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Covariant description of canonical formalism in geometrical theories

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Quantum field theories are usually studied either by means of path integrals or by means of canonical quantization. Path integral quantization has the great virtue of explicitly maintaining all relevant symmetries, such as Poincaré invariance. The canonical approach is usually interpreted as an approach that ruins Poincaré invariance from the beginning through an explicit choice of a 'time' coordinate. This is not necessarily so, however. The essence of the canonical formalism can be developed in a way that manifestly preserves all relevant symmetries, including Poincaré invariance (Witten, 1986; Zuckerman, 1986). The purpose of the present paper is to carry this out in the case of non-abelian gauge theories and general relativity.

In the canonical formalism of a theory with N degrees of freedom, one usually introduces coordinates and momenta p^i and q^j, $i,j=1,\dots,N$. One then defines the two-form

$$\omega = \mathrm{d}p^i \wedge \mathrm{d}q^i. \tag{1}$$

It is convenient to combine the p^i and q^j in a variable Q^I, $I=1,\dots,2N$, with $Q^i=p^i$ for $i\leqslant N$ and $Q^i=q^{i-N}$ for $i>N$. One can think of ω as an antisymmetric $2N\times 2N$ matrix ω_{IJ} whose non-zero matrix elements are $\omega_{i,i+N}=-\omega_{i+N,i}=1$. This matrix is invertible; we will denote the inverse matrix as ω^{IJ}. One defines the Poisson bracket of any two functions $A(Q^I)$ and $B(Q^I)$ by

$$[A,B]=\omega^{IJ}\frac{\partial A}{\partial Q^I}\frac{\partial B}{\partial Q^J}. \tag{2}$$

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Given the form of (1), this is easily seen to coincide with the usual definitions of Poisson brackets. The advantage of the definition in (2) is that, as is well known (see e.g. Abraham and Marsden, 1967), the essential features of ω can be described in an invariant way. Let Z be the phase space of the theory under discussion, that is, the space on which the ps and qs are coordinates. If one interprets ω as a two-form on Z, then it is clearly a closed two-form,

$$d\omega = 0, \tag{3}$$

since its components are constant in the coordinate system used in (1). What is more, we have already noted that the matrix ω is invertible. The converse to this is as follows. Let ω be any two-form on a manifold Z (which for us will be the phase space of a physical theory). Suppose that ω is closed (obeys (3)), and is non-degenerate in the sense that at each point $z \in Z$, the matrix $\omega_{ij}(z)$ is invertible. Then it is a classical theorem that locally one can introduce coordinates on Z to put ω in the standard form (1). (This is not true globally in theories with interesting geometrical content.) A non-degenerate closed two-form is called a symplectic structure. Thus, to describe the canonical formalism of a theory it is not at all necessary to find or choose ps and qs; the essence of the matter is to describe a symplectic structure on the classical phase space.

Clearly, the notion of a 'symplectic structure on phase space' is a more intrinsic concept than the idea of choosing ps and qs. However, at first sight it might appear that the very concept of phase space is a non-covariant concept, tied to a non-covariant, Hamiltonian description. This is not really so. The whole idea, classically, of picking ps and qs is that the initial values of the ps and qs determine a solution of the classical equations. More precisely, classical solutions of any given physical theory, in any given coordinate system, are in one-to-one correspondence with the values of the ps and qs at time zero. This simple consideration leads us to a manifestly covariant definition of what we mean by classical phase space: *in a given physical theory, classical phase space is the space of solutions of the classical equations.* We can always, if we wish, pick a coordinate system and identify the classical solutions with the initial data in that coordinate system, but there is no necessity to make such a non-covariant choice.

Given a relativistic field theory, such as the scalar field theory with Lagrangian

$$L = \int_M \left(\frac{1}{2} \partial_\alpha \phi \, \partial^\alpha \phi - V(\phi) \right), \tag{4}$$

(here M is spacetime, and ϕ is a scalar field), how do we go about finding a covariant description of a symplectic structure on the space, Z, of classical solutions? A point in Z is a solution of the classical equations

$$0=\Delta\phi - V'(\phi), \tag{5}$$

with $\Delta = -\partial_\alpha \partial^\alpha$ being the standard Laplacian. In order to construct a symplectic structure on Z, we will need to discuss functions, tangent vectors, differential forms, and exterior derivatives on Z.

Functions on Z are, of course, the easiest to discuss. Among the most important are the following. Let $x \in M$ be a spacetime point. If ϕ is a solution of (5), its value $\phi(x)$ at the point x is a real number. The mapping from the *function* ϕ to the *number* $\phi(x)$ is a real valued function on Z. We will denote this function by $\phi(x)$.

Now we will consider tangent vectors. At a point $p \in Z$, corresponding to a solution of (5), a tangent vector would be a small displacement in ϕ which preserves (5). Thus, a tangent vector is the same as a solution of the linearized equations which we obtain by expanding around a solution of (5). Requiring that $\tilde{\phi} = \phi + \delta\phi$ should obey (5) to lowest order in $\delta\phi$, we find

$$0=\Delta\,\delta\phi - V''(\phi)\,\delta\phi. \tag{6}$$

A solution of (6) is a tangent vector at the point in phase space corresponding to the solution ϕ. The tangent space is the vector space T of solutions of (6).

Now, how do we describe one-forms on Z? The space of one-forms is, of course, the dual of the tangent space which we have just described; it is the space of linear functionals on T. Most important for our purposes are certain one-forms on Z which we will now describe. Let $x \in M$ be a spacetime point. For every solution $\delta\phi$ of (6), its value $\delta\phi(x)$ at the point x is, of course, a number. The transformation from the function $\delta\phi$ to the number $\delta\phi(x)$ is a one-form on phase space which we will call $\delta\phi(x)$. More generally, we can make k-forms as wedge functions of the one-forms $\delta\phi(x)$:

$$A = \int \mathrm{d}x_1 \ldots \mathrm{d}x_n\, \alpha_{x_1 \ldots x_n}(\phi)\, \delta\phi(x_1)\, \delta\phi(x_2) \ldots \delta\phi(x_n). \tag{7}$$

Here, for each n-tuple of spacetime points $x_1 \ldots x_n$, $\alpha_{x_1 \ldots x_n}(\phi)$ is an arbitrary zero form – an arbitrary real valued function on Z. On the right-hand side of (7), the product of the $\delta\phi(x_i)$ is understood to be a wedge product. In particular, as the wedge product of one-forms is anticommuting, we interpret the $\delta\phi(x)$ as anticommuting objects:

$$\delta\phi(x)\,\delta\phi(y) = -\delta\phi(y)\,\delta\phi(x). \tag{8}$$

The general n-form on Z can be expanded as in (7); this expansion, however, is not unique, since the $\delta\phi(x)$ are not linearly independent, being subject to (6).

Finally, we need an exterior derivative on Z, which we will call δ and which must map k forms to $(k+1)$-forms. It should obey

$$\delta^2=0 \tag{9}$$

and the Leibniz rule

$$\delta(AB)=\delta A\cdot B+(-1)^A A\cdot\delta B. \tag{10}$$

We define δ by saying that acting on the zero-form (function) $\phi(x)$, δ gives the one-form that we have called $\delta\phi(x)$. The Leibniz rule then determines the action of δ on an arbitrary zero-form Γ:

$$\delta(\Gamma(\phi))=\int \mathrm{d}x\frac{\delta\Gamma}{\delta\phi(x)}\,\delta\phi(x), \tag{11}$$

where $\delta\Gamma/\delta\phi(x)$ is just the variational derivative of Γ with respect to $\phi(x)$. Equation (11) is a familiar formula to which we are giving, perhaps, a slightly novel meaning. To act with δ on k-forms of $k>0$, one must bear in mind, first of all, that as the one-form $\delta\phi(x)$ is the exterior derivative of the zero-form $\phi(x)$, it must be closed:

$$\delta(\delta\phi(x))=0. \tag{12}$$

Also, using the Leibniz rule, we then have the exterior derivative of a general k form (7):

$$\delta A=\int \mathrm{d}x_0\ldots\mathrm{d}x_n\frac{\delta\alpha_{x_1\ldots x_n}(\phi)}{\delta\phi(x_0)}\,\delta\phi(x_0)\,\delta\phi(x_1)\ldots\delta\phi(x_n). \tag{13}$$

Although our definition of δ has been rather formal, one can readily see that it possesses the standard properties of the exterior derivative. Thus, if V is a vector field, Λ a zero-form, and i_V the operation of contraction with V, we have $V(\Lambda)=i_V\,\delta\Lambda$.

Having defined the relevant concepts, how are we to find a symplectic structure in, say, the scalar field theory (4)? The idea (Witten, 1986; Zuckerman, 1986) is to consider the 'symplectic current'

$$J_\alpha(x)=\delta\phi(x)\,\partial_\alpha\,\delta\phi(x). \tag{14}$$

At each spacetime point, (14) is a two-form on Z; but in its dependence on x, J_α is a conserved current:

$$\partial_\alpha J^\alpha(x)=0. \tag{15}$$

To verify (15), one needs the equation of motion (6) and the fact that $\delta\phi$ is anticommuting. As J_α is conserved, its integral over an initial value hypersurface Σ,

$$\omega = \int_\Sigma d\Sigma_\alpha J^\alpha, \tag{16}$$

is independent of the choice of Σ and so in particular is Poincaré invariant. The two-form ω is our desired symplectic structure on Z. It is evidently closed, in view of (12), and it is easy to see that upon picking Σ to be the standard initial value surface $t=0$, (16) reduces to

$$\omega = \int_\Sigma \delta\phi\, \delta\dot{\phi}, \tag{17}$$

which (as in (1)) is the standard formula.

Our goal in the present paper is to implement this procedure in the case of Yang–Mills theory and general relativity. The main novelty that arises is the need to establish gauge invariance as well as Poincaré invariance of the symplectic structure. We will consider the two cases in turn.

16.1 Yang–Mills theory

Considering first Yang–Mills theory, let A be the gauge connection and F the Yang–Mills curvature or field strength. The covariant derivative of a charged field Λ is $D_\alpha\Lambda = \partial_\alpha\Lambda + [A_\alpha, \Lambda]$. The Yang–Mills equation of motion is

$$0 = [D^\mu, F_{\mu\nu}] = \partial^\mu F_{\mu\nu} + [A^\mu, F_{\mu\nu}]. \tag{18}$$

The variation of (18) is

$$D^\mu\, \delta F_{\mu\nu} + [\delta A^\mu, F_{\mu\nu}] = 0. \tag{19}$$

We define the symplectic current $J_\alpha(x)$ as

$$J_\alpha = \mathrm{Tr}[\delta A^\mu\, \delta F_{\mu\alpha}], \tag{20}$$

where

$$\delta F_{\mu\alpha} = D_\mu\, \delta A_\alpha - D_\alpha\, \delta A_\mu. \tag{21}$$

Let us show that J_α is conserved. We have

$$\begin{aligned} \partial^\alpha J_\alpha &= \mathrm{Tr}[D^\alpha\, \delta A^\mu\, \delta F_{\alpha\mu}] + \mathrm{Tr}[\delta A^\mu D^\alpha\, \delta F_{\alpha\mu}] \\ &= \tfrac{1}{2}\mathrm{Tr}[\delta F^{\alpha\mu}\, \delta F_{\alpha\mu}] - \mathrm{Tr}[\delta A^\mu[\delta A^\alpha, \delta F_{\alpha\mu}]] = 0. \end{aligned} \tag{22}$$

On the last line of (22), the second term vanishes because δA is anticommuting, and the first because δF is anticommuting. So the two-form

$$\omega = \int_\Sigma \mathrm{Tr}[\delta A^\mu\, \delta F_{\alpha\mu}]\, d\Sigma^\alpha \tag{23}$$

is Poincaré invariant.

It is easy to see that ω is also closed; δA^μ, being the exterior derivative of the zero-form A^μ, is closed, while, in view of (21) and the anticommutativity of δ, we have

$$\begin{aligned}\delta(\delta F_{\mu\alpha}) &= \delta(D_\mu \delta A_\alpha - D_\alpha \delta A_\mu) = \delta([A_\mu, \delta A_\alpha] - [A_\alpha, \delta A_\mu]) \\ &= \{\delta A_\mu, \delta A_\alpha\} - \{\delta A_\alpha, \delta A_\mu\} = 0. \end{aligned} \tag{24}$$

Therefore, ω is closed.

It remains to discuss the behavior of ω under gauge transformations. First of all, the gauge transformation law for the gauge field is

$$A_\mu \to A_\mu + [D_\mu, \varepsilon] = A_\mu + \partial_\mu \varepsilon + [A_\mu, \varepsilon]. \tag{25}$$

Varying (25), we find that under gauge transformations, δA transforms as

$$\delta A_\mu \to \delta A_\mu + [\delta A_\mu, \varepsilon]. \tag{26}$$

In particular, δA transforms homogeneously under gauge transformations. And δF transforms in the same way:

$$\delta F_{\mu\alpha} \to \delta F_{\mu\alpha} + [\delta F_{\mu\alpha}, \varepsilon]. \tag{27}$$

Consequently, J^α and ω are gauge invariant.

This is an important step in the right direction, but it is not the end of the story. Let $\hat{Z}$ be the space of solutions of (18), and let Z be the space of solutions of (18) modulo gauge transformations. Thus, $Z = \hat{Z}/G$, with G being the group of gauge transformations. So far we have defined a gauge-invariant closed two-form ω on $\hat{Z}$. What we want is a gauge-invariant closed two-form on Z. We would like to show that the differential form ω that we have defined on $\hat{Z}$ is the pullback from Z to $\hat{Z}$ of a differential form on Z, which we will also call ω. This will be so if the following condition is obeyed. If V is any vector field tangent to the G orbits on $\hat{Z}$, and i_V is the operation of contraction with V, the requirement is $i_V\omega = 0$. This is a fancy way of saying that the components of ω in the gauge directions are zero; one must require this since the gauge directions are eliminated in passing from $\hat{Z}$ to Z, so a differential form on Z cannot have non-zero components in those directions.

In Yang–Mills theory, the gauge directions in field space are simply

$$\delta A_\mu = D_\mu \varepsilon. \tag{28}$$

More generally, we can consider a field variation

$$\delta A_\mu = \delta' A_\mu + D_\mu \varepsilon, \tag{29}$$

which has a gauge component $D_\mu \varepsilon$, and another component $\delta' A_\mu$ which is not pure gauge. To verify that ω has vanishing components in the gauge directions, we must show that if we insert (29) in the definition of ω, the term proportional to ε drops out. The expression which must vanish is

$$\Delta\omega=\int d\Sigma_\alpha \operatorname{Tr}[D_\mu\varepsilon\,\delta F^{\alpha\mu}+\delta A_\mu[F^{\alpha\mu},\varepsilon]]=\int d\Sigma_\alpha\,\partial_\mu \operatorname{Tr}(\varepsilon F^{\alpha\mu}). \tag{30}$$

In the last step, we have used anticommutativity of ε and $\delta F^{\alpha\mu}$ and the equation of motion (19). Indeed, (30) vanishes, being the integral of a total derivative, and this completes the construction of a symplectic structure on the gauge-invariant space Z.

16.2 General relativity

We now turn our attention to general relativity. The discussion is similar, although somewhat more complicated. Let $g_{\mu\nu}$ be the metric tensor of spacetime, and $R_{\mu\nu\alpha\beta}$ the corresponding Riemann tensor. The equation of motion of pure general relativity is the Einstein equation $R_{\mu\nu}=0$. To find its variation, remember that for any vector field V, one has

$$[D_\mu, D_\nu]V^\alpha=R^\alpha{}_{\lambda\mu\nu}V^\lambda. \tag{31}$$

One finds for the variation of the Einstein equation

$$0=D_\alpha\,\delta\Gamma^\alpha_{\mu\nu}-D_\mu\,\delta\Gamma^\alpha_{\nu\alpha}. \tag{32}$$

Here

$$\delta\Gamma^\alpha_{\mu\nu}=\tfrac{1}{2}g^{\alpha\beta}(D_\mu\,\delta g_{\nu\beta}+D_\nu\,\delta g_{\mu\beta}-D_\beta\,\delta g_{\mu\nu}) \tag{33}$$

is the variation of the Levi-Civita connection Γ. Note that $\delta\Gamma$ transforms as a tensor. We define the symplectic current

$$J^\alpha=\delta\Gamma^\alpha_{\mu\nu}[\delta g^{\mu\nu}+\tfrac{1}{2}g^{\mu\nu}\,\delta\ln g]-\delta\Gamma^\nu_{\mu\nu}[\delta g^{\alpha\mu}+\tfrac{1}{2}g^{\alpha\mu}\,\delta\ln g], \tag{34}$$

where $\delta\ln g\equiv\delta\ln\det(g_{\mu\nu})=g^{\mu\nu}\,\delta g_{\mu\nu}=-g_{\mu\nu}\,\delta g^{\mu\nu}$. Also keep in mind that $\delta g^{\mu\nu}=-g^{\mu\alpha}g^{\nu\beta}\,\delta g_{\alpha\beta}$.

We have to show that $D_\alpha J^\alpha=0$. Terms with $\delta\ln g$ give

$$\tfrac{1}{2}(D_\alpha\,\delta\Gamma^\alpha_{\mu\nu})g^{\mu\nu}\,\delta\ln g-\tfrac{1}{2}(D_\alpha\,\delta\Gamma^\nu_{\mu\nu})g^{\alpha\mu}\,\delta\ln g+\tfrac{1}{2}\,\delta\Gamma^\alpha_{\mu\nu}g^{\mu\nu}D_\alpha\,\delta\ln g$$
$$-\tfrac{1}{2}\,\delta\Gamma^\nu_{\mu\nu}g^{\alpha\mu}D_\alpha\,\delta\ln g=\delta\Gamma^\alpha_{\mu\nu}g^{\mu\nu}\,\delta\Gamma^\lambda_{\alpha\lambda}, \tag{35}$$

where we have used (33), (32), the fact that the metric tensor is covariantly constant, and the anticommutativity of $\delta\Gamma$. The remaining terms are

$$(D_\alpha\,\delta\Gamma^\alpha_{\mu\nu})\,\delta g^{\mu\nu}+\delta\Gamma^\alpha_{\mu\nu}D_\alpha\,\delta g^{\mu\nu}-(D_\alpha\,\delta\Gamma^\nu_{\mu\nu})\,\delta g^{\alpha\mu}-\delta\Gamma^\nu_{\mu\nu}D_\alpha\,\delta g^{\alpha\mu}$$
$$=-\delta\Gamma^\nu_{\mu\nu}D_\alpha\,\delta g^{\alpha\mu}=\delta\Gamma^\nu_{\mu\nu}\,\delta\Gamma^\mu_{\alpha\beta}g^{\alpha\beta},$$

which, together with (35), gives $D_\alpha J^\alpha=0$. This makes

$$\omega=\int_\Sigma d\Sigma_\alpha\sqrt{g}\,J^\alpha$$

Poincaré invariant.

Let us show that ω is closed, $\delta\omega=0$. We have

$$\delta\omega=\int_{\Sigma} d\Sigma_\alpha(\delta\sqrt{g}\,J^\alpha+\sqrt{g}\,\delta J^\alpha)$$

$$\delta J^\alpha=-\tfrac{1}{2}\,\delta\Gamma^\alpha_{\mu\nu}\,\delta g^{\mu\nu}\,\delta\ln g+\tfrac{1}{2}\,\delta\Gamma^\nu_{\mu\nu}\,\delta g^{\alpha\mu}\,\delta\ln g.$$

Remembering that $\delta\ln g$ is an anticommuting one-form whose square is zero, we have

$$\delta J^\alpha=-\tfrac{1}{2}J^\alpha\,\delta\ln g.$$

As $\delta\sqrt{g}=\frac{1}{2}\sqrt{g}\,\delta\ln g$, $\delta\omega=0$. Thus, ω is closed.

It now remains to investigate gauge invariance of ω. The fact that ω is invariant under diffeomorphisms is relatively trivial; it follows from the fact that all ingredients in the definition of ω, including $\delta\Gamma$, transform homogeneously, like tensors. As in the Yang–Mills case, the more delicate point is to show that we obtain a closed two-form not just on the space $\hat{Z}$ of solutions of the Einstein equations, but also on the subtler space $Z=\hat{Z}/G$, with G being the group of diffeomorphisms. We must show that components of ω tangent to the G orbits vanish. Under a diffeomorphism $x_\mu\rightarrow x_u+\varepsilon_\mu$, the metric changes by

$$g_{\mu\nu}\rightarrow g_{\mu\nu}+D_\mu\varepsilon_\nu+D_\nu\varepsilon_\mu.$$

We assume ε has compact support or, more generally, is asymptotic at infinity to a Killing vector field. It we write

$$\delta g_{\mu\nu}=\delta' g_{\mu\nu}+D_\mu\varepsilon_\nu+D_\nu\varepsilon_\mu, \tag{36}$$

where the $D\varepsilon$ terms are pure gauge but $\delta' g$ is not, then our task is to show that, with (36) inserted in ω, the $D\varepsilon$ terms do not contribute. It is useful first to rewrite J^α as

$$\begin{aligned}J^\alpha=\tfrac{1}{2}g^{\alpha\beta}(D_\mu\,\delta g_{\nu\beta}+D_\nu\,\delta g_{\mu\beta}-D_\beta\,\delta g_{\mu\nu})\,\delta g^{\mu\nu}\\ -\tfrac{1}{2}D_\mu\,\delta g^{\mu\alpha}\,\delta\ln g+\tfrac{1}{2}\,\delta g^{\mu\alpha}D_\mu\,\delta\ln g-\tfrac{1}{2}(D^\alpha\,\delta\ln g)\,\delta\ln g.\end{aligned} \tag{37}$$

Inserting (36) in (37), the term linear in ε is (after dropping the $'$ from $\delta' g$)

$$\begin{aligned}\Delta J^\alpha=[D_\mu,D^\alpha]\varepsilon_\nu\,\delta g^{\mu\nu}+[D_\nu,D_\mu]\,\delta g^{\alpha\mu}\varepsilon^\nu\\ +D_\nu[(D_\mu\,\delta g^{\mu\nu}+D^\nu\,\delta\ln g)\varepsilon^\alpha+D^\nu\,\delta g^{\mu\alpha}\varepsilon_\mu\\ +D_\mu\varepsilon^\alpha\,\delta g^{\mu\nu}+\tfrac{1}{2}D^\nu\varepsilon^\alpha\,\delta\ln g-(\alpha\leftrightarrow\nu)],\end{aligned} \tag{38}$$

where we have used (33), (32), and the identity $[D_\mu,D_\nu]\varepsilon^\mu=0$, which follows from $R_{\mu\nu}=0$. We are entitled to discard from ΔJ^μ terms of the form $D_\sigma X^{\mu\sigma}$, with $X^{\mu\sigma}$ being an antisymmetric tensor; such terms will vanish when inserted in the integral for ω. Discarding such total derivatives from (38), we are left with

$$R_{\nu\lambda\mu}{}^{\alpha}\varepsilon^{\lambda}\,\delta g^{\mu\nu}+(R^{\mu}{}_{\lambda\nu\mu}\,\delta g^{\lambda\alpha}+R^{\alpha}{}_{\lambda\nu\mu}\,\delta g^{\mu\lambda})\varepsilon^{\nu}$$
$$=(R_{\nu\lambda\mu}{}^{\alpha}+R^{\alpha}{}_{\nu\mu\lambda})\varepsilon^{\lambda}\,\delta g^{\mu\nu}=g^{\alpha\rho}R_{\rho\lambda\mu\nu}\varepsilon^{\lambda}\,\delta g^{\mu\nu}=0.$$

Therefore, components of ω tangent to the action of the diffeomorphism group are zero.

In conclusion, we have described in a manifestly covariant way the foundations of the canonical formalism of Yang–Mills theory and general relativity. Since a similar treatment of string theory has been given elsewhere, it seems that such an approach is possible for all of the geometrical theories in physics.

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