, denoted $range_{c_i}(\mathbf{c})$, is then given by the set $\{m \in [0, k] \colon \sigma_{c_i}(m) > 0\}$, which is worth observing, is an interval. Let $\#S$ denote the cardinality of a set S and $[c_i]$ the set of candidates in $\mathbf{c}$ sharing the position c_i, i.e., $[c_i] = \{1 \leq j \leq n \colon c_i = c_j\}$. We now define the *utility* $u_i(\mathbf{c})$ of party P_i on profile $\mathbf{c}$ as:

$$u_i(\mathbf{c}) = \frac{1}{\#[c_i]} \sum_{m \in [0,k]} \sigma_{c_i}(m) \cdot f(m)$$

Observe that in the current setting each voter is attracted to one candidate and hence $\sum_{P_i \in P} u_i(c) = V(f)$.

It is worth noting that our setting defines a class of strategic games, where parties are players and their available candidates are their strategies. Hence, in the remainder of the paper we will study the properties of game-theoretic solution concepts induced by party choices, specifically Nash equilibrium.

Given a strategy profile $\mathbf{c} = (c_1, \ldots, c_n)$, a party P_i and a candidate $c_i' \in P_i$, we denote $(c_i', \mathbf{c}_{-i}) = (c_1, \ldots, c_i', \ldots, c_n)$, i.e., the strategy profile in which the candidate of P_i is c_i' and all other parties select the same candidate as in $\mathbf{c}$. We say that profile $\mathbf{c}$ is a *(pure) Nash equilibrium* (NE), if $u_i(\mathbf{c}) \geqslant u_i(c_i', \mathbf{c}_{-i})$ for all i and $c_i' \in P_i$.In words, a profile is a Nash equilibrium if no party can improve their utility by switching their choice unilaterally.

Input representation. As we are concerned with the computational complexity of decision problems within this framework, it is useful to clarify the input representation. In particular, it is important to notice that we can represent a game with voters situated on the line $[0, k]$ with a distribution given by a function $f : [0, k] \to \mathbb{N}_0$ and a set of parties $P = \{P_1, \ldots, P_n\}$ as a $(n + 1) \times (k + 1)$ table, where entry $(1, i)$ specifies the number of voters at position $i - 1$ and for $j > 1$, entry (j, i) specifies if party P_{j-1} has a potential candidate at position $i - 1$. So, the representation of the game has size bounded by $(k + 1) \cdot n + (k + 1) \cdot \log \max_f$ bits, where $\max_f$ denotes the maximum number of voters at any point on line $[0, k]$.

We will also consider a natural class of games, where we have parties with non-overlapping ranges, which we call a *sector structure.* Intuitively, this is a scenario in which each two parties are one at the left of the other. Formally, for line $[0, k]$ with set of parties $P = \{P_1, \ldots, P_n\}$, we say that P has the *sector structure* if $i \neq j$ implies $[\min(P_i), \max(P_i)] \cap [\min(P_j), \max(P_j)] = \emptyset$. We will assume, without loss of generality, that, given a set of parties $P = \{P_1, \ldots, P_n\}$ with a sector structure and parties P_i, P_j such that $i > j$, we have that $c_i > c_j$ for each $c_i \in P_i$ and $c_j \in P_j$. Notice that, if we only have two groups P_1, P_2, the parties have the sector structure if and only if either for every $c_1 \in P_1$ and $c_2 \in P_2$, $c_1 > c_2$ or for every $c_1 \in P_1$ and $c_2 \in P_2$, $c_2 > c_1$.

3 EQUILIBRIA

In this section we carry out an analysis of Nash equilibria, focusing on existence and computation. We start with games with two parties, then we generalise to the n-party case.

3.1 Games with two parties

3.1.1 Existence. Our model is a generalisation of the discrete version of Hotelling-Downs model where, as argued earlier on, a Nash equilibrium is guaranteed to exist.

To see this in our model, consider a $[0, k]$ line and two parties, P_1 and P_2. It is well known (see e.g. [23]) that if $P_1 = P_2 = [0, k]$, k is even and f is uniform, then the game has a unique NE, which can be computed in $k - 1$ rounds of iterated elimination of strictly dominated strategies. In this equilibrium both players choose the central position, getting utility of $\frac{V(f)}{2}$ each. When k is odd instead, then any outcome in which players select a central position is an NE.

With voters that are potentially non uniformly distributed this fact is still true, provided the notion of central position is replaced by that of *median* position, which we will define next and also need later on. A position $\mathbf{m} \in [0, k]$ is called *median* if $f(\mathbf{m}) > 0$, $\sum_{n \leqslant \mathbf{m}} f(n) \geqslant \frac{V(f)}{2}$ and $\sum_{n \geqslant \mathbf{m}} f(n) \geqslant \frac{V(f)}{2}$. In words, a median is a non-empty position such that half of the voters is located there or on the left of it, and a half there or on the right of it. Given a line, we denote as $\mathbf{m}_L$ its smallest median position, and as $\mathbf{m}_R$ its largest. If the median position is unique, we simply refer to it as $\mathbf{m}$.

Median positions always exist but they need not be unique. Moreover, there are cases in which median positions do not come consecutively. To see this, consider a $[0, 4]$ line and the distribution of voters f such that $f(0) = f(4) = 1$ and for $i \in [1, 3]$, $f(i) = 0$. Then, both 0 and 4 are the only median positions and are not consecutive.

Note also that we immediately have that in any election there are at most two median positions and that, given positions $\mathbf{m}_L, \mathbf{m}_R$, if $n \in [\mathbf{m}_L + 1, \mathbf{m}_R - 1]$, then $f(n) = 0$.

We now use the notion of a median position to show that an NE is guaranteed to exist if the parties' choices are intervals, i.e., for each party P there are p_ℓ, p_r such that $P = [p_\ell, p_r]$. Incidentally, this encodes the action space as studied in [28], under a very basic "voting" rule.

Let us fix the following definition. For candidates $p_1, p_1' \in P_1$ and $p_2 \in P_2$ we say that p_1' *is strictly closer to* p_2 *than* p_1 whenever either $p_1 < p_1' < p_2$ or $p_2 < p_1' < p_1$. This lemma follows directly.

Lemma 3.1. *Let* $\mathbf{c} = (p_1, p_2)$ *be a strategy profile with* $p_1 \in P_1$ *and* $p_2 \in P_2$*. Then, if there exists* $p_1' \in P_1$ *such that* p_1' *is strictly closer to* p_2 *than* p_1*, we have that* $u_1(p_1', p_2) \geqslant u_1(p_1, p_2)$*.*

Now we are ready to show the existence of NE in interval models.

Proposition 3.2. *Let* f *be the distribution of voters and* P_1, P_2 *be parties. If* P_1 *and* P_2 *are intervals, then there are* $c_1 \in P_1, c_2 \in P_2$ *such that* (c_1, c_2) *is an NE.*

Proof. Let P_1 and P_2 be intervals. We show there exists an NE profile $\mathbf{c}$=(c_1, c_2). Recall that $\mathbf{m}_L$ and $\mathbf{m}_R$ denote the median positions. If there is a single median position, then $\mathbf{m} = \mathbf{m}_L = \mathbf{m}_R$.

Case 1: $\{\mathbf{m}_L, \mathbf{m}_R\} \cap P_1 \neq \emptyset$ **and** $\{\mathbf{m}_L, \mathbf{m}_R\} \cap P_2 \neq \emptyset$**.** Then, take $\mathbf{m}_1 \in \{\mathbf{m}_L, \mathbf{m}_R\}$ such that $\mathbf{m}_1 \in P_1$ and $\mathbf{m}_2 \in \{\mathbf{m}_L, \mathbf{m}_R\}$ such that $\mathbf{m}_2 \in P_2$. Then, let us show that $(\mathbf{m}_1, \mathbf{m}_2)$ is an NE. Suppose it is not and that w.l.o.g. there is a candidate $c_1' \in P_1$ such that $u_1(c_1', \mathbf{m}_2) > u_1(\mathbf{m}_1, \mathbf{m}_2)$. Observe that by the definition of a median position $u_1(\mathbf{m}_1, \mathbf{m}_2) \geqslant \frac{V(f)}{2}$. But also $u_2(\mathbf{m}_1, \mathbf{m}_2) \geqslant \frac{V(f)}{2}$, which implies that $u_2(\mathbf{m}_1, \mathbf{m}_2) \leqslant \frac{V(f)}{2}$. So, $u_1(\mathbf{m}_1, \mathbf{m}_2) = u_2(\mathbf{m}_1, \mathbf{m}_2) = \frac{V(f)}{2}$. It is then easy to see that, by the properties of median, for every value of c_1', we have that $u_1(c_1', \mathbf{m}_2) \leqslant \frac{V(f)}{2}$. Hence, c_1' is not a profitable deviation which contradicts the assumptions.

Case 2: $\{\mathbf{m}_L, \mathbf{m}_R\} \cap P_1 = \emptyset$ **or** $\{\mathbf{m}_L, \mathbf{m}_R\} \cap P_2 = \emptyset$**.** W.l.o.g. let $\{\mathbf{m}_L, \mathbf{m}_R\} \cap P_1 = \emptyset$. Notice that as P_1 is an interval, either (1) for all $p \in P_1$, $p < \mathbf{m}_L$, or (2) for all $p \in P_1$, $p > \mathbf{m}_R$, or (3) for all $p \in P_1$, $p \in [\mathbf{m}_L + 1, \mathbf{m}_R - 1]$.

Suppose that (1) is the case. We first consider the case in which for some $p \in P_2$, $p > \max(P_1)$. Then, take the smallest such $p \in P_2$ (denote it as s_2). Notice that $\max(P_1)$ is a best response to s_2 by Lemma 3.1. Also, if $\min(P_2) > \max(P_1)$, then s_2 is a best response by Lemma 3.1. Moreover, $u_2(\max(P_1), s_2) \geqslant \frac{V(f)}{2}$ if $\min(P_2) \leqslant \max(P_1)$, and $u_2(\max(P_1), c_2) \leqslant \frac{V(f)}{2}$ for every $c_2 \leqslant s_2$, as $\max(P_1) < \mathbf{m}_L$. So, $(\max(P_1), s_2)$ is an NE. If instead it is not true that for some $p \in P_2$, $p > \max(P_1)$, we either have that for all $p \in P_2$, $p < \max(P_1)$, or that $\max(P_2) = \max(P_1)$. If for all $p \in P_2$, $p < \max(P_1)$, take the smallest $p \in P_1$ such that $p > \max(P_2)$ (denote it s_1). Notice that $(s_1, \max(P_2)$ is an NE by the definition of a median position and Lemma 3.1, similarly to how we argued above. Also, if $\max(P_1) = \max(P_2)$, the profile $(\max(P_1), \max(P_2))$ is an NE by the definition of a median position, as $max(P_1) < \mathbf{m}_L$.

The reasoning for case (2) is symmetric, and the claim can be shown for case (3) along similar lines. ◻

Critically, the existence of Nash equilibria is no longer guaranteed in the general setting where parties are not necessarily given by intervals, which is arguably a more realistic representation of nomination processes.

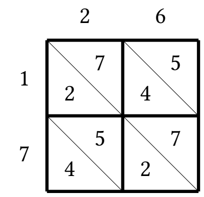


Figure 2: Normal form representation, with rows representing P_1's choices and column P_2's. The matrix entries encode the utilities as a function of k and f.

Proposition 3.3. *There are games with two parties and no Nash equilibria even when the distribution of voters is uniform.*

Proof. Take the [0,8] line of uniformly distributed voters and two parties P_1, P_2, with $P_1 = \{1, 7\}$, $P_2 = \{2, 6\}$. For simplicity, we model the instance as the normal form game in Figure 2. It is straightforward to check that this game has no NE. ◻

Notice that the existence of a Nash equilibrium is not guaranteed in our framework even in the simplest case of games with two parties and a uniform distribution of voters. This motivates the need for an algorithmic analysis of deciding the existence of Nash equilibria in our framework.

3.1.2 Computation. We now move to studying the complexity of checking whether a Nash equilibrium exists or not in a given two-player game. While it is straightforward to see that a polynomial-time algorithm exists for this problem (simply try all possible profiles and check if any is an NE), we will give a linear-time algorithm. Moreover, the procedure we present will also return an equilibrium profile whenever one exists. Before presenting it and proving its soundness, we show that, if elections with two parties admit an NE, they also admit an NE in which one of the parties selects a candidate close to the median position.

We start with the concept of *most central candidates.* Consider a party P_i on a line [0,k]. We call P_i's *most central candidates* the set $C_i = \{L_i^L, L_i^R, R_i^L, R_i^R\}$, where $L_i^L, L_i^R, R_i^L, R_i^R \in P_i$ and

$$L_i^L = \underset{\{p \in P_i \colon p \leqslant \mathbf{m}_L\}}{\operatorname{argmin}} |\mathbf{m}_L - p|$$

$$L_i^R = \underset{\{p \in P_i \colon p \in [\mathbf{m}_L + 1, \mathbf{m}_R - 1]\}}{\operatorname{argmin}} |p - \mathbf{m}_L|$$

$$R_i^L = \underset{\{p \in P_i \colon p \in [\mathbf{m}_L + 1, \mathbf{m}_R - 1]\}}{\operatorname{argmin}} |\mathbf{m}_R - p|$$

$$R_i^R = \underset{\{p \in P_i \colon p \geqslant \mathbf{m}_R\}}{\operatorname{argmin}} |p - \mathbf{m}_R|$$

In words, P_i's most central candidates are those that are closest to the left (i.e., L_i^L and L_i^R) and the right (i.e., R_i^L and R_i^R) median voter positions. Notice that while every party P has at least one most central position, the cardinality of C_i may vary. For instance, if the median voter position is unique and a party has a candidate that is exactly there, then the set of most central candidates of a party is a singleton. Figure 3 illustrates the central candidates of a party P_1 made up by candidates p_1, p_2, p_3 and p_4 on a line [0,6].

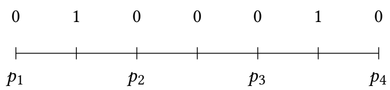


Figure 3: p_1, p_2, p_3 and p_4 from party P_1 are the L_1^L, L_1^R, R_1^L and R_1^R candidates respectively, as $\mathbf{m}_L = 1$ and $\mathbf{m}_R = 5$.

We show that if an NE exists, then there is an NE in which at least one of the two parties selects a most central candidate. This allows us to show that a strategy profile is an NE, if both parties select candidates located between median positions.

Lemma 3.4. *Let $[0,k]$ be a line, f be a distribution of voters, P_1, P_2 be parties, and (c_1, c_2) a strategy profile such that $c_1, c_2 \in [\boldsymbol{m}_L, \boldsymbol{m}_R]$. Then, (c_1, c_2) is an NE.*

Proof. Notice that by properties of median, for every profile (c_1', c_2') such that $c_1', c_2' \in [\mathbf{m}_L, \mathbf{m}_R]$, $u_1(c_1', c_2') = u_2(c_1', c_2') = \frac{V(f)}{2}$. So, we can assume w.l.o.g that $c_1 \leqslant c_2$. Let us show further that c_1 is a best response to c_2, i.e. that for every $c_1' \in P_1$, $u_1(c_1', c_2) \leqslant u_1(c_1, c_2)$. Indeed, if $c_1' < \mathbf{m}_L < c_1$, then the claim holds by Lemma 3.1. Also, if $c_1' \in [\mathbf{m}_L, \mathbf{m}_R]$, then $u_1(c_1', c_2) = u_2(c_1, c_2) = \frac{V(f)}{2}$. Finally, if $c_1' > \mathbf{m}_R$, then by properties of median $u_1(c_1', c_2) \leqslant \frac{V(f)}{2}$, so the claim follows. Analogously it can be shown that c_2 is a best response to c_1. □

We will further show that if there is an NE in given elections, then there is an NE in which at least one of the parties selects a most central candidate.

Lemma 3.5. *For every line $[0,k]$, distribution of voters f and parties P_1, P_2, if there is an NE (c_1, c_2), then there is an NE profile (c_1', c_2') such that $c_1' \in C_1$ or $c_2' \in C_2$.*

Proof. Take the line $[0,k]$, distribution of voters f and parties P_1, P_2. Suppose that there is an NE profile $\mathbf{c} = (c_1, c_2)$. Let us show that there is an NE profile (c_1', c_2') such that $c_1' \in C_1$ or $c_2' \in C_2$. W.l.o.g. assume that $c_1 \leqslant c_2$.

Case 1: $c_1 \in C_1$ or $c_2 \in C_2$. The claim follows immediately.

Case 2: $c_1 \notin C_1$ and $c_2 \notin C_2$. Consider the following cases.

(i) $c_1, c_2 \in [\mathbf{m}_L+1, \mathbf{m}_R-1]$: then, (L_1^R, R_2^L) is an NE by Lemma 3.4.

(ii) $c_1, c_2 < \mathbf{m}_L$: If $c_1 = c_2$, notice that $u_1(c_1, c_2) = u_2(c_1, c_2) = \frac{V(f)}{2}$. Further, consider the position L_2^L. We will show that (c_1, L_2^L) is a NE. Observe that it exists since $c_2 < \mathbf{m}_L$ and $c_2 \notin C_2$. Notice that as $\mathbf{c}$ is an NE, $u_2(c_1, L_2^L) \leqslant u_2(c_1, c_2)$. So, by properties of median $u_2(c_1, L_2^L) = u_2(c_1, c_2) = \frac{V(f)}{2}$. Therefore, as c_2 is a best response to c_1, so is L_2^L. To show that c_1 is a best response to L_2^L as well, assume for contradiction that $u_1(c_1', L_2^L) > u_1(c_1, L_2^L)$ for some $c_1' \in P_1$. Observe that $u_1(c_1, L_2^L) = \frac{V(f)}{2}$. As $c_1 < L_2^L \leq \mathbf{m}_L$, by the definition of median we then get that $c_1' > L_2^L$. But then, $u_1(c_1', L_2^L) \leqslant u_1(c_1', c_2)$ by Lemma 3.1. Hence, $u_1(c_1', c_2) > u_1(c_1, c_2)$, as $u_1(c_1, c_2) = \frac{V(f)}{2}$. This is however impossible as (c_1, c_2) is an NE.

Assume further that $c_1 \neq c_2$. Then, consider position L_1^L and the profile (L_1^L, c_2). L_1^L exists since $c_1 < \mathbf{m}_L$ and $c_1 \notin C_1$. Observe that, as c_1 is a best response to c_2, so is L_1^L. Indeed, if $L_1^L < c_2$, then L_1^L is also a best response by Lemma 3.1. Also notice that, as $c_1 < c_2 < \mathbf{m}_L$, we have $u_1(c_1, c_2) \leqslant \frac{V(f)}{2}$. So, as if $L_1^L \geqslant c_2$, then $u_1(L_1^L, c_2) \geqslant \frac{V(f)}{2}$, so L_1^L is a best response. We can further show that (L_1^L, c_2) is an NE analogously to the case in which $c_1 = c_2$.

(iii) $c_1, c_2 > \mathbf{m}_R$: reasoning is symmetric to (ii).

(iv) $c_1 < \mathbf{m}_L < c_2$: w.l.o.g. let $\mathbf{m}_L \in range_{c_1}(\mathbf{c})$. Hence, $u_1(\mathbf{c}) \geqslant u_2(\mathbf{c})$. Consider position L_2^R, if it exists, and R_2^R otherwise. W.l.o.g. we assume the that L_2^R exists and show that (c_1, L_2^R) is an NE. As (c_1, c_2) is an NE and by Lemma 3.1 $u_2(c_1, c_2) \leqslant u_2(c_1, L_2^R)$, we have that $u_2(c_1, c_2) = u_2(c_1, L_2^R) \leqslant \frac{V(f)}{2}$. So, $u_1(c_1, L_2^R) \geqslant \frac{V(f)}{2}$. Further, as c_2 is a best response to c_1, so is L_2^R by Lemma 3.1. To show that c_1 is a best response to L_2^R as well, assume for contradiction that there is a $c_1' \in P_1$ such that $u_1(c_1', L_2^R) > u_1(c_1, L_2^R)$. By the properties of median and Lemma 3.1, $u_1(c_1', L_2^R) \leq u_1(c_1, L_2^R)$, if $c_1' < c_1$ or $c_1' \geq L_2^R$. Hence, $c_1 < c_1' < L_2^R$. But then $u_1(c_1', c_2) > u_1(c_1, c_2)$ again by Lemma 3.1. This, however, leads to a contradiction, since (c_1, c_2) had been assumed to be an NE.

(v) $c_1 < \mathbf{m}_R < c_2$: reasoning is symmetric to (iv). □

We can now show that checking if an NE exists can be done in linear time. Given parties P_1, P_2 and a candidate $c_1 \in P_1$, $c_2 \in P_2$ is a best response to c_1 if and only if $c_2 \in \operatorname{argmax}_{p \in P_2} u_2(c_1, p)$. Notice that a profile (c_1, c_2) is an NE if and only if c_1 is a best response to c_2 and c_2 is a best response to c_1.

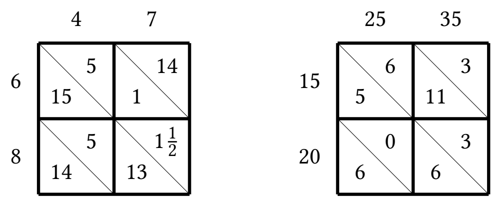
Lemma 3.6. *Take a line $[0,k]$ and $c_1 \in P_1$. Then, there is a best response c_2 to c_1 such that one of the following holds: (1) $c_2 = c_1$, (2) $c_1 < c_2$ and for every $c_2' \in P_2$ such that $c_2' > c_1$, $|c_2' - c_1| > |c_2 - c_1|$, or (3) $c_1 > c_2$ and for every $c_2' \in P_2$ such that $c_1 > c_2'$, $|c_1 - c_2'| > |c_1 - c_2|$.*

Thanks to Lemma 3.6 we have that given a choice of one of two parties, we only need to check three choices of the second to find its best response. This and the previous facts allows us for a simple procedure for checking if an NE exists and, if it does, constructing a profile witnessing it.

Theorem 3.7. *If only two parties are present, then checking if an NE exists is linear-time solvable.*

Proof. Consider the line $[0,k]$, distribution of voters f and two parties P_1, P_2. Then, compute the sets of most central candidates, C_1 and C_2. Notice that we can do it in linear time, having computed the median positions which is also possible in linear time. Also, by Lemma 3.5, we know that if there is an NE in the considered game, then there is a profile $\mathbf{c} = (c_1, c_2)$ which is an NE and $c_1 \in C_1$ or $c_2 \in C_2$.

Given an i in C_1, let $S(i)$ be the set of $c_2 \in P_2$ such that either (1) $c_2 = i$, (2) $i < c_2$ and for every $c_2' \in P_2$ s.t $c_2' > i$, $c_2' - i > c_2 - i$, or (3) $i > c_2$ and for every $c_2' \in P_2$ s.t $i > c_2$, $i - c_2' > i - c_2$. Notice that $\#S(i) \leqslant 3$ and that by Lemma 3.6 there is $c_2 \in \operatorname{argmax}_{p \in P_2} u_2(i, p)$ such that $c_2 \in S(i)$. Then, we can easily compute $\operatorname{argmax}_{p \in P_2} u_2(i, p)$. We now check symmetrically if for



(a) Game in the proof of Proposition 3.8 **(b) Game in Example 3.10**

Figure 4: In the games above, P_1 chooses rows, P_2 columns.

some $c_2 \in \text{argmax}_{p \in P_2} u_2(i, p)$, $i \in \text{argmax}_{p \in P_1} u_1(p, c_2)$. If yes, we found an NE. Repeat this procedure for all i in $C_1 \cup C_2$. As $\#C_1 + \#C_2$ is bounded by 8, we can find an NE, if it exists, in linear time. So, our linear time algorithm computes the sets of most central candidates and subsequently checks, for each member of these sets, whether they can be extended to a NE profile. □

3.2 Games with many parties

Let us observe that there are instances without NE for games with any number of parties present, even when the setting is restricted to a uniform distribution of voters. This constitutes a major difference between the studied framework and the classical Hotelling-Downs model, where the existence of NE depends on the number of agents (see [12]). In that framework, for instance, an NE equilibrium exists with four agents competing, but not with three.

Proposition 3.8. *For every $n \geqslant 2$, there is a game with a uniform distribution of voters and n parties, that has no NE.*

Proof. Suppose that $n > 2$. Then, take the line [0,20] and a set of parties $P = \{P_1, P_2, \ldots, P_n\}$. Then, let $P_1 = \{4, 7\}$, $P_2 = \{6, 8\}$ and for every P_i such that $i > 2$ and $i \leqslant n$, $P_i = \{5\}$. Note that in all strategy profiles parties other than P_1 and P_2 select 5. Let us then consider utilities of parties P_1 and P_2 in all strategy profiles. The utilities of these parties are shown in the Figure 4 (a). It is routine to check that this game has no NE. Finally notice that by Proposition 3.3 there are instances without NE also for the two party case. □

Interestingly, there are cases without NEs, even if there is a party that is guaranteed to get the majority of votes. This would be impossible, however, if parties were only concerned with winning the elections rather than with attracting voters.

Example 3.9. Consider the elections with voters uniformly distributed on the line $[0, 100]$. Also, consider the parties $P_1 = \{70\}$, $P_2 = \{73, 89\}$, $P_3 = \{88, 90\}$, $P_4 = \{88, 90\}$, $P_5 = \{75\}$, $P_6 = \{100\}$. Notice that, by construction, under any strategy profile in this game, P_1 receives at least 70 out of 101 votes. Let us show now that there is no NE in this game. As actions of P_1, P_5 and P_6 are fixed, we focus on the utilities of parties P_2, P_3 and P_4. Table 1 gives the utilities of these parties in all strategy profiles. It is straightforward to check that there is no NE in this game.

	(73,88,88)	(89,88,88)	(89,88,90)	(73,88,90)	(89,90,88)	(73,90,88)	(73,90,90)	(89,90,90)
u_2	$2\frac{1}{2}$	6	1	$2\frac{1}{2}$	1	$2\frac{1}{2}$	$2\frac{1}{2}$	$7\frac{1}{2}$
u_3	$6\frac{1}{4}$	$3\frac{1}{2}$	7	$7\frac{1}{2}$	$5\frac{1}{2}$	6	$6\frac{1}{4}$	$2\frac{3}{4}$
u_4	$6\frac{1}{4}$	$3\frac{1}{2}$	$5\frac{1}{2}$	6	7	$7\frac{1}{2}$	$6\frac{1}{4}$	$2\frac{3}{4}$

Table 1: Utility of parties P_2, P_3, P_4 in all strategy profiles. Given a profile (c_2, c_3, c_4), c_2 is the choice of party P_2, c_3 is the choice of P_3 and c_4 is the choice of P_4.

Moreover, it turns out that there are instances without NE even if parties have the sector structure.

Observation 1. *There are games where P has the sector structure but which have no Nash equilibria.*

Example 3.10. Take a line $[0, 36]$ and distribution of voters such that $f(11) = 5$, $f(21) = 6$, $f(35) = 3$ and for every $i \in [0, 36] \setminus \{11, 21, 35\}$, $f(i) = 0$. Also, let $P = \{P_1, P_2, P_3, P_4\}$ with $P_1 = \{15, 20\}$, $P_2 = \{25, 35\}$, $P_3 = \{5\}$ and $P_4 = \{36\}$. Notice that P has the sector structure. The strategic game between P_1 and P_1 is depicted in Figure 4(b). It is routine to check that this game has no NE, as P_3 and P_4 have only one action available.

Interestingly, this observation does not hold if the distribution of voters is uniform. This is due to an interesting observation that if a representative of a party is located between two neighbours, selecting an alternative which is also located between these neighbours is not profitable. Given a strategy profile $\mathbf{c}$ and a uniform distribution of voters, a candidate $c_i \in P_i$ in $\mathbf{c}$ and $c'_i \in P_i$, we say that c_i and c'_i *have the same neighbourhood* if the following conditions hold: (i) both $L(c_i)$ and $R(c_i)$ exist; (ii) $L(c_i) < c'_i < R(c_i)$, and (iii) c_i and c'_i do not share their position with any other party's candidates in $\mathbf{c}$. I.e., $c_i \neq c_j$ and $c'_i \neq c_j$ for all c_j in $\mathbf{c}$ with $j \neq i$. Notice that this notion is only defined for candidates which are not leftmost or rightmost in a strategy profile.

Example 3.11. Consider the game in Figure 5. This is an example of a game where p_2^1 and p_2^2 have the same neighbourhood.

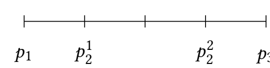


Figure 5: A game with the line of uniformly distributed voters of length 4 and parties $P_1 = \{0\}$, $P_2 = \{1, 3\}$ and $P_3 = \{4\}$.

Let us state now the following lemma.

Lemma 3.12. *When voters are uniformly distributed, then, for every strategy profile $\mathbf{c}$ and a party P_i, with $c_i, c'_i \in P_i$, we have that if c_i and c'_i have the same neighbourhood, then $u_i(\mathbf{c}) = u_i(c'_i, \mathbf{c}_{-i})$.*

Proof. Take a strategy profile $\mathbf{c}$, party P_i and a candidate $c'_i \in P_i$ such that $c'_i > L(c_i)$ and $c'_i < R(c_i)$. From the definition of utilities, we get by calculation that $u_i(\mathbf{c}) = u_i(c'_i, \mathbf{c}_{-i}) = \frac{R(c_i) - L(c_i)}{2}$. □

This leads us to the existence of NE in games with parties with the sector structure and uniform distribution of voters.

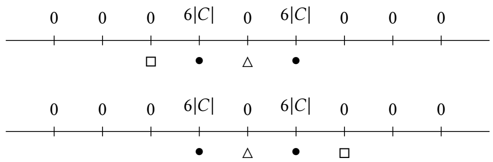


Figure 6: Variable segments $[9i, 9i + 8]$ (above) and $[9j, 9j + 8]$ (below) for a variables x_i and x_j such that $\neg x_i \in C_k$ and $x_j \in C_k$. Variable parties P_{x_i} and P_{x_j} are indicated by the bullets in respectively the top and bottom segment. Choosing the left candidate corresponds to setting variable x_i, respectively x_j, to true, and choosing the right candidate corresponds to setting variable x_i, respectively x_j, to false. The clause party P_{C_k} has candidates at the positions indicated by the boxes. (If neither x_m nor $\neg x_m$ occurs in C_k, party P_{C_k} has no candidates in segment $[9m, 9m+8]$). The triangles denotes the solitary candidates of parties $P_{x'_i}$ and $P_{x'_j}$.

Proposition 3.13. *For every line $[0, k]$, distribution of voters f and a set of parties $P = \{P_1, \ldots, P_n\}$, if f is uniform and P has the sector structure, then there exists an NE.*

Proof. Take a line of k uniformly distributed voters and set of parties $P = \{P_1, \ldots, P_n\}$ with the sector structure. Consider any strategy profile $\mathbf{c}$ such that $c_1 = \max(P_1)$ and $c_n = \min(P_n)$. Notice that, as c_1 is the leftmost position in $\mathbf{c}$ and c_n is the rightmost position in $\mathbf{c}$, by Lemma 3.1, P_1 and P_n cannot improve their utilities unilaterally. But also, as P has the sector structure, for every other party P_i and a pair $c_i, c'_i \in P_i$, c_i and c'_i have the same neighbourhood. So, by Lemma 3.12, no other party $P_i \in P$ can improve their utility. Hence, $\mathbf{c}$ is an NE. □

Given that there are instances without NE even in elections with a relatively simple structure, it is natural to study the complexity of checking whether there is an NE in a given game. While it is not difficult to see that the problem is solvable in polynomial time when the number of parties is bounded by a constant, we find that the general case is NP-complete.

Theorem 3.14. *Given line $[0, k]$, distribution of voters f, and a set of parties $P = \{P_1, \ldots, P_n\}$, deciding whether the game on $[0, k]$, P, and f has a Nash equilibrium is NP-complete. The problem remains NP-hard even if the party size is bounded by a constant not smaller than 5.*

Proof. First notice that the problem is in NP: given a strategy profile $\mathbf{c} = (c_1, \ldots, c_n)$, we can check in polynomial time whether $u_i(\mathbf{c}) \geq u_i(\mathbf{c}_{-i}, c'_i)$ for every party P_i and strategy $c'_i \in P_i$.

We prove NP-hardness by a reduction from the satisfiability problem 3SAT. Let an instance φ of 3SAT be given by a set $C = \{C_0, \ldots, C_{|C|-1}\}$ of clauses, where, for each $0 \leq k < |C|$, clause C_k is given by a set of three distinct literals $\{\ell_0^k, \ell_1^k, \ell_2^k\}$ over a set of variables X. Let C'_k denote a copy of C_k and x'_i a copy of x_i.

We now construct the game on the **line** $[0, 9(|X| + |C|) - 1]$, which we can conveniently think of as being composed of $|X| + |C|$ segments of length 9. Figures 6 and 7 illustrate our construction.

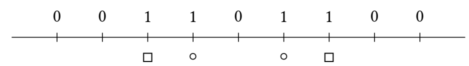


Figure 7: Clause segment $[9(|X|+k), 9(|X|+k)+8]$ for clause C_k. Party P_{C_k} has candidates at the locations indicated by the boxes, but has no candidates in any other clause segments. Party $P_{C'_k}$ has two candidates at the locations indicated by the circles.

Thus, we have for each variable x_i ($0 \leq i < |X|$) a *variable segment* $[9i, 9i + 8]$ and for each clause C_k ($0 \leq k < |C|$) a *clause segment* $[9(|X| + k), 9(|X| + k) + 8]$. Thus all positions $n < 9|X|$ lie in variable segment whereas all positions $n \geq 9|X|$ lie in a clause segment. Now define the **distribution function** f such that, for every $0 \leq n < 9(|X| + |C|)$,

$$f(n) = \begin{cases} 6|C| & \text{if} \quad n < 9|X| \quad \text{and} \quad n \bmod 9 \in \{3, 5\}, \\ 1 & \text{if} \quad n \geq 9|X| \quad \text{and} \quad n \bmod 9 \in \{2, 3, 5, 6\}, \\ 0 & \text{otherwise.} \end{cases}$$

As **parties** we have for every variable x_i ($0 \leq i < |X|$), and every clause C_k and its copy C'_k ($0 \leq k < |C|$),

$$\begin{aligned} P_{x_i} &= \{9i + 3, 9i + 5\} \\ P_{x'_i} &= \{9i + 4\} \\ P_{C_k} &= \{9i + 6 \colon x_i \in C_k\} \cup \{9i + 2 \colon \neg x_i \in C_k\} \cup \\ &\quad \{9(|X| + k) + 2, 9(|X| + k) + 6\} \\ P_{C'_k} &= \{9(|X| + k) + 3, 9(|X| + k) + 5\}. \end{aligned}$$

Importantly, observe that the distribution function has been chosen in such a way that a party can attract voters from the segment within which its representative is positioned. Also observe that the size of each party P_{x_i} and each party $P_{C'_k}$ is 2, whereas the size of each party P_{C_k} is 5.

We prove that this game has an NE if and only if φ is satisfiable.

First assume that φ is satisfiable and let $\alpha : X \longrightarrow \{\top, \bot\}$ be a satisfying assignment, that is, α satisfies at least one literal in each clause. Given assignment α, we consider profiles $\mathbf{c} = (c_{x_0}, \ldots, c_{C'_K})$, which we will refer to as **proto-equilibria**, that are such that for every variable x_i with $0 \leq i < |X|$,

$$c_{x_i} = \begin{cases} 9i + 3 & \text{if } \alpha(x_i) = \top, \\ 9i + 5 & \text{if } \alpha(x_i) = \bot. \end{cases}$$

Moreover, to qualify as a proto-equilibrium, for every clause C_k there has to be some literal ℓ in C_k that is satisfied by α such that

$$c_{C_k} = \begin{cases} 9j + 6 & \text{if } \alpha(x_j) = \top \text{ and } \ell = x_j, \\ 9j + 2 & \text{if } \alpha(x_j) = \bot \text{ and } \ell = \neg x_j. \end{cases}$$

We furthermore require $c_{C'_k} = 9(|X| + k) + 3$ for all $0 \leq k < |C|$. Obviously, $c_{x'_i} = 9i + 4$ for $0 \leq i < |X|$.

By means of the following potential argument, we now show that among the proto-equilibria for α, there must be at least one NE. To this end, let $\lambda_i^{\mathbf{c}}$, for each proto-NE $\mathbf{c}$ for α, and each $0 \leq i < |X|$, be the number of clause players that choose their representative from the variable segment $[9i, 9i+8]$ under $\mathbf{c}$, that is,

$$\lambda_i^{\mathbf{c}} = |\{C_k \in C \colon c_{C_k} \in [9i, 9i+8]\}|.$$

Let $\lambda^{\mathbf{c}} = (\lambda_{i_0}^{\mathbf{c}}, \ldots, \lambda_{i_{|X|-1}}^{\mathbf{c}})$ be a sequence of the values $\lambda_0^{\mathbf{c}}, \ldots, \lambda_{|X-1|}^{\mathbf{c}}$ ordered in non-decreasing order. We argue that any proto-NE $\mathbf{c}$ for which the sequence $\lambda^{\mathbf{c}}$ is *lexicographically maximal* is also an NE.[1]

To this end, let $\mathbf{c}^*$ be a proto-equilibrium for which $\lambda^{\mathbf{c}^*}$ is lexicographically maximal. Then, for every variable party P_{x_i} it holds that $u_{x_i}(\mathbf{c}^*_{-x_i}, 9i+3) = u_{x_i}(\mathbf{c}^*_{-x_i}, 9i+5) = 6|C|$, and it follows that P_{x_i} does not want to deviate from $\mathbf{c}^*$. Obviously, the singleton parties $P_{x_i'}$ $(0 \le i < |X|)$ cannot profitably deviate from $\mathbf{c}^*$ either.

Moreover, for every party $P_{C_k'}$ we have,

$$u_{C_k'}(\mathbf{c}^*_{-C_k'}, 9(|X|+k)+3) = u_{C_k'}(\mathbf{c}^*_{-C_k'}, 9(|X|+k)+5) = 4,$$

because $\mathbf{c}^*_{C_k} \notin [9(|X|+k), 9(|X|+k)+9]$, and, therefore, $P_{C_k'}$ does not want to deviate from $\mathbf{c}^*$ either.

Now, consider an arbitrary clause party P_{C_k}. As $\mathbf{c}^*$ is a proto-equilibrium, we find that there is some $0 \le i < |X|$ such that $c_{C_k} = 9i+2$, if $c_{x_i} = 9i+5$, and $c_{C_k} = 9i+6$, if $c_{x_i} = 9i+3$. In either case, $u_{C_k}(\mathbf{c}^*) = \frac{3|C|}{\lambda_i^{\mathbf{c}^*}}$. As $\lambda_i^{\mathbf{c}^*} \le |C|$, it follows that $u_{C_k}(\mathbf{c}^*) \ge 3$. Observe that, if P_{C_k} were to deviate and choose its representative in another variable segment $[9j, 9j+8]$ such that either $c'_{C_k} = 9j+2$ and $\mathbf{c}^*_{x_j} = 9j+3$, or $c'_{C_k} = 9j+6$ and $\mathbf{c}^*_{x_j} = 9j+5$, then $u_{C_k}(\mathbf{c}^*) = 0$. Moreover, if P_{C_k} were to deviate to a position in clause segment $[9(|X|+k), 9(|X|+k)+8]$, then both $u_{C_k}(\mathbf{c}^*_{-C_k}, 9(|X|+k)+2) \le 2$ and $u_{C_k}(\mathbf{c}^*_{-C_k}, 9(|X|+k)+6) \le 2$; again party P_{C_k} does not want to deviate from $\mathbf{c}^*$. Finally, assume for contradiction P_{C_k} would profit from deviating to a position c'_{C_k} in a variable segment $[9j, 9j+8]$ with $0 \le j \le |X|$ different from $[9i, 9i+8]$ such that either $c'_{C_k} = 9j+2$ and $\mathbf{c}^*_{x_j} = 9j+5$, or $c'_{C_k} = 9j+6$ and $\mathbf{c}^*_{x_j} = 9j+3$. Let $\mathbf{c}^{**} = (\mathbf{c}^*_{C_k}, c'_{C_k})$. Notice that $\mathbf{c}^{**}$ is a proto-equilibrium. Moreover, $u_{C_k}(\mathbf{c}^{**}) > u_{C_k}(\mathbf{c}^*)$, that is, $\frac{3|C|}{\lambda_j^{\mathbf{c}^{**}}} > \frac{3|C|}{\lambda_j^{\mathbf{c}^*}}$. Hence, $\lambda_j^{\mathbf{c}^{**}} < \lambda_i^{\mathbf{c}^*}$. Observing that $\lambda_i^{\mathbf{c}^{*}*} = \lambda_i^{\mathbf{c}^*} - 1$ and $\lambda_j^{\mathbf{c}^{*}*} = \lambda_j^{\mathbf{c}^*} + 1$, we find that $\lambda_j^{\mathbf{c}^*} < \lambda_i^{\mathbf{c}^*}$, $\lambda_j^{\mathbf{c}^{**}} \le \lambda_i^{\mathbf{c}^{**}}$, and $\lambda_k^{\mathbf{c}^*} = \lambda_k^{\mathbf{c}^{*}*}$ for all $k \ne i, j$. It follows that $\lambda^{c^{**}}$ is lexicographically greater than λ^{c^*}, a contradiction.

For the opposite direction, assume that φ is not satisfiable. Consider an arbitrary profile $\mathbf{c} = (c_{x_0}, \ldots, c_{C_K'})$, and assume for contradiction that $\mathbf{c}$ is an NE. Let $\alpha_{\mathbf{c}}$ be the assignment such that for every $0 \le i < |X|$,

$$\alpha_{\mathbf{c}}(x_i) = \begin{cases} \top & \text{if } c_{x_i} = 9i+3, \\ \bot & \text{if } c_{x_i} = 9i+5. \end{cases}$$

Then, there is some clause C_k $(0 \le k < |C|)$ such that $\alpha_{\mathbf{c}}$ evaluates every literal in C_k to false. Accordingly, if c_{C_k} is in a variable segment $[9i, 9i+8]$ with $0 \le i < |X|$, then either both $c_{C_k} = 9i+2$ and $c_{x_i} = 9i+3$, or both $c_{C_k} = 9i+6$ and $c_{x_i} = 9i+5$. In either case $u_{C_k}(\mathbf{c}) = 0$. Now, consider $d_{C_k} = 9(|X|+k)+2$. Then, $u_{C_k}(\mathbf{c}_{-C_k}, d_{C_k}) \ge 1$. Hence, $\mathbf{c}$ is not an NE, a contradiction.

To conclude, assume c_{C_k} is in the segment $[9(|X|+k), 9(|X|+k)+8]$. Observe that, if $c_{C_k} = 9(|X|+k)+2$ and $c_{C_k'} = 9(|X|+k)+3$, party P_{C_k} would deviate to $d_{C_k} = 9(|X|+k)+6$. If $c_{C_k} = 9(|X|+k)+6$ and $c_{C_k'} = 9(|X|+k)+3$, party $P_{C_k'}$ deviates to $d_{C_k'} = 9(|X|+k)+5$, and, if $c_{C_k} = 9(|X|+k)+6$ and $c_{C_k'} = 9(|X|+k)+5$, party P_{C_k} deviates to $d_{C_k} = 9(|X|+k)+2$. Finally, if $c_{C_k'} = 9(|X|+k)+5$ and $c_{C_k} = 9(|X|+k)+2$, then party $P_{C_k'}$ deviates to $d_{C_k'} = 9(|X|+k)+3$. It follows that $\mathbf{c}$ is not an NE, a contradiction. We may conclude that the game does not allow for any NE. ◻

4 DISCUSSION AND CONCLUSIONS

We studied a variation of the Hotelling-Downs model where political parties compete for voters located on a left-to-right political spectrum, by selecting a representative within their pool of potential nominees, who in turn have fixed political stances and attract the closer voters. Even if restricting ourselves to a finite number of positions, the framework can be directly generalised in various ways, for example to scenarios where finitely many voters are placed on real intervals, preserving utilities.

Our results indicate that predicting nominee selection can be a computationally hard problem. In particular, we have shown games without NE even with two parties (Proposition 3.3). Also, we have established that NE computation is NP-complete with more than two parties (Theorem 3.14). Conversely, computing NE becomes easy in two-party systems (Theorem 3.7).

Our contribution suggests a number of directions for future research. In particular, it is worth studying solution concepts other than NE, such as dominant strategy equilibria (DSE). This concept is especially interesting from the perspective of predicting parties' actions. Finding a dominant strategy for a given party strongly suggests their choice regardless of other parties' selections. We believe that in the setting studied in the current paper checking the existence of a DSE is algorithmically easier than verifying the existence of an NE. Furthermore, even though the NP-hardness of NE existence shows the difficulty of this problem in the general case, it is natural to study classes of elections in which this is tractable.

Establishing the parametrised complexity of checking the existence of NE is an important follow up. We saw that if the number of parties is fixed at a constant, then the problem can actually be solved in polynomial time. Indeed, if the line is given by $[0, k]$ the number of parties is n, then one can enumerate the at most k^n possible strategies, and for each one of them, check whether it is a NE in polynomial time. This corresponds to an XP algorithm parameterised by the number of parties. However, obtaining a fixed-parameter algorithm parameterised by the number of parties, i.e., an $f(n)k^c$ time algorithm, where c is a constant independent of k and n, appears to be a challenging problem.

Another interesting direction involves the modelling assumptions, starting with the role of information, e.g., taking into account the uncertainty of voters participating in the election. In this paper, we did not take into account the possibility of strategic behaviour from the side of voters, either, who might misrepresent their position on the political spectrum to get a better outcome in the end. Also, in the context of voting, it is certainly interesting to study parties that are concerned with winning the elections under various voting rules, rather than accumulating utility only. Likewise, we assumed that each party only selects one candidate, leaving open the problem of parties choosing sets of candidates, instead.

[1]Here we understand that the lexicographic order with respect to the normal relation $\le$ on the integers, and where, for instance, $(0, 1, 3, 4, 7, 9)$ is lexicographically greater than $(0, 1, 2, 7, 8, 8)$.

REFERENCES

[1] Hee-Kap Ahn, Siu-Wing Cheng, Otfried Cheong, Mordecai Golin, and Rene Van Oostrum. 2004. Competitive facility location: the Voronoi game. *Theoretical Computer Science* 310 (2004), 457–467.

[2] Sayan Bandyapadhyay, Aritra Banik, Sandip Das, and Hirak Sarkar. 2015. Voronoi game on graphs. *Theoretical Computer Science* 562 (2015), 270–282.

[3] Vittorio Bilò, Michele Flammini, and Cosimo Vinci. 2020. The Quality of Content Publishing in the Digital Era. In *24th European Conference on Artificial Intelligence (ECAI)*, Vol. 325. 35–42.

[4] Meena Boppana, Rani Hod, Michael Mitzenmacher, and Tom Morgan. [n.d.]. Voronoi Choice Games. In *43rd International Colloquium on Automata, Languages, and Programming (ICALP 2016)*. 23:1–23:13.

[5] Allan Borodin, Omer Lev, Nisarg Shah, and Tyrone Strangway. 2019. Primarily about Primaries. In *Proceedings of the 33rd AAAI Conference on Artificial Intelligence (AAAI)*. 1804–1811.

[6] Markus Brill and Vincent Conitzer. 2015. Strategic voting and strategic candidacy. In *Twenty-Ninth AAAI Conference on Artificial Intelligence*. 819–826.

[7] Sandro Brusco, Marcin Dziubiński, and Jaideep Roy. 2012. The Hotelling–Downs model with runoff voting. *Games and Economic Behavior* 74, 2 (2012), 447–469.

[8] William Cross and André Blais. 2012. Who selects the party leader? *Party Politics* 18, 2 (2012), 127–150.

[9] Anthony Downs. 1957. An economic theory of democracy. (1957).

[10] Christoph Dürr and Nguyen Kim Thang. 2007. Nash Equilibria in Voronoi Games on Graphs. In *Algorithms – ESA 2007*, Lars Arge, Michael Hoffmann, and Emo Welzl (Eds.).

[11] Bhaskar Dutta, Matthew O Jackson, and Michel Le Breton. 2001. Strategic candidacy and voting procedures. *Econometrica* 69, 4 (2001), 1013–1037.

[12] B Curtis Eaton and Richard G Lipsey. 1975. The principle of minimum differentiation reconsidered: Some new developments in the theory of spatial competition. *The Review of Economic Studies* 42, 1 (1975), 27–49.

[13] Horst A. Eiselt. 2011. Equilibria in competitive location models. In *Foundations of location analysis*, Horst A Eiselt and Vladimir Marianov (Eds.). Springer, 139–162.

[14] H. A. Eiselt, Gilbert Laporte, and Jacques-François Thisse. 1993. Competitive Location Models: A Framework and Bibliography. *Transportation Science* 27, 1 (1993), 44–54.

[15] Edith Elkind, Evangelos Markakis, Svetlana Obraztsova, and Piotr Skowron. 2015. Equilibria of plurality voting: Lazy and truth-biased voters. In *International Symposium on Algorithmic Game Theory*. Springer, 110–122.

[16] Hülya Eraslan and Andrew McLennan. 2004. Strategic candidacy for multivalued voting procedures. *Journal of Economic Theory* 117, 1 (2004), 29–54.

[17] Piotr Faliszewski, Laurent Gourvès, Jérôme Lang, Julien Lesca, and Jérôme Monnot. 2016. How Hard is it for a Party to Nominate an Election Winner?. In *Proceedings of the Twenty-Fifth International Joint Conference on Artificial Intelligence, (IJCAI)*. 257–263.

[18] Michal Feldman, Amos Fiat, and Iddan Golomb. 2016. On voting and facility location. In *Proceedings of the 2016 ACM Conference on Economics and Computation, (EC)*. 269–286.

[19] Michal Feldman, Amos Fiat, and Svetlana Obraztsova. 2016. Variations on the Hotelling-Downs model. In *Thirtieth AAAI Conference on Artificial Intelligence*. 496–501.

[20] Gaëtan Fournier. 2019. General distribution of consumers in pure Hotelling games. *International Journal of Game Theory* 48, 1 (2019), 33–59.

[21] Harold Hotelling. 1929. Stability in Competition. *The Economic Journal* 39, 153 (1929), 41–57.

[22] Marios Mavronicolas, Burkhard Monien, Vicky G. Papadopoulou, and Florian Schoppmann. 2008. Voronoi Games on Cycle Graphs. In *Mathematical Foundations of Computer Science 2008*. 503–514.

[23] Reshef Meir. 2018. *Strategic Voting*. Morgan & Claypool.

[24] Hervé Moulin. 1980. On strategy-proofness and single peakedness. *Public Choice* 35, 4 (1980), 437–455.

[25] Matías Núñez and Marco Scarsini. 2016. Competing over a finite number of locations. *Economic Theory Bulletin* 4, 2 (2016), 125–136.

[26] Matías Núñez and Marco Scarsini. 2017. Large spatial competition. In *Spatial Interaction Models*. Springer, 225–246.

[27] Svetlana Obraztsova, Edith Elkind, Maria Polukarov, and Zinovi Rabinovich. 2015. Strategic candidacy games with lazy candidates. In *Proceedings of the Twenty-Fourth International Joint Conference on Artificial Intelligence, (IJCAI)*. 610–616.

[28] Itay Sabato, Svetlana Obraztsova, Zinovi Rabinovich, and Jeffrey S Rosenschein. 2017. Real candidacy games: A new model for strategic candidacy. In *Proceedings of the 16th Conference on Autonomous Agents and MultiAgent Systems, (AAMAS)*. 867–875.

[29] Abhijit Sengupta and Kunal Sengupta. 2008. A hotelling–downs model of electoral competition with the option to quit. *Games and Economic Behavior* 62, 2 (2008), 661–674.

[30] Donald E. Stokes. 1963. Spatial Models of Party Competition. *American Political Science Review* 57, 2 (1963), 368–377.

[31] Emiel Christiaan Henrik Veendorp and Anjum Majeed. 1995. Differentiation in a two-dimensional market. *Regional Science and Urban Economics* 25, 1 (1995), 75–83.

[32] Bram Wauters. 2010. Explaining participation in intra-party elections: evidence from Belgian political parties. *Party Politics* 16, 2 (2010), 237–259.