

An introduction to moduli spaces of curves and their intersection theory

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Abstract

This paper is an introduction to moduli spaces of Riemann surfaces, their Deligne–Mumford compactifications, natural cohomology classes that they carry, and the intersection numbers between these classes.

The paper is meant to be accessible, but reasonably short, with an emphasis on examples, exercises, and methods of computations. The foundations of the theory of moduli spaces on the other hand are replaced by informal explanations.

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Introduction

This paper is an introduction to intersection theory on moduli spaces of curves. It is meant to be as elementary as possible, but still reasonably short.

The intersection theory of an algebraic variety M looks for answers to the following questions: What are the interesting cycles (algebraic subvarieties) of M and what cohomology classes do they represent? What are the interesting vector bundles over M and what are their characteristic classes? Can we describe the full cohomology ring of M and identify the above classes in this ring? Can we compute their intersection numbers? In the case of moduli space, the full cohomology ring is still unknown. We are going to study its subring called the “tautological ring” which contains the classes of most interesting cycles and the characteristic classes of most interesting vector bundles.

Although it is known that for large g and n the rank of the tautological ring is much smaller than that of the full cohomology ring of $\overline{\mathcal{M}}_{g,n}$, most natural geometrically defined cohomology classes happen to be tautological and it is actually not so simple to construct examples of nontautological cohomology classes, see [15]. Tautological rings were conjectured to possess an interesting structure by C. Faber and R. Pandharipande in [9], [10]. Some of these conjectures are proved [26], [14], [16], while others are still open. Another motivation to study the tautological rings is that they are sufficient for all applications related to the Gromov–Witten theory.

To give a sense of purpose to the reader, we assume the following goal: after reading this chapter, one should be able to write a computer program evaluating all intersection numbers between the tautological classes on the moduli space of stable curves, and to understand the foundation of every step of these computations. A program like that was first written by C. Faber [8], but our approach is a little different.

Other good introductions to moduli spaces include [18] and [33].

Section 1 is an informal introduction to moduli spaces of smooth and stable curves. It contains many definitions and theorems and lots of examples, but no proofs.

In Section 2 we define the tautological cohomology classes on moduli spaces. Simplest computations of intersection numbers are carried out.

In Section 3 we explain how to reduce the computations of all intersection numbers of all tautological classes to those involving only the so-called ψ -

classes. This involves a variety of useful techniques from algebraic geometry, in particular the Grothendieck–Riemann–Roch formula.

Finally, in Section 4 we formulate Witten’s conjecture (Kontsevich’s theorem) that allows one to compute all intersection numbers among the ψ -classes. Explaining the proof of Witten’s conjecture is beyond the scope of this exposition.

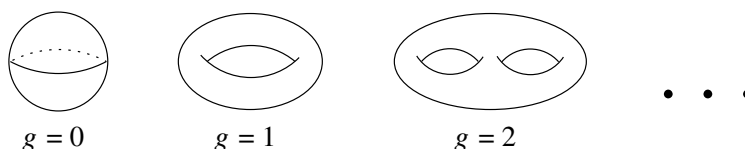
Acknowledgement. The paper is based on a series of three lectures for graduate students that the author gave at the *Journées mathématiques de Glanon* in July 2006. I am deeply grateful to the organizers for the invitation. I would also like to thank M. Kazarian whose unpublished notes on moduli spaces largely inspired the third section of the paper. This work was partially supported by the NSF grant 0905809 “String topology, field theories, and the topology of moduli spaces”.

1 From Riemann surfaces to moduli spaces

1.1 Riemann surfaces

Terminology. The main objects of our study are the *smooth compact complex curves* also called *Riemann surfaces* with n marked numbered pairwise distinct points. Unless otherwise specified they are assumed to be connected.

Every compact complex curve has an underlying structure of a 2-dimensional oriented smooth compact surface, that is uniquely characterized by its genus g .



Example 1.1. The sphere possesses a unique structure of Riemann surface up to isomorphism: that of a complex projective line \mathbb{CP}^1 (see [11], IV.4.1). A complex curve of genus 0 is called a *rational curve*. The automorphism group of \mathbb{CP}^1 is $\mathrm{PSL}(2, \mathbb{C})$ acting by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}.$$

Exercise 1.2. Check that this is, indeed, a group action.

Exercise 1.3. Prove that the automorphism group $\mathrm{PSL}(2, \mathbb{C})$ of \mathbb{CP}^1 allows one to send any three distinct points x_1, x_2, x_3 to 0, 1, and ∞ respectively in a unique way.

Example 1.4. Up to isomorphism every structure of Riemann surface on the torus is obtained by factorizing \mathbb{C} by a lattice, that is, a discrete additive subgroup $L \simeq \mathbb{Z}^2$ of \mathbb{C} (see [11], IV.6.1). A complex curve of genus 1 is called an *elliptic curve*. The automorphism group $\mathrm{Aut}(E)$ of any elliptic curve E contains a subgroup isomorphic to E itself acting by translations.

Exercise 1.5. Prove that two elliptic curves \mathbb{C}/L_1 and \mathbb{C}/L_2 are isomorphic if and only if $L_2 = aL_1$, $a \in \mathbb{C}^*$.

1.2 Moduli spaces

Moduli spaces of Riemann surfaces of genus g with n marked points can be defined as *smooth Deligne–Mumford stacks* (in the algebraic-geometric setting) or as *smooth complex orbifolds* (in an analytic setting). The latter notion is simpler and will be discussed in the next section. For the time being we define moduli spaces as sets.

Definition 1.6. For $2 - 2g - n < 0$, the *moduli space* $\mathcal{M}_{g,n}$ is the set of isomorphism classes of Riemann surfaces of genus g with n marked points.

Remark 1.7. The automorphism group of any Riemann surface satisfying $2 - 2g - n < 0$ is finite (see [11], V.1.2, V.1.3). On the other hand, every Riemann surface with $2 - 2g - n \geq 0$ has an infinite group of marked point preserving automorphisms. For reasons that will become clear in Section 1.3, this makes it impossible to define the moduli spaces $\mathcal{M}_{0,0}$, $\mathcal{M}_{0,1}$, $\mathcal{M}_{0,2}$, and $\mathcal{M}_{1,0}$ as orbifolds. (They still make sense as sets, but this is of little use.)

Example 1.8. Let $g = 0$, $n = 3$. Every rational curve (C, x_1, x_2, x_3) with three marked points can be identified with $(\mathbb{CP}^1, 0, 1, \infty)$ in a unique way. Thus $\mathcal{M}_{0,3} =$ one point.

Example 1.9. Let $g = 0$, $n = 4$. Every curve (C, x_1, x_2, x_3, x_4) can be uniquely identified with $(\mathbb{CP}^1, 0, 1, \infty, t)$. The number $t \neq 0, 1, \infty$ is determined by the positions of the marked points on C . It is called the *modulus* and gave rise to the term “moduli space”. The moduli space $\mathcal{M}_{0,4}$ is the set of values of t , that is $\mathcal{M}_{0,4} = \mathbb{CP}^1 \setminus \{0, 1, \infty\}$.

Exercise 1.10. The rational curve $(\mathbb{CP}^1, x_1, x_2, x_3, x_4)$ with pairwise distinct x_i ’s is isomorphic to $(\mathbb{CP}^1, 0, 1, \infty, t)$. Compute t as function of x_1, x_2, x_3, x_4 .

Example 1.11. Generalizing the previous example, take $g = 0$ and an arbitrary n . Each curve (C, x_1, \dots, x_n) can be uniquely identified with $(\mathbb{CP}^1, 0, 1, \infty, t_1, \dots, t_{n-3})$. The moduli space $\mathcal{M}_{0,n}$ is given by

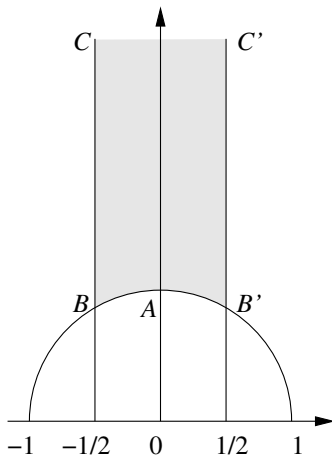
$$\mathcal{M}_{0,n} = \{(t_1, \dots, t_{n-3}) \in (\mathbb{CP}^1)^{n-3} \mid t_i \neq 0, 1, \infty, t_i \neq t_j\}.$$

Example 1.12. According to Example 1.4, every elliptic curve is isomorphic to the quotient of \mathbb{C} by a rank 2 lattice L . The image of $L \subset \mathbb{C}$ is a natural marked point on E . Thus $\mathcal{M}_{1,1} = \{\text{lattices}\}/\mathbb{C}^*$. Consider a direct basis

(z_1, z_2) of a lattice L . Multiplying L by $1/z_1$ we obtain a lattice with basis $(1, \tau)$, where τ lies in the upper half-plane \mathbb{H} . Choosing another basis of the same lattice we obtain another point $\tau' \in \mathbb{H}$. Thus the group $\mathrm{SL}(2, \mathbb{Z})$ of direct base changes in a lattice acts on \mathbb{H} . This action is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

We have $\mathcal{M}_{1,1} = \mathbb{H}/\mathrm{SL}(2, \mathbb{Z})$. The matrix $-\mathrm{Id} \in \mathrm{SL}(2, \mathbb{Z})$ acts trivially on \mathbb{H} . The group $\mathrm{PSL}(2, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z})/\pm \mathrm{Id}$ has a fundamental domain shown in the next figure. The moduli space $\mathcal{M}_{1,1}$ is obtained from the fundamental domain by identifying the arcs AB and AB' and the half-lines BC and $B'C'$.



Note that the orbifold structure of $\mathcal{M}_{1,1}$ corresponds to the quotient $\mathbb{H}/\mathrm{SL}(2, \mathbb{Z})$ rather than $\mathbb{H}/\mathrm{PSL}(2, \mathbb{Z})$.

Example 1.13. Let $g = 2$, $n = 0$. By Riemann–Roch’s theorem, every Riemann surface of genus g carries a g -dimensional vector space Λ of abelian differentials (that is, holomorphic differential 1-forms). Each abelian differential has $2g - 2$ zeroes. (See [11], III.4.)

For $g = 2$, we have $\dim \Lambda = 2$. Let (α, β) be a basis of Λ , and consider the map $f: C \rightarrow \mathbb{CP}^1$ given by the quotient $f = \alpha/\beta$. (In intrinsic terms the image of f is the projectivization of the dual vector space of Λ . Choosing a basis in Λ identifies it with \mathbb{CP}^1 .) The map f is of degree at most 2, because both α and β have two zeroes and no poles. But f cannot be a constant (because then α and β would be proportional to each other and would not

form a basis of Λ) and cannot be of degree 1 (because then it would establish an isomorphism between its genus 2 domain and its genus 0 target, which is not possible). Thus $\deg f = 2$. The involution of C that interchanges the two sheets of f or, in other words, interchanges the two zeroes of every holomorphic differential, is called the *hyperelliptic involution*. By an Euler characteristic count (or applying the Riemann–Hurwitz formula, which is the same) we obtain that f must have six ramification points, that is, six distinct points in \mathbb{CP}^1 that have one double preimage rather than two simple preimages. The six preimages of these points on C are the fixed points of the hyperelliptic involution and are called *Weierstrass points*.

Summarizing, we see that giving a genus 2 Riemann surface is equivalent to giving six distinct nonnumbered points on a rational curve. Thus $\mathcal{M}_{2,0} = \mathcal{M}_{0,6}/S_6$, where S_6 is the symmetric group. However, this equality only holds for sets. The moduli spaces $\mathcal{M}_{2,0}$ and $\mathcal{M}_{0,6}/S_6$ actually have different orbifold structures, because every genus 2 curve has an automorphism that the genus 0 curve with six marked points does not have: namely, the hyperelliptic involution.

1.3 Orbifolds

Here we give a minimal set of definitions necessary for our purposes. Readers interested in learning more about orbifolds and stacks are referred to [3], [25], [34].

1.3.1 Orbifold charts and atlases

A smooth complex (respectively, real) n -dimensional orbifold is locally isomorphic to an open ball in \mathbb{C}^n (respectively, \mathbb{R}^n) factorized by a finite group action. Let us give a precise definition.

Let X be a topological space.

Definition 1.14. An *complex (resp., real) orbifold chart* on X is the following data:

$$U/G \xrightarrow{\varphi} V \subset X,$$

where $U \subset \mathbb{C}^n$ (resp, $U \subset \mathbb{R}^n$) is a contractible open set endowed with a bi-holomorphic (resp., smooth) action of a finite group G , $V \subset X$ is an open set, and φ is a homeomorphism from U/G to V .

We shall call an orbifold chart a *chart*.

Sometimes the chart will be denoted simply by V if this does not lead to ambiguity. Note that the action of G is not necessarily faithful, that is, a nontrivial subgroup of G can act trivially on U .

Definition 1.15. A chart

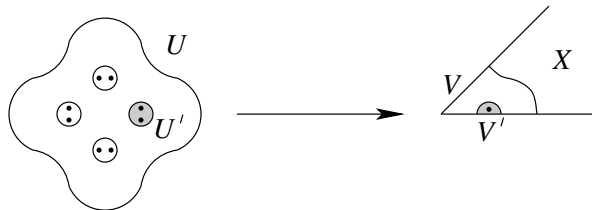
$$U'/G' \xrightarrow{\varphi'} V' \subset X$$

is called a *subchart* of

$$U/G \xrightarrow{\varphi} V \subset X$$

if V' is a subset of V and there is a group homomorphism $G' \rightarrow G$ and a holomorphic (resp., smooth) embedding $U' \hookrightarrow U$ such that (i) the embedding and the group morphism commute with the group actions; (ii) the G' -stabilizer of every point in U' is isomorphic to the G -stabilizer of its image in U ; (iii) the embedding commutes with the isomorphisms φ and φ' .

The following figure shows a typical example of a sub-chart.



Exercise 1.16. Two points $x_1 \in V_1$ and $x_2 \in V_2$ in two orbifold charts are called *locally equivalent* if they are contained in two isomorphic subcharts $x_1 \in V'_1 \subset V_1$ and $x_2 \in V'_2 \subset V_2$ and the isomorphism sends x_1 to x_2 . Classify points in 1-dimensional complex and real orbifolds up to local equivalence.

Definition 1.17. Two orbifold charts V_1 and V_2 are called *compatible* if every point of $V_1 \cap V_2$ is contained in some chart V_3 that is a subchart of both V_1 and V_2 .

Note that any attempt to define a chart $V_1 \cap V_2$ would lead to problems, because $V_1 \cap V_2$ is, in general, not connected and the preimages of $V_1 \cap V_2$ in U_1 and in U_2 are not necessarily contractible.

Definition 1.18. An *atlas* on a topological space X is a family of compatible charts entirely covering X . A *maximal atlas* is an atlas that cannot be increased by adding more charts. A *smooth complex orbifold* is a topological space X together with a maximal atlas.

Definition 1.19. Let X be an orbifold and $x \in X$ a point. The *stabilizer* of x is the stabilizer in G of a preimage of x in U under φ in some chart. (By definition, if we choose another chart or another preimage we will get an isomorphic stabilizer, though the isomorphism is not canonical.)

Exercise 1.20. Let M is a smooth complex manifold endowed with an action of a finite group G . Construct the natural orbifold atlas on $X = M/G$.

1.3.2 Maps, vector bundles, differential forms

All notions related to manifolds and possessing a local definition can be automatically extended to orbifolds.

For instance, a differential form α on a chart V is defined as a G -invariant differential form α_U on U . The integral of α over a chain $C \subset V$ is defined as

$$\frac{1}{|G|} \int_{\varphi^{-1}(C)} \alpha_U.$$

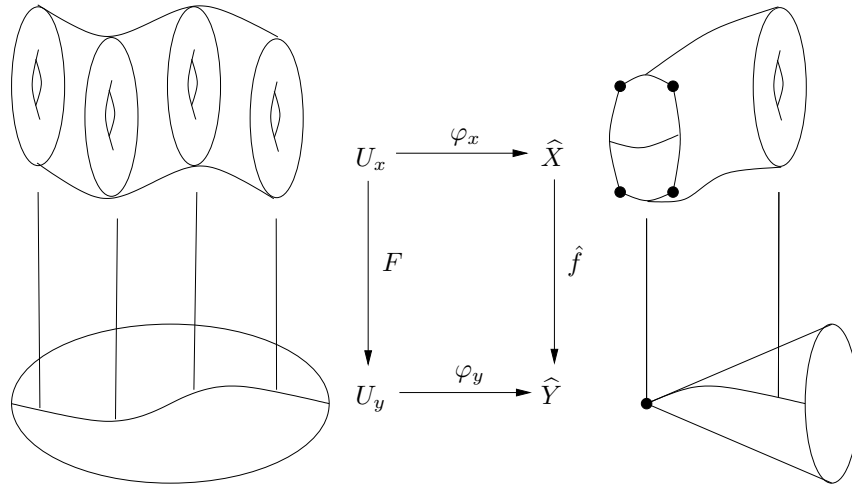
Further, a vector bundle over a chart V is defined as a vector bundle over the open set U together with a fiberwise linear lifting of the G -action to the total space of the bundle. We can define a connection on a vector bundle and the curvature of the connection in the natural way.

Exercise 1.21. What is a vector bundle over the orbifold $[\text{point}]/G$?

Defining a morphism of orbifolds in general would lead us to new technical difficulties. However it is easy to define a morphism of orbifolds in the case where the fibers of the morphism are manifolds. This will be enough for our purposes. The definition is similar to that of vector bundles.

Definition 1.22. A *map of orbifolds* $f: X \rightarrow Y$ with *manifold fibers* is a continuous map of the underlying topological spaces $\hat{f}: \hat{X} \rightarrow \hat{Y}$ together with the choice for every $y \in Y$ of a chart $\varphi_y: U_y/G \rightarrow V_y$ containing y , a holomorphic map $F: U_x \rightarrow U_y$, a lifting of the G -action on U_y to U_x commuting with F and an isomorphism φ_x of U_x/G with an open suborbifold of X , such that $\varphi_y \circ F = \hat{f} \circ \varphi_x$.

The figure on the next page represents a morphism of orbifolds whose fibers are tori and whose image is a curve inside Y . Note that the fibers of \hat{f} are finite quotients of the fibers of F . When we talk about the fibers of a map of orbifolds f we mean the latter and not the former.



1.3.3 Euler characteristic

This section will not be used in the sequel and can be skipped in first reading.

Definition 1.23. The *Euler characteristic* $\chi(X)$ of an orbifold X is defined by

$$\chi(X) = \sum_G \frac{1}{|G|} \chi(X_G),$$

where X_G is the locus of points with stabilizer G .

For instance, the Euler characteristic of \mathbb{R}^n with trivial G -action is $(-1)^n/|G|$.

Exercise 1.24. Show that for a smooth manifold X endowed with a G -action we have $\chi(X/G) = \chi(X)/|G|$.

Exercise 1.25. Let X be a compact complex connected 1-dimensional orbifold such that the stabilizer of its generic point is trivial. Find the orbifold like that whose Euler characteristic is negative, but as close to 0 as possible.

Exercise 1.26. Deduce from the previous exercise that if a finite group G faithfully acts on a Riemann surface of genus $g \geq 2$ by automorphisms then $|G| \leq 84(g-1)$.

1.3.4 Homology and cohomology over \mathbb{Q} .

Defining global characteristics of orbifolds, for instance, their cohomology rings or their homotopy groups, is more delicate. For instance, the homotopy type of $[\text{point}]/G$ is $K(G, 1)$. It is possible to define the ring $H^*(X, \mathbb{Z})$ for an orbifold X , but we will not do it here. Instead, we content ourselves with the straightforward definition of the cohomology ring over \mathbb{Q} .

Definition 1.27. The *homology*, resp. *cohomology groups of an orbifold over \mathbb{Q}* are defined as the homology, resp. cohomology groups of its underlying topological space (also over \mathbb{Q}).

Theorem 1.28 ([4]). *Poincaré duality holds for homology and cohomology groups over \mathbb{Q} of smooth compact complex or real oriented orbifolds.*

Remark 1.29. Let X be an orbifold and Y an irreducible sub-orbifold. Denote by \hat{X} and \hat{Y} the underlying topological spaces. By convention, the homology class $[Y] \in H_*(X, \mathbb{Q}) = H_*(\hat{X}, \mathbb{Q})$ is equal to $\frac{1}{|G_Y|}[\hat{Y}] \in H_*(\hat{X}, \mathbb{Q})$, where G_Y is the stabilizer of a generic point of Y .

Example 1.30. Consider the action of $\mathbb{Z}/k\mathbb{Z}$ on $\mathbb{C}P^1$ by rotations and let X be the quotient orbifold. Then the class $[0] \in H_0(X, \mathbb{Q})$ is $1/k$ times the class of a generic point.

It turns out that the moduli space $\mathcal{M}_{g,n}$ (for $2 - 2g - n < 0$) possesses a natural structure of a smooth complex $(3g - 3 + n)$ -dimensional orbifold. Moreover, the stabilizer of a point $t \in \mathcal{M}_{g,n}$ is equal to the automorphism group of the corresponding Riemann surface with n marked points C_t . Let us explain how to endow the moduli space with an orbifold structure.

We say that $p: \mathcal{C} \rightarrow B$ is a *family of genus g Riemann surfaces with n marked points* if p is endowed with n disjoint sections $s_i: B \rightarrow \mathcal{C}$ (so that $p \circ s_i = \text{Id}$) and every fiber of p is a smooth Riemann surface. The intersections of the sections with every fiber of p are the marked points of the fiber. If we have two families $p_1: \mathcal{C}_1 \rightarrow B_1$ and $p_2: \mathcal{C}_2 \rightarrow B_2$ and a subset $B'_2 \subset B_2$, we say that *the restriction of p_2 to B'_2 is a pull-back of p_1* if there exists a morphism $\varphi: B'_2 \rightarrow B_1$ such that \mathcal{C}_2 restricted to B'_2 is isomorphic to the pull-back of \mathcal{C}_1 under φ .

Theorem 1.31 ([18], 2.C). *Let C be a genus g Riemann surface with n marked points, $2 - 2g - n < 0$. Let G be its (finite) isomorphism group. There exists*

(a) an open bounded simply connected domain $U \subset \mathbb{C}^{3g-3+n}$;
 (b) a family $p: \mathcal{C} \rightarrow U$ of genus g Riemann surfaces with n marked points;
 (c) a G -action on \mathcal{C} descending to a G -action on U and satisfying the following conditions:

1. The fiber C_0 over $0 \in \mathbb{C}^{3g-3+n}$ is isomorphic to C .
2. The action of G preserves C_0 and acts as the symmetry group of C_0 .
3. For any family of smooth curves with n marked points $\mathcal{C}_B \rightarrow B$ such that C_b is isomorphic to C for some $b \in B$, there exists an open subset $B' \subset B$ containing b and a map $\varphi: B' \rightarrow U$, unique up to a composition with the action of G , such that the restriction of the family $\mathcal{C}_B \rightarrow B$ to B' is the pull-back by φ of the family $\mathcal{C} \rightarrow U$.

Theorem 1.31 leads to a construction of two smooth orbifolds.

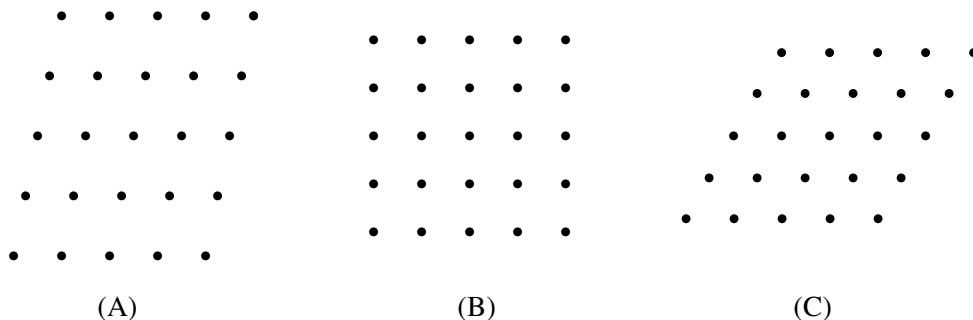
The first one, $\mathcal{M}_{g,n}$, covered by the charts U/G , is the moduli space. It follows from the theorem that the stabilizer of $t \in \mathcal{M}_{g,n}$ is isomorphic to the symmetry group of the surface C_t .

The second one, $\mathcal{C}_{g,n}$ is covered by the open sets \mathcal{C} (these are not charts, because they are not simply connected, but it is easy to subdivide them into charts). There is an orbifold morphism $p: \mathcal{C}_{g,n} \rightarrow \mathcal{M}_{g,n}$ between the two.

Definition 1.32. The map $p: \mathcal{C}_{g,n} \rightarrow \mathcal{M}_{g,n}$ is called the *universal curve* over $\mathcal{M}_{g,n}$.

The fibers of the universal curve are Riemann surfaces with n marked points, and each such surface appears exactly once among the fibers. If we consider the induced map of underlying topological spaces $\hat{p}: \mathcal{C}_{g,n} \rightarrow \mathcal{M}_{g,n}$, then its fibers are of the form C/G , where C is a Riemann surface and G its automorphism group.

Example 1.33. As we explained in Example 1.12, the moduli space $\mathcal{M}_{1,1}$ is isomorphic to $\mathbb{H}/\mathrm{SL}(2, \mathbb{Z})$. The stabilizer of a lattice L in $\mathrm{SL}(2, \mathbb{Z})$ is the group of basis changes of L that amount to homotheties of \mathbb{C} . These can be viewed as isomorphisms of the elliptic curve \mathbb{C}/L . Thus the stabilizer of a point in the moduli space is indeed isomorphic to the automorphism group of the corresponding curve.



The stabilizer of a generic lattice L (case A) is the group $\mathbb{Z}/2\mathbb{Z}$ composed of the identity and the central symmetry.

The stabilizer of the lattice $\mathbb{Z} + i\mathbb{Z}$ (case B) is the group $\mathbb{Z}/4\mathbb{Z}$ of rotations by multiples of 90° .

The stabilizer of the lattice $\mathbb{Z} + \frac{1+i\sqrt{3}}{2}\mathbb{Z}$ (case C) is the group $\mathbb{Z}/6\mathbb{Z}$ of rotations by multiples of 60° .

By abuse of language we will often “forget” that the moduli spaces are orbifolds and treat them as manifolds, bearing in mind the above definitions.

Exercise 1.34. For $g \geq 2$, a genus g surface with a Riemannian metric of constant curvature -1 can be obtained by gluing together the opposite sides of $4g$ -gon on the hyperbolic plane. For this to be possible the lengths of the opposite sides of the $4g$ -gon should be pairwise equal and its angles should add up to 2π . What is the dimension of the space of $4g$ -gons like that? Compare it to the dimension of $\mathcal{M}_{g,0}$.

Exercise 1.35. Compute the Euler characteristic of $\mathcal{M}_{1,1}$.

Exercise 1.36. Construct a family of genus 1 curves over \mathbb{C}^* with one marked point such that all the fibers are isomorphic to a given elliptic curve E , but the family is not isomorphic to $\mathbb{C}^* \times E$. (This family shows the difficulties one has to overcome to define maps between orbifolds in general. Indeed, the family determines a map from \mathbb{C}^* to $\mathcal{M}_{1,1}$ in the sense of orbifolds. This map is non-trivial even though its image is just one \mathbb{C} -point.)

1.4 Stable curves and the Deligne–Mumford compactification

As the examples of Section 1.2 show, the moduli space $\mathcal{M}_{g,n}$ is, in general, not compact. We are now going to compactify it by adding new points that correspond to the so-called “stable curves”. Let us start with an example.

1.4.1 The case $g = 0, n = 4$

As explained in Example 1.9, the moduli space $\mathcal{M}_{0,4}$ is isomorphic to $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$. A point $t \in \mathbb{CP}^1 \setminus \{0, 1, \infty\}$ encodes the following curve C_t :

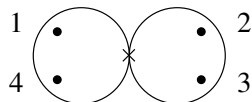
$$(C, x_1, x_2, x_3, x_4) \simeq (\mathbb{CP}^1, 0, 1, \infty, t).$$

What will happen as $t \rightarrow 0$? At first sight, we will simply obtain a curve with four marked points, two of which coincide: $x_1 = x_4$. However, such an approach is unjust with respect to the points x_1 and x_4 . Indeed, without changing the curve C_t , we can change its local coordinate via the map $x \mapsto x/t$ and obtain the curve

$$(C, x_1, x_2, x_3, x_4) \simeq (\mathbb{CP}^1, 0, 1/t, \infty, 1).$$

What we now see in the limit is that x_1 and x_4 do not glue together any longer, but this time x_2 and x_3 do tend to the same point.

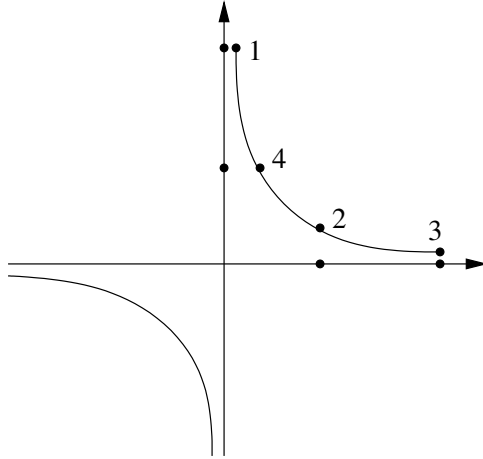
Since there is no reason to prefer one local coordinate to the other, neither of the pictures is better than the other one. Thus the right thing to do is to include *both* limit curves in the description of the limit:



The right-hand component corresponds to the initial local coordinate x , while the left-hand component corresponds to the local coordinate x/t . The components intersect transversely (contrary to what the picture might suggest).

It can be, at first, difficult to imagine, how a sphere can possibly tend to a curve consisting of two spheres. To make this more visual, consider the following example. Let $xy = t$ (or $xy = tz^2$ in homogeneous coordinates) be a family of curves in \mathbb{CP}^2 parameterized by t . On each of these curves we mark the following points:

$$\begin{aligned} [x_1 : y_1 : z_1] &= [0 : 1 : 0], & [x_2 : y_2 : z_2] &= [1 : t : 1], \\ [x_3 : y_3 : z_3] &= [1 : 0 : 0], & [x_4 : y_4 : z_4] &= [t : 1 : 1]. \end{aligned}$$



Then, for $t \neq 0$, the curve is isomorphic to \mathbb{CP}^1 with four marked points, while for $t = 0$ it degenerates into a curve composed of two spheres (the coordinate axes) with two marked points on each sphere.

Now we go back to the general case.

1.4.2 Stable curves

Stable curves are complex algebraic curves that are allowed to have exactly one type of singularities, namely, simple nodes. The simplest example of a curve with a node is the plane curve given by the equation $xy = 0$, that has a node at the origin. The neighborhood of a node is diffeomorphic to two discs with identified centers. A node can be desingularized in two different ways. We say that a node is *normalized* if the two discs with identified centers that form its neighborhood are unglued, *i.e.*, replaced by disjoint discs. A node is *smoothened* if the two discs with identified centers that form its neighborhood are replaced by a cylinder.

Definition 1.37. A *stable curve* C with n marked points x_1, \dots, x_n is a compact complex algebraic curve satisfying the following conditions. (i) The only singularities of C are simple nodes. (ii) The marked points are distinct and do not coincide with the nodes. (iii) The curve (C, x_1, \dots, x_n) has a finite number of automorphisms.

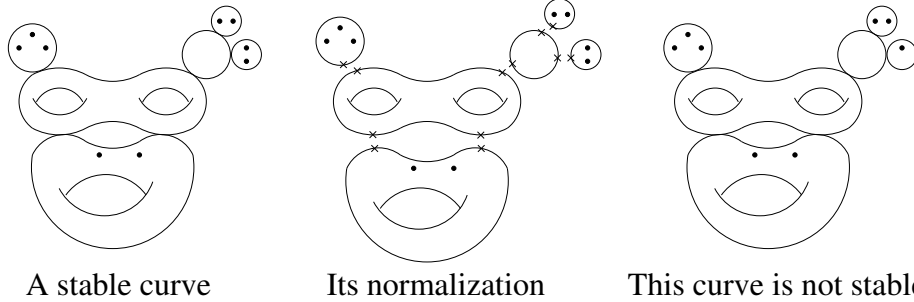
Unless stated otherwise, stable curves are assumed to be connected.

The *genus* of a stable curve C is the genus of the surface obtained from C by smoothening all its nodes.

The *normalization* of a stable curve C is the smooth not necessarily connected curve obtained from C by normalizing all its nodes.

Condition (iii) in the above definition can be reformulated as follows. Let C_1, \dots, C_k be the connected components of the normalization of C . Let g_i be the genus of C_i and n_i the number of *special points*, *i.e.*, marked points and preimages of the nodes on C_i . Then Condition (iii) is satisfied if and only if $2 - 2g_i - n_i < 0$ for all i . In this form, the condition is, of course, much easier to check.

The stable curve in the picture below is of genus 4.



Exercise 1.38. Let C be a stable curve of genus g with n marked points. Then the Euler characteristic of $C \setminus$ (marked points and nodes) equals $2 - 2g - n$.

Exercise 1.39. Deduce from the previous exercise that there is only a finite number of topological types of stable curves of genus g with n marked points.

Theorem 1.40 ([18], Chapter 4). *There exists a smooth compact complex $(3g-3+n)$ -dimensional orbifold $\overline{\mathcal{M}}_{g,n}$, a smooth compact complex $(3g-2+n)$ -dimensional orbifold $\overline{\mathcal{C}}_{g,n}$, and a map $p: \overline{\mathcal{C}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$ such that*

- (i) $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ is an open dense sub-orbifold and $\mathcal{C}_{g,n} \subset \overline{\mathcal{C}}_{g,n}$ its preimage under p ;
- (ii) the fibers of p are stable curves of genus g with n marked points;
- (iii) each stable curve is isomorphic to exactly one fiber;
- (iv) the stabilizer of a point $t \in \overline{\mathcal{M}}_{g,n}$ is isomorphic to the automorphism group of the corresponding stable curve C_t .

Definition 1.41. The space $\overline{\mathcal{M}}_{g,n}$ is called the *Deligne–Mumford compactification* of the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces. The family $p: \overline{\mathcal{C}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$ is called the *universal curve*.

This compactification was constructed by Deligne and Mumford [5] for $n = 0$ and by Knudsen [23] in general. Sometimes it is also called the Deligne–Mumford–Knudsen compactification.

Exercise 1.42. Find the limit in $\overline{\mathcal{M}}_{0,6}$ of the following family of smooth curves:

$$(C, x_1, x_2, x_3, x_4, x_5, x_6) = (\mathbb{CP}^1, 0, 1, \infty, t, t^2, 2t^2)$$

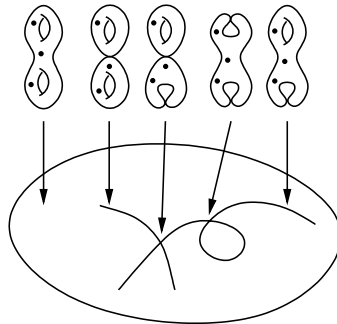
as $t \rightarrow 0$.

Exercise 1.43. Let C be a fixed curve of genus $g \geq 1$ with two marked points x and y . Find the limit of C in $\overline{\mathcal{M}}_{g,2}$ as y tends to x while the curve stays the same.

Exercise 1.44. Let $C_\varepsilon \subset \mathbb{CP}^2$ be the curve given by $y^2 = x^3 - \varepsilon x$, or $zy^2 = x^3 - \varepsilon xz^2$ in homogeneous coordinates. The point $p = [0 : 1 : 0]$ will be marked. Find the limit in $\overline{\mathcal{M}}_{1,1}$ of the family of curves (C_ε, p) as $\varepsilon \rightarrow 0$.

Definition 1.45. The set $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ parametrizing singular stable curves is called the *boundary* of $\overline{\mathcal{M}}_{g,n}$.

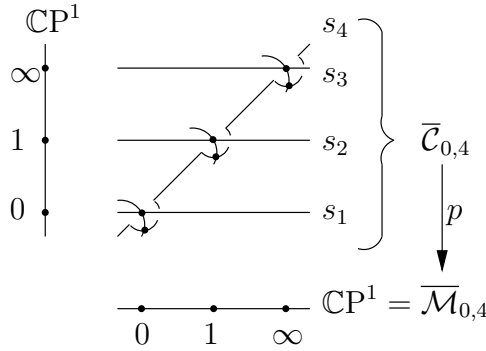
The boundary is a sub-orbifold of $\overline{\mathcal{M}}_{g,n}$ of codimension 1, in other words, a divisor. The term “boundary” may lead one to think that $\overline{\mathcal{M}}_{g,n}$ has a singularity at the boundary, but this is not true: as we have already stated, $\overline{\mathcal{M}}_{g,n}$ is a smooth orbifold, and the boundary points are as smooth as any other points of $\overline{\mathcal{M}}_{g,n}$. A generic point of the boundary corresponds to a stable curve with exactly one node. If a point t of the boundary corresponds to a stable curve C_t with k nodes, there are k local components of the boundary that intersect transversally at t . Each of these components is obtained by smoothening $k - 1$ out of k nodes of C_t . Thus the boundary is a divisor with normal crossings in $\overline{\mathcal{M}}_{g,n}$. The figure below shows two components of the boundary divisor in $\overline{\mathcal{M}}_{g,n}$ and the corresponding stable curves.



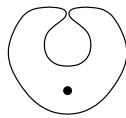
1.4.3 Examples

Example 1.46. We have $\overline{\mathcal{M}}_{0,3} = \mathcal{M}_{0,3} = \text{a point}$. Indeed, the unique stable genus 0 curve with three marked points is smooth.

Example 1.47. Consider the projection $\tilde{p}: \mathbb{CP}^1 \times \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ on the first factor. Consider further four distinguished sections $\tilde{s}_i: \mathbb{CP}^1 \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$: $\tilde{s}_1(t) = (t, 0)$, $\tilde{s}_2(t) = (t, 1)$, $\tilde{s}_3(t) = (t, \infty)$, $\tilde{s}_4(t) = (t, t)$. Now take the blow-up X of $\mathbb{CP}^1 \times \mathbb{CP}^1$ at the three points $(0, 0)$, $(1, 1)$, and (∞, ∞) where the fourth section intersects the three others. We obtain a map $p: X \rightarrow \mathbb{CP}^1$ endowed with four nonintersecting sections. Its fiber over $t \in \mathbb{CP}^1 \setminus \{0, 1, \infty\}$ is the Riemann sphere with four marked points $0, 1, \infty$, and t . The three special fibers over $0, 1$, and ∞ are singular stable curves. Thus the map $p: X \rightarrow \mathbb{CP}^1$ is actually the universal curve $\overline{\mathcal{C}}_{0,4} \rightarrow \overline{\mathcal{M}}_{0,4}$.



Example 1.48. The moduli space $\overline{\mathcal{M}}_{1,1}$ is obtained from $\mathcal{M}_{1,1}$ by adding one point corresponding to the singular stable curve:



We do not prove the compactness of $\overline{\mathcal{M}}_{g,n}$ here, but illustrate it with some more examples.

Exercise 1.49. Following exercise 1.43 let (C, x) be a fixed curve of genus $g \geq 1$ with one marked point and let $D \in C$ be a small disc centered at x . Over the punctured disc D^* we have a natural family of smooth curves with two marked points, namely, to $y \in D^*$ we assign the curve (C, x, y) . Extend this family to a family of stable curves over D .

Exercise 1.50. As in Exercise 1.42 consider the following family of smooth curves over the punctured unit disc:

$$(C, x_1, x_2, x_3, x_4, x_5, x_6) = (\mathbb{CP}^1, 0, 1, \infty, t, t^2, 2t^2),$$

for $t \in D^*$. Extend this family to a family of stable curves over D .

Exercise 1.51. Following Exercise 1.44 consider the family of smooth curves over $\varepsilon \in D^*$ given by $C_\varepsilon : y^2 = x^3 - \varepsilon^3 x$, or $zy^2 = x^3 - \varepsilon xz^2$ in homogeneous coordinates. The point $p = [0 : 1 : 0]$ will be marked. Extend this family to a family of stable curves over D . Show that the family $y^2 = x^3 - \varepsilon x$ cannot be extended to a family of stable curves over D .

Exercise 1.52. In Example 1.13 we constructed a map $\mathcal{M}_{2,0} \rightarrow \mathcal{M}_{0,6}/S_6$ that assigns to a genus 2 curve its quotient under the hyperelliptic involution with 6 marked branch points. Does this map extend to a map $\overline{\mathcal{M}}_{2,0} \rightarrow \overline{\mathcal{M}}_{0,6}/S_6$?

Exercise 1.53. Construct a natural one-to-one correspondence between the orbifolds $\overline{\mathcal{M}}_{g,n+1}$ and $\overline{\mathcal{C}}_{g,n}$. Don't forget to consider the case of a stable curve with $n + 1$ marked points that becomes unstable if we erase the $(n + 1)$ st marked point. (The statement of this exercise is reformulated and proved in Proposition 2.2.)

1.4.4 The universal curve at the neighborhood of a node

As in Section 1.4.1, consider the map $p: \mathbb{C}^2 \rightarrow \mathbb{C}$ given by $(x, y) \mapsto t = xy$. Then the fibers of p over $t \neq 0$ are smooth (and isomorphic to \mathbb{C}^*) while the fiber over $t = 0$ has a node (and is isomorphic to two copies of \mathbb{C} glued together at the origin).

It turns out that this example gives a local model for every node in every universal curve.

Proposition 1.54 (See [18], 3.B, Deformations of stable curves). *Let $p: \overline{\mathcal{C}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$ be the universal curve and $z \in \overline{\mathcal{C}}_{g,n}$ a node in a singular fiber. Then there is a neighborhood of z in $\overline{\mathcal{C}}_{g,n}$ with a system of local coordinates $T_1, \dots, T_{3g-4+n}, x, y$ and a neighborhood of $p(z)$ in $\overline{\mathcal{M}}_{g,n}$ with a system of local coordinates t_1, \dots, t_{3g-3+n} such that in these coordinates p is given by*

$$t_i = T_i \quad (1 \leq i \leq 3g - 4 + n), \quad t_{3g-3+n} = xy.$$

2 Cohomology classes on $\overline{\mathcal{M}}_{g,n}$

In this section we introduce several natural cohomology classes on the moduli spaces. The ring generated by these classes is called the *tautological cohomology ring* of $\overline{\mathcal{M}}_{g,n}$. Although it is known that for large g and n the rank of the tautological ring is much smaller than that of the full cohomology ring of $\overline{\mathcal{M}}_{g,n}$, most natural geometrically defined cohomology classes happen to be tautological and it is actually not so simple to construct examples of nontautological cohomology classes [15].

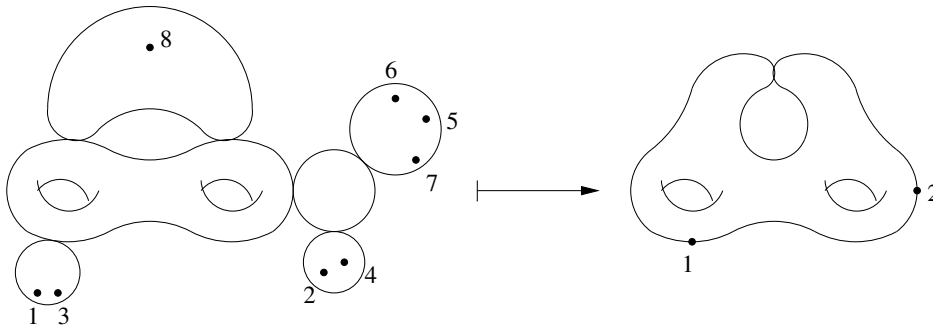
2.1 Forgetful and attaching maps

2.1.1 Forgetful maps

The idea of a forgetful map is to assign to a genus g stable curve (C, x_1, \dots, x_{n+m}) the curve (C, x_1, \dots, x_n) , where we have “forgotten” m marked points out of $n + m$. The main problem is that the resulting curve (C, x_1, \dots, x_n) is not necessarily stable. Assume that $2 - 2g - n < 0$. Then, either the curve (C, x_1, \dots, x_n) is stable, or it has at least one genus 0 component with one or two special points. In the latter case this component can be contracted into a point. For the curve thus obtained we can once again ask ourselves if it is stable or not, and if not find another component to contract. Since the number of irreducible components decreases with each operation, in the end we will obtain a stable curve $(\hat{C}, \hat{x}_1, \dots, \hat{x}_n)$ together with a *stabilization map* $(C, x_1, \dots, x_{n+m}) \rightarrow (\hat{C}, \hat{x}_1, \dots, \hat{x}_n)$.

Definition 2.1. The *forgetful map* $p: \overline{\mathcal{M}}_{g,n+m} \rightarrow \overline{\mathcal{M}}_{g,n}$ is the map that assigns to a curve (C, x_1, \dots, x_{n+m}) the stabilization of the curve (C, x_1, \dots, x_n) .

The following picture illustrates the action of $p: \overline{\mathcal{M}}_{3,8} \rightarrow \overline{\mathcal{M}}_{3,2}$.



Proposition 2.2. *The universal curve $\bar{\mathcal{C}}_{g,n} \rightarrow \bar{\mathcal{M}}_{g,n}$ and the forgetful map $\bar{\mathcal{M}}_{g,n+1} \rightarrow \bar{\mathcal{M}}_{g,n}$ are isomorphic as families over $\bar{\mathcal{M}}_{g,n}$.*

Proof. A point $t \in \bar{\mathcal{M}}_{g,n+1}$ encodes a stable curve (C, x_1, \dots, x_{n+1}) . Denote by $(\hat{C}, \hat{x}_1, \dots, \hat{x}_n)$ the stabilization of (C, x_1, \dots, x_n) and by $y \in \hat{C}$ the image of x_{n+1} under the stabilization. Then $(\hat{C}, \hat{x}_1, \dots, \hat{x}_n)$ is a point in $\bar{\mathcal{M}}_{g,n}$ and the pair $((\hat{C}, \hat{x}_1, \dots, \hat{x}_n), y)$ is an element of $\bar{\mathcal{C}}_{g,n}$.

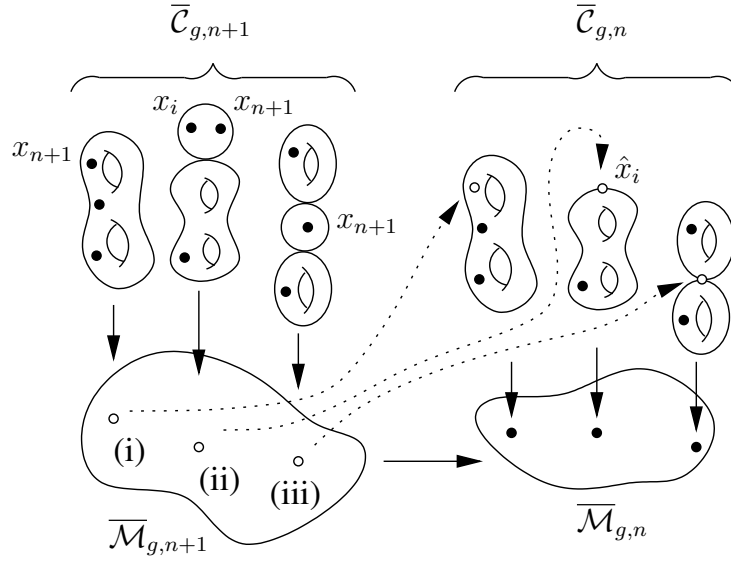
To understand this isomorphism more precisely, let us distinguish three cases.

(i) Suppose the curve (C, x_1, \dots, x_n) is stable. Then

$$(\hat{C}, \hat{x}_1, \dots, \hat{x}_n) = (C, x_1, \dots, x_n).$$

In this case $y = x_{n+1}$ on the curve $C = \hat{C}$.

(ii) Suppose x_{n+1} lies on a genus 0 component C_0 of C that contains another marked point x_i , a node, and no other special points. Then \hat{C} is obtained from C by contracting the component C_0 , and \hat{x}_i is the image of C_0 . In this case, $y = \hat{x}_i$.



(iii) Finally, suppose x_{n+1} lies on a genus 0 component C_0 of C that, in addition, contains two nodes, and no other special points. Then \hat{C} is obtained from C by contracting the component C_0 . In this case, y is the image of C_0 and it is a node of \hat{C} .

It is easy to construct the inverse map and thus to prove that we have constructed an isomorphism.

The above figure shows three points in $\overline{\mathcal{M}}_{g,n+1}$ and their images in $\overline{\mathcal{C}}_{g,n}$. \square

The following exercises show some uses of forgetful maps.

Exercise 2.3. Show the following cohomological relation in $H^2(\overline{\mathcal{M}}_{0,6}, \mathbb{Q})$:

$$\begin{aligned}
 & \left[\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \end{array} \right] + \left[\begin{array}{c} \text{Diagram 4} \\ \text{Diagram 5} \\ \text{Diagram 6} \end{array} \right] + \left[\begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \\ \text{Diagram 9} \end{array} \right] + \left[\begin{array}{c} \text{Diagram 10} \\ \text{Diagram 11} \\ \text{Diagram 12} \end{array} \right] \\
 = & \left[\begin{array}{c} \text{Diagram 13} \\ \text{Diagram 14} \\ \text{Diagram 15} \end{array} \right] + \left[\begin{array}{c} \text{Diagram 16} \\ \text{Diagram 17} \\ \text{Diagram 18} \end{array} \right] + \left[\begin{array}{c} \text{Diagram 19} \\ \text{Diagram 20} \\ \text{Diagram 21} \end{array} \right] + \left[\begin{array}{c} \text{Diagram 22} \\ \text{Diagram 23} \\ \text{Diagram 24} \end{array} \right]
 \end{aligned}$$

Here each picture represents the boundary divisor whose points encode the curves as shown or more degenerate stable curves. The equality is between the Poincaré dual cohomology classes of these divisors.

Exercise 2.4. Let $\delta_{\text{irr}} \in \overline{\mathcal{M}}_{1,n}$ be the boundary divisor consisting of stable curves with at least one non-separating node:

$$\delta_{\text{irr}} = \left[\begin{array}{c} \text{Diagram of a genus 1 curve with a non-separating node and marked points} \end{array} \right]$$

By abuse of notation we also write $\delta_{\text{irr}} \in H^2(\overline{\mathcal{M}}_{1,n})$ for the Poincaré dual cohomology class. Show that $\delta_{\text{irr}}^2 = 0$.

2.1.2 Attaching maps

Let $I \sqcup J$ be a partition of the set $\{1, \dots, n+2\}$ into two disjoint subsets such that $n+1 \in I$, $n+2 \in J$. Choose two integers g_1 and g_2 in such a way that $g_1 + g_2 = g$. Denote by $\overline{\mathcal{M}}_{g_1, I}$ the moduli space of stable curves whose marked points are labeled by the elements of I , and likewise for $\overline{\mathcal{M}}_{g_2, J}$.

Definition 2.5. The *attaching map of separating kind* $q: \overline{\mathcal{M}}_{g_1, I} \times \overline{\mathcal{M}}_{g_2, J} \rightarrow \overline{\mathcal{M}}_{g, n}$ assigns to two stable curves the stable curve obtained by identifying the marked points with numbers $n + 1$ and $n + 2$.

The *attaching map of nonseparating kind* $q: \overline{\mathcal{M}}_{g-1, n+2} \rightarrow \overline{\mathcal{M}}_{g, n}$ assigns to a stable curve the stable curve obtained by identifying the marked points with numbers $n + 1$ and $n + 2$.

2.1.3 Tautological rings: preliminaries

We will now start introducing tautological classes. The term “tautological classes” was introduced by D. Mumford [31] along with a definition of the λ -classes and κ -classes for moduli spaces without marked points. These classes (that we introduce in Section 2.3) were later re-defined by S. Morita in the topological setting [30]. The ψ -classes (that we introduce in Section 2.2) were first defined by E. Miller in [27] and became truly important after E. Witten formulated his conjecture [35] on their intersection numbers. We explain this conjecture in Section 4.

Before going into details let us give a definition that motivates the appearance of these classes.

Definition 2.6. The minimal family of subrings $R^*(\overline{\mathcal{M}}_{g, n}) \subset H^*(\overline{\mathcal{M}}_{g, n})$ stable under the push-forwards by all forgetful and attaching maps is called the family of *tautological rings* of the moduli spaces of stable curves.

Thus $1 \in H^0(\overline{\mathcal{M}}_{g, n})$ lies in the tautological ring (since a subring contains the unit element by definition), the classes represented by boundary strata lie in the tautological ring (since they are images of 1 under attaching maps), the self-intersection of a boundary stratum lies in the tautological ring, and so on. Now we will give an explicit construction of other tautological classes.

The relative cotangent line bundle. Let $p: \overline{\mathcal{C}}_{g, n} \rightarrow \overline{\mathcal{M}}_{g, n}$ be the universal curve and $\Delta \subset \overline{\mathcal{C}}_{g, n}$ the set of nodes in the singular fibers. Over $\overline{\mathcal{C}}_{g, n} \setminus \Delta$ there is a holomorphic line bundle \mathcal{L} cotangent to the fibers of the universal curve. We are going to extend this line bundle to the whole universal curve. To do that, it is enough to consider the local picture $p: (x, y) \mapsto xy$ (see Section 1.4.4). In coordinates (x, y) , the line bundle \mathcal{L} is generated by the sections $\frac{dx}{x}$ and $\frac{dy}{y}$ modulo the relation $\frac{d(xy)}{xy} = \frac{dx}{x} + \frac{dy}{y} = 0$. Since the restriction of the 1-form $d(xy)$ on every fiber of p vanishes, the line bundle

thus obtained is indeed identified with the cotangent line bundle to the fibers of $\overline{\mathcal{C}}_{g,n}$.

Definition 2.7. The line bundle \mathcal{L} extended to the whole universal curve is called the *relative cotangent line bundle*.

The restriction of \mathcal{L} to a fiber C of the universal curve is a line bundle over C . If C is smooth, then $\mathcal{L}|_C$ is the cotangent line bundle and its holomorphic sections are the abelian differentials, that is, the holomorphic differential 1-forms. By extension, the holomorphic sections of $\mathcal{L}|_C$ are called abelian differentials for any stable curve. They can be described as follows.

Definition 2.8. An *abelian differential on a stable curve C* is a meromorphic 1-form α on each component of C satisfying the following properties: (i) the only poles of α are at the nodes of C , (ii) the poles are at most simple, (iii) the residues of the poles on two branches meeting at a node are opposite to each other.

Example 2.9. The figure in Example 1.48 represents a stable curve obtained by identifying two points of the Riemann sphere. On the Riemann sphere we introduce the coordinate z such that the marked point is situated at $z = 1$, while the identified points are $z = 0$ and $z = \infty$. In this coordinate, the abelian differentials on the curve have the form $\lambda \frac{dz}{z}$. The residues of this differential at 0 and ∞ equal λ and $-\lambda$ respectively.

Exercise 2.10. Show that the space of abelian differentials on any genus g stable curve with marked points has dimension g as in the case of a smooth curve.

It follows from standard algebraic-geometric arguments (see [19], Exercise 5.8) that, since the space of abelian differentials has the same dimension for each fiber of the universal curve, these spaces actually form a rank g holomorphic vector bundle over $\overline{\mathcal{M}}_{g,n}$.

Definition 2.11. The *Hodge bundle* Λ is the rank g vector bundle over $\overline{\mathcal{M}}_{g,n}$ whose fiber over $t \in \overline{\mathcal{M}}_{g,n}$ is constituted by the abelian differentials on the curve C_t .

Remark 2.12. More generally, when we speak about meromorphic forms on a stable curve with poles of orders k_1, \dots, k_n at the marked points x_1, \dots, x_n , we will actually mean meromorphic sections of \mathcal{L} with poles as above, or, in

algebraic-geometric notation, the sections of $\mathcal{L}(\sum k_i x_i)$. In other words, in addition to the poles at the marked points, we allow the 1-forms to have simple poles at the nodes with opposite residues on the two branches.

Exercise 2.13. Given a list of n nonnegative integers k_1, \dots, k_n find the dimension of the space of sections of the line bundle $\mathcal{L}(\sum k_i x_i)$ restricted to a stable curve C .

2.2 The ψ -classes

Definition of ψ -classes. First we construct n holomorphic line bundles $\mathcal{L}_1, \dots, \mathcal{L}_n$ over $\overline{\mathcal{M}}_{g,n}$. The fiber of \mathcal{L}_i over a point $x \in \overline{\mathcal{M}}_{g,n}$ is the *cotangent line to the curve C_x at the i th marked point*. More precisely, let $s_i: \overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{C}}_{g,n}$ be the section corresponding to the i th marked point (so that $p \circ s_i = \text{Id}$). Then $\mathcal{L}_i = s_i^*(\mathcal{L})$.

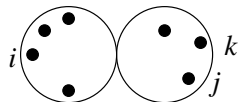
Definition 2.14. The ψ -classes are the first Chern classes of the line bundles \mathcal{L}_i ,

$$\psi_i = c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

2.2.1 Expressing ψ_i as a sum of divisors for $g = 0$

Over $\overline{\mathcal{M}}_{0,n}$ it is possible to construct an explicit section of the line bundle \mathcal{L}_i and to express its first Chern class ψ_i as a linear combination of divisors.

For pairwise distinct $i, j, k \in \{1, \dots, n\}$, denote by $\delta_{i|jk}$ the set of stable genus 0 curves with a node separating the i th marked point from the j th and k th marked points.



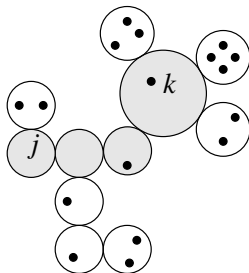
The set $\delta_{i|jk}$ is a divisor on $\overline{\mathcal{M}}_{0,n}$ and we denote by $[\delta_{i|jk}] \in H^2(\overline{\mathcal{M}}_{0,n})$ its Poincaré dual cohomology class.

Proposition 2.15. On $\overline{\mathcal{M}}_{0,n}$ we have $\psi_i = [\delta_{i|jk}]$ for any j, k .

Proof. We construct an explicit meromorphic section α of the dual cotangent line bundle \mathcal{L} over the universal curve. Its restriction to the i th section s_i of the universal curve will give us a holomorphic section of \mathcal{L}_i . The class ψ_i is then represented by the divisor of its zeroes.

The meromorphic section α of \mathcal{L} is constructed as follows. On each fiber of the universal curve (*i.e.*, on each stable curve) there is a unique meromorphic 1-form (in the sense of Remark 2.12) with simple poles at the j th and the k th marked points with residues 1 and -1 respectively. This form gives us a section of \mathcal{L} on each stable curve. Their union is the section α of \mathcal{L} over the whole universal curve.

In order to determine the zeroes of the restriction $\alpha|_{s_i}$ let us study α in more detail. A stable curve C of genus 0 is a tree of spheres. One of the spheres contains the j th marked point, another one (possibly the same) contains the k th marked point. There is a chain of spheres connecting these two spheres (shown in grey in the figure).



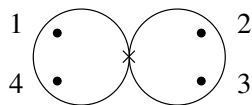
On every sphere of the chain, the 1-form $\alpha|_C$ has two simple poles: one with residue 1 (at the j th marked point or at the node leading to the j th marked point) and one with residue -1 (at the k th marked point or at the node leading to the k th marked point). The 1-form vanishes on the spheres that do not belong to the chain. Thus α determines a nonvanishing cotangent vector at the i th marked point if and only if the i th marked point lies on the chain. In other words, $\alpha|_{s_i}$ vanishes if and only if the curve C contains a node that separates the i th marked point from the j th and the k th marked points. But this is precisely the description of $\delta_{i|jk}$. By a local coordinates computation it is possible to check that $\alpha|_{s_i}$ has a *simple* zero along $\delta_{i|jk}$.

We conclude that the divisor $\delta_{i|jk}$ represents the class ψ_i . \square

Example 2.16. We have

$$\int_{\mathcal{M}_{0,4}} \psi_1 = 1,$$

because the divisor $\delta_{1|23}$ is composed of exactly one point corresponding to the curve:



Example 2.17. Let us compute the integral

$$\int_{\overline{\mathcal{M}}_{0,5}} \psi_1 \psi_2.$$

It is possible to express both classes in divisors and then study the intersection of these divisors, but this method is rather complicated, because it involves a struggle with self-intersections. A better idea is to express the ψ -classes in terms of divisors one at a time. We have

$$\psi_1 = [\delta_{1|23}] = \left[\begin{array}{c} 4 \\ \circ \quad \circ \\ 1 \quad 2 \\ \circ \quad \circ \\ 5 \quad 3 \end{array} \right] + \left[\begin{array}{c} 4 \\ \circ \quad \circ \\ 1 \quad 5 \\ \circ \quad \circ \\ 5 \quad 3 \end{array} \right] + \left[\begin{array}{c} 4 \\ \circ \quad \circ \\ 1 \quad 4 \\ \circ \quad \circ \\ 5 \quad 3 \end{array} \right]$$

Now we must compute the integral of ψ_2 over $\delta_{1|23}$. Each of the three components of $\delta_{1|23}$ is isomorphic to $\overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{0,4}$, and we see that ψ_2 is the pull-back of a ψ -class either from $\overline{\mathcal{M}}_{0,3}$ (for the first component) or from $\overline{\mathcal{M}}_{0,4}$ (for the second and the third components). In the first case, the integral of ψ_2 vanishes, while in the second and the third cases it is equal to 1 according to Example 2.16. We conclude that

$$\int_{\overline{\mathcal{M}}_{0,5}} \psi_1 \psi_2 = 2.$$

Exercise 2.18. Let X be a smooth algebraic manifold, $D \subset X$ a divisor and $L \rightarrow X$ and line bundle. Show that $c_1(L) \cap D = c_1(L|_D)$. This property was implicitly used in the last computation.

Exercise 2.19. Compute the integrals

$$\int_{\overline{\mathcal{M}}_{0,5}} \psi_1^2; \quad \int_{\overline{\mathcal{M}}_{0,6}} \psi_1^3; \quad \int_{\overline{\mathcal{M}}_{0,6}} \psi_1^2 \psi_2; \quad \int_{\overline{\mathcal{M}}_{0,6}} \psi_1 \psi_2 \psi_3.$$

Exercise 2.20. Show that

$$\int_{\overline{\mathcal{M}}_{0,n}} \psi_1^{n-3} = 1.$$

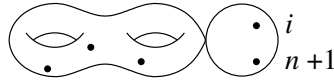
for any $n \geq 3$.

Exercise 2.21. Show that in $H^2(\overline{\mathcal{M}}_{0,n})$ we have

$$(n-1) \sum_{i=1}^n \psi_i = \sum_{I \sqcup J = \{1, \dots, n\}} |I| \cdot |J| \cdot [\delta_{I|J}].$$

Here the sum is taken over all unordered decompositions of the set $\{1, \dots, n\}$ into two subsets I and J with at least two elements; $\delta_{I|J}$ is the boundary divisor of curves with a node separating the markings in I from the markings in J ; and $[\delta_{I|J}]$ is its Poincaré dual cohomology class. Hint: compute the sum $\sum_{i,j,k} [\delta_{i|jk}]$ in two different ways.

Exercise 2.22. Let p be the forgetful map $p: \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$. Consider the set of stable curves that contain a spherical component with exactly three special points: a node and the marked points number i and $n+1$.



The points encoding such curves form a divisor $\delta_{(i,n+1)}$ of $\overline{\mathcal{M}}_{g,n+1}$. Now we can consider the class ψ_i ($1 \leq i \leq n$) both on $\overline{\mathcal{M}}_{g,n}$ and on $\overline{\mathcal{M}}_{g,n+1}$. We have

$$\psi_i - p^*(\psi_i) = [\delta_{(i,n+1)}].$$

Exercise 2.23. Let D_i be the divisor of the i th special section in the universal curve $p: \overline{\mathcal{C}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$. Then we have $p_*(D_i^{k+1}) = (-\psi_i)^k$.

The last two exercises are discussed in [35], Section 2b.

2.2.2 Modular forms and the class ψ_1 on $\overline{\mathcal{M}}_{1,1}$

Recall that a lattice $L \subset \mathbb{C}$ is a discrete additive subgroup of \mathbb{C} isomorphic to \mathbb{Z}^2 .

Definition 2.24. A *modular form of weight* $k \in \mathbb{N}$ is a function F on the set of lattices such that (i) $F(cL) = F(L)/c^k$ for $c \in \mathbb{C}^*$ and (ii) the function $f(\tau) = F(\mathbb{Z} + \tau\mathbb{Z})$ is holomorphic on the upper half-plane $\text{Im } \tau > 0$, and (iii) $f(\tau)$ is bounded on the half-plane $\text{Im } \tau \geq C$ for any positive constant C .

Since the lattice $\mathbb{Z} + \tau\mathbb{Z}$ is the same as $\mathbb{Z} + (\tau + 1)\mathbb{Z}$, the function f is periodic with period 1. Therefore there exists a function $\varphi(q)$, holomorphic on the open punctured unit disc, such that $f(\tau) = \varphi(e^{2\pi i\tau})$. This function is bounded at the neighborhood of the origin, therefore it can be extended to a holomorphic function on the whole unit disc and expanded into a power series in q at 0, which is the usual way to represent a modular form.

Since $-L = L$ for every lattice L , we see that there are no nonzero modular forms of odd weight. On the other hand, there exists a nonzero modular form of any even weight $k \geq 4$, given by

$$E_k(L) = \sum_{z \in L \setminus \{0\}} \frac{1}{z^k}.$$

(For odd k this sum vanishes, while for $k = 2$ it is not absolutely convergent.) The value of the corresponding function $\varphi_k(q)$ at $q = 0$ is equal to

$$\varphi_k(0) = \lim_{\text{Im}\tau \rightarrow \infty} E_k(\mathbb{Z} + \tau\mathbb{Z}) = \sum_{z \in \mathbb{Z} \setminus \{0\}} \frac{1}{z^k} = 2\zeta(k).$$

The relation between modular forms and the ψ -class on $\overline{\mathcal{M}}_{1,1}$ comes from the following proposition.

Proposition 2.25. *The space of modular forms of weight k is naturally identified with the space of holomorphic sections of $\mathcal{L}_1^{\otimes k}$ over $\overline{\mathcal{M}}_{1,1}$.*

Proof. Let F be a modular form of weight k . We claim that $F(L)dz^k$ is a well-defined holomorphic section of $\mathcal{L}_1^{\otimes k}$ over $\overline{\mathcal{M}}_{1,1}$.

First of all, if \mathbb{C}/L is any elliptic curve, then the value of $F(L)dz^k$ at the marked point (the image of $0 \in \mathbb{C}$) is indeed a differential k -form, that is, an element of the fiber of $\mathcal{L}_1^{\otimes k}$. If we apply a homothety $z \mapsto cz$, replacing L by cL , we obtain an isomorphic elliptic curve. However, the k -form $F(L)dz^k$ does not change, because $F(L)$ is divided by c^k , while dz^k is multiplied by c^k . Thus $F(L)dz^k$ is a well-defined section of $\mathcal{L}_1^{\otimes k}$.

The fact that this section is holomorphic over $\mathcal{M}_{1,1}$ follows from the fact that $f(\tau)$ is holomorphic. The fact that it is also holomorphic at the boundary point follows from the fact the function $\varphi(q)$ is holomorphic at $q = 0$.

Conversely, if s is a holomorphic section of $\mathcal{L}_1^{\otimes k}$, then taking the value of s over the curve \mathbb{C}/L and dividing by dz^k , we obtain a function on lattices L . The same argument as above shows that it is a modular form of weight k . \square

Proposition 2.26. *We have*

$$\int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = \frac{1}{24}.$$

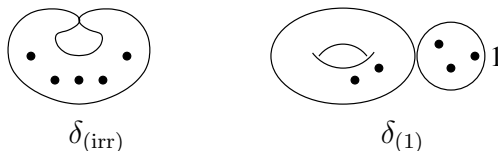
Proof. We are going to give three similar computations leading to the same result. Denote by $f_k(\tau)$ and $\varphi_k(q)$ the functions associated with the modular form E_k . One can check (see, for instance [32], chapter VII) that in the modular figure (*i.e.*, on $\mathcal{M}_{1,1}$) the function f_4 has a unique simple zero at $\tau = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, while f_6 has a unique simple zero at $\tau = i$. Further, the function

$$\left(\frac{\varphi_4}{2\zeta(4)} \right)^3 - \left(\frac{\varphi_6}{2\zeta(6)} \right)^2$$

has a unique zero at $q = 0$. The stabilizers of the corresponding points in $\overline{\mathcal{M}}_{1,1}$ are $\mathbb{Z}/6\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, and $\mathbb{Z}/2\mathbb{Z}$ respectively (see Example 1.33). Thus the first Chern class of $\mathcal{L}_1^{\otimes 4}$ equals $1/6$, that of $\mathcal{L}_1^{\otimes 6}$ equals $1/4$, that of $\mathcal{L}_1^{\otimes 12}$ equals $1/2$. In every case we find that the first Chern class of \mathcal{L}_1 equals $\psi_1 = 1/24$. \square

Exercise 2.27. Show that $\tau = i$ is indeed a zero of f_6 while $\tau = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ indeed a zero of f_4 .

Exercise 2.28. Denote by $\delta_{(\text{irr})}, \delta_{(1)} \subset \overline{\mathcal{M}}_{1,n}$ the divisors



In other words, the points of $\delta_{(\text{irr})}$ encode curves with at least one nonseparating node; the points of $\delta_{(1)}$ encode curves with a separating node dividing the curve into a stable curve of genus 1 and a stable curve of genus 0 containing the marked point number 1. Show that in $H^2(\overline{\mathcal{M}}_{1,n})$ we have the equality

$$\psi_1 = \frac{1}{12}[\delta_{(\text{irr})}] + [\delta_{(1)}].$$

2.3 Other tautological classes

All cohomology classes we consider are with rational coefficients.

2.3.1 The classes on the universal curve

On the universal curve we define the following classes.

- D_i is the divisor given by the i th section of the universal curve. In other words, the intersection of D_i with a fiber C of $\bar{\mathcal{C}}_{g,n}$ is the i th marked point on C . By abuse of notation we denote by $D_i \in H^2(\bar{\mathcal{C}}_{g,n})$ the cohomology class Poincaré dual to the divisor.
- $D = \sum_{i=1}^n D_i$.
- $K = c_1(\mathcal{L}^{\log}) \in H^2(\bar{\mathcal{C}}_{g,n})$, where $\mathcal{L}^{\log} = \mathcal{L}(D)$ is the line bundle \mathcal{L} twisted by the divisor D .
- Δ is the codimension 2 subvariety of $\bar{\mathcal{C}}_{g,n}$ consisting of the nodes of the singular fibers. By abuse of notation, $\Delta \in H^4(\bar{\mathcal{C}}_{g,n})$ will also denote the Poincaré dual cohomology class.
- Let N be the normal vector bundle to Δ in $\bar{\mathcal{C}}_{g,n}$. Then we set

$$\Delta_{k,l} = (-c_1(N))^k \Delta^{l+1}.$$

To simplify the notation, we introduce two symbols ν_1 and ν_2 with the convention $\nu_1 + \nu_2 = -c_1(N)$, $\nu_1\nu_2 = c_2(N)$. In other words, ν_1 and ν_2 are the Chern roots of N^\vee (see Definition 3.3). Since $c_2(N)\Delta = \Delta^2$, we also identify $\nu_1\nu_2$ with Δ . Thus, even though the symbols ν_1 and ν_2 separately are meaningless, every symmetric polynomial in ν_1 and ν_2 divisible by $\nu_1\nu_2$ determines a well-defined cohomology class. For instance, we have

$$\Delta_{k,l} = \Delta \cdot (\nu_1 + \nu_2)^k (\nu_1\nu_2)^l = (\nu_1 + \nu_2)^k (\nu_1\nu_2)^{l+1}.$$

Since Δ is the set of nodes in the singular fibers of $\bar{\mathcal{C}}_{g,n}$, it has a natural 2-sheeted (unramified) covering $p: \tilde{\Delta} \rightarrow \Delta$ whose points are couples (node + choice of a branch). Over $\tilde{\Delta}$ we can define two natural line bundles \mathcal{L}_α and \mathcal{L}_β cotangent, respectively, to the first and to the second branch at the node. The pull-back p^*N of N to $\tilde{\Delta}$ is naturally identified with $\mathcal{L}_\alpha^\vee \oplus \mathcal{L}_\beta^\vee$. Thus, if $P(\nu_1, \nu_2)$ is a symmetric polynomial, we have $p^*(\Delta \cdot P(\nu_1, \nu_2)) = \tilde{\Delta} \cdot P(c_1(\mathcal{L}_\alpha), c_1(\mathcal{L}_\beta))$.

2.3.2 Intersecting classes on the universal curve

Proposition 2.29. *For all $1 \leq i, j \leq n$, $i \neq j$ we have*

$$KD_i = D_i D_j = K\Delta = D_i \Delta = 0 \in H^*(\bar{\mathcal{C}}_{g,n}).$$

Proof. The divisors D_i and D_j do not intersect, so the intersection of the corresponding classes vanishes. Similarly, the divisor D_i does not meet Δ , so their intersection vanishes. The restriction of the line bundle \mathcal{L}^{\log} to D_i is trivial. Indeed, the sections of \mathcal{L}^{\log} are 1-forms with simple poles at the marked points, and the fiber at the marked point is the line of residues, so it is canonically identified with \mathbb{C} . The intersection KD_i is the first Chern class of the restriction of \mathcal{L}^{\log} to D_i . Therefore it vanishes. The restriction of \mathcal{L}^{\log} to Δ is not necessarily trivial. However its pull-back to the double-sheeted covering $\tilde{\Delta}$ is trivial (because the fiber is the line of residues identified with \mathbb{C}). Alternatively, one can say that $(\mathcal{L}^{\log})^{\otimes 2}$ is trivial. Therefore $K\Delta = 0$. \square

Remark 2.30. A line bundle whose tensor power is trivial is called *rationally trivial*. Although it is not necessarily trivial itself, all its characteristic classes over \mathbb{Q} vanish. This is the case of $\mathcal{L}|_{\Delta} = \mathcal{L}^{\log}|_{\Delta}$.

Corollary 2.31. *Every polynomial in the classes D_i , K , $\Delta_{k,l}$ on $\bar{\mathcal{C}}_{g,n}$ can be written in the form*

$$P_K(K) + \sum_{i=1}^n P_i(D_i) + \Delta \cdot P_{\Delta}(\nu_1, \nu_2),$$

while P_K and P_i , $1 \leq i \leq n$ are arbitrary polynomials, while P_{Δ} is a symmetric polynomial with the convention $\nu_1 \nu_2 = \Delta$, $\nu_1 + \nu_2 = -c_1(N)$.

Proof. Given a polynomial in D_i , K , $\Delta_{k,l}$, we can, according to the proposition, cross out the “mixed terms”, that is, the monomials containing products $D_i D_j$, $D_i K$, $K \Delta_{k,l}$ or $D_i \Delta_{k,l}$. We end up with a sum of powers of D_i , powers of K , and products of $\Delta_{k,l}$. Now, by definition, $\Delta_{k_1, l_1} \Delta_{k_2, l_2} = \Delta_{k_1+k_2, l_1+l_2+1}$. Therefore a polynomial in the variables $\Delta_{k,l}$ can be rewritten in the form $\Delta P_{\Delta}(\nu_1, \nu_2)$, where P_{Δ} is a symmetric polynomial. \square

2.3.3 The classes on the moduli space

Let $p: \overline{\mathcal{C}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$ be the universal curve. On the moduli space $\overline{\mathcal{M}}_{g,n}$ we define the following classes.

- $\kappa_m = p_*(K^{m+1}) \in H^{2m}(\overline{\mathcal{M}}_{g,n})$.
- $\psi_i = -p_*(D_i^2) \in H^2(\overline{\mathcal{M}}_{g,n})$.
- $\delta_{k,l} = p_*(\Delta_{k,l}) \in H^{k+2l+1}(\overline{\mathcal{M}}_{g,n})$.
- $\lambda_i = c_i(\Lambda) \in H^{2i}(\overline{\mathcal{M}}_{g,n})$, where Λ is the Hodge bundle and c_i the i th Chern class.

Thus, with the exception of the λ -classes, the tautological classes on $\overline{\mathcal{M}}_{g,n}$ are push-forwards of tautological classes on $\overline{\mathcal{C}}_{g,n}$ and their products.

Note that the new definition of ψ -classes coincides with Definition 2.14 by Exercise 2.23. Also note that our definition of κ -classes follows the convention of Arbarello and Cornalba [1].

Exercise 2.32. Describe the class $\delta_{0,0}$ more explicitly.

Exercise 2.33. Show that

$$\int_{\overline{\mathcal{M}}_{1,1}} \lambda_1 = \frac{1}{24}.$$

Theorem 2.34. *The classes ψ_i , κ_m , $\delta_{k,l}$, and λ_i lie in the tautological ring in the sense of Definition 2.6.*

Proof. Let $p: \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$ be the forgetful map. Then $\psi_i = -p_*(\delta_{(i,n+1)}^2)$, where $\delta_{(i,n+1)}$ is defined in Exercise 2.22, while $\kappa_m = p_*(\psi_{n+1}^{m+1})$. The class $\delta_{k,l}$ is the sum of push-forwards under the attaching maps of the class $(\psi_{n+1} + \psi_{n+2})^k (\psi_{n+1} \psi_{n+2})^l$. Thus all these classes lie in the tautological ring. The class λ_i is expressed via the ψ -, κ -, and δ -classes in Theorem 3.27. It follows that it too lies in the tautological ring. \square

3 Algebraic geometry on moduli spaces

In the previous section we introduced a wide range of tautological classes on the moduli space $\overline{\mathcal{M}}_{g,n}$, namely, the ψ -, κ -, δ -, and λ -classes. Now we would like to learn how to compute all possible intersection numbers between these classes.

This is done in three steps.

First, by applying the Grothendieck–Riemann–Roch (GRR) formula we express λ -classes in terms of ψ -, κ -, and δ -classes. This gives us an opportunity to introduce the GRR formula and to give an example of its application in a concrete situation.

Second, by studying the pull-backs of the ψ -, κ -, and δ -classes under attaching and forgetful maps, we will be able to eliminate one by one the κ - and δ -classes from intersection numbers.

The remaining problem of computing intersection numbers of the ψ -classes is much more difficult. The answer was first conjectured by E. Witten [35]. It is formulated below in Theorems 4.4 and 4.5. Witten’s conjecture now has at least 5 different proofs (the most accessible to a non-specialist is probably [22]), and all of them use nontrivial techniques. In this note we will not prove Witten’s conjecture, but give its formulation and say a few words about how it appeared.

3.1 Characteristic classes and the GRR formula

In this section we present the Grothendieck–Riemann–Roch (GRR) formula. But first we recall the necessary information on characteristic classes of vector bundles, mostly without proofs.

3.1.1 The first Chern class

Definition 3.1. Let $L \rightarrow B$ be a holomorphic line bundle over a complex manifold B . Let s be a nonzero meromorphic section of L and $Z - P$ the associated divisor: the set of zeroes minus the set of poles of s . Then $[Z] - [P] \in H^2(B, \mathbb{Z})$ is called the *first Chern class* of L and denoted by $c_1(L)$.

Exercise 3.2. Show that the first Chern class is well-defined, *i.e.*, it does not depend on the choice of the section.

It turns out that $c_1(L)$ is a *topological* invariant of L . In other words, it only depends on the topological type of L and B , but not on the complex structure of B nor on the holomorphic structure of L . Actually, there exists a different definition of first Chern classes using connections and their curvatures. This definition does not involve the holomorphic structure at all. In particular, it implies that if a vector bundle can be endowed with a flat connection, then all its characteristic classes vanish.

3.1.2 Total Chern class, Todd class, Chern character

Let $V \rightarrow B$ be a vector bundle of rank k .

Definition 3.3. We say that V can be *exhausted* by line bundles if we can find a line subbundle L_1 of V , then a line subbundle L_2 of the quotient $V_1 = V/L_1$, then a line subbundle L_3 of the quotient $V_2 = V_1/L_2$, and so on, until the last quotient is itself a line bundle L_k . This is equivalent to asking that V has a complete flag of subbundles, with graded pieces L_1, \dots, L_k . The simplest case is when $V = \bigoplus L_i$.

If V is exhausted by line bundles, the first Chern classes $r_i = c_1(L_i)$ are called the *Chern roots* of V .

Definition 3.4. Let V be a vector bundle with Chern roots r_1, \dots, r_k . Its *total Chern class* is defined by

$$c(V) = \prod_{i=1}^k (1 + r_i);$$

its *Todd class* is defined by

$$\text{td}(V) = \prod_{i=1}^k \frac{r_i}{e^{r_i} - 1};$$

its *Chern character* is defined by

$$\text{ch}(V) = \sum_{i=1}^k e^{r_i}.$$

The homogeneous parts of degree i of these classes are denoted by c_i , td_i , and ch_i respectively.

If we know the total Chern class of a vector bundle, we can compute its Todd class and Chern character (except ch_0 , which is equal to the rank of the bundle). For instance, let us compute ch_3 . We have

$$\begin{aligned} \text{ch}_3 &= \frac{1}{6} \sum r_i^3 \\ &= \frac{1}{6} \left(\sum r_i \right)^3 - \frac{1}{2} \left(\sum r_i \right) \left(\sum_{i < j} r_i r_j \right) + \frac{1}{2} \sum_{i < j < k} r_i r_j r_k \\ &= \frac{1}{6} c_1^3 - \frac{1}{2} c_1 c_2 + \frac{1}{2} c_3. \end{aligned}$$

Exercise 3.5. Compute the expressions for ch_1 , ch_2 , td_1 , and td_2 in terms of Chern classes c_1 and c_2 .

Not every vector bundle can be exhausted by line bundles. The characteristic classes are defined in full generality by using the following proposition, also called the *splitting principle*.

Proposition 3.6. *For every vector bundle $V \rightarrow B$ there exists a proper map of complex manifolds $p: B' \rightarrow B$ such that $p^*(V)$ can be exhausted by line bundles and the induced morphism $p^*: H^*(B, \mathbb{Z}) \rightarrow H^*(B', \mathbb{Z})$ in cohomology is an injection.*

Sketch of proof. Let $V \rightarrow B$ be a vector bundle and $p: \mathbb{P}(V) \rightarrow B$ its projectivization. Then the tautological line bundle over $\mathbb{P}(V)$ is a subbundle of $p^*(V)$ and p induces an injection from $H^*(B, \mathbb{Z})$ to $H^*(\mathbb{P}(V), \mathbb{Z})$. To prove the proposition we apply this construction several times. \square

Thus, for instance, $c(V)$ is uniquely defined by the condition $p^*(c(V)) = c(p^*V)$, and similarly for the Todd class and the Chern character. This way of defining characteristic classes dates back to Grothendieck.

Remark 3.7. In general, given a power series f , one can assign to every vector bundle the corresponding *character* and *class*. The character is given by $\sum f(r_i)$, while the class is given by $\prod f(r_i)$. Thus for an exact sequence

$$0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow 0$$

of vector bundles, we have

$$\text{class}(V_2) = \text{class}(V_1) \cdot \text{class}(V_3),$$

$$\text{character}(V_2) = \text{character}(V_1) + \text{character}(V_3).$$

For instance, the Todd class is defined using the series

$$f(x) = \frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{n \geq 1} \frac{B_{2n}}{(2n)!} x^{2n},$$

where B_{2n} are the Bernoulli numbers:

$$B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730}, \dots$$

One of the standard references for characteristic classes is [28]; for an algebraic-geometric presentation see [13], Chapter 4.

Exercise 3.8. Let $\delta_{\text{irr}} \subset \overline{\mathcal{M}}_{g,n}$ be the divisor of curves with at least one nonseparating node. Show that $\lambda_g|_{\delta_{\text{irr}}} = 0$.

Exercise 3.9. Find the mistake in the following “proof”.

Let C be a smooth curve. It is well-known that the fiber of Λ^\vee over C contains the lattice $H_1(C, \mathbb{Z})$, because every cycle in $H_1(C, \mathbb{Z})$ represents a linear form on the space of holomorphic differentials on C . The presence of this lattice insures that there is a flat connection on the vector bundle Λ^\vee over $\mathcal{M}_{g,n}$. (The lattice is no longer well-defined on the boundary of $\overline{\mathcal{M}}_{g,n}$, so the flat connection does not extend to the whole $\overline{\mathcal{M}}_{g,n}$.) It follows that the restrictions of all the classes λ_i to $\mathcal{M}_{g,n}$ vanishes.

Exercise 3.10. Show that the restriction of λ_g to $\mathcal{M}_{g,n}$ vanishes using a corrected version of the wrong proof from the previous exercise.

3.1.3 Cohomology spaces of vector bundles

Our use of cohomology spaces of vector bundles is very limited, so we only introduce them briefly. See [17], Chapter 0 for more details.

To a holomorphic vector bundle V over a smooth compact complex algebraic variety B one assigns its Dolbeault cohomology spaces, $H^k(B, V)$, $k = 0, 1, \dots, \dim B$. These are the cohomology groups $\text{Ker } \bar{\partial} / \text{Im } \bar{\partial}$ of the complex

$$0 \xrightarrow{\bar{\partial}} \mathcal{A}^{0,0}(V) \xrightarrow{\bar{\partial}} \mathcal{A}^{0,1}(V) \xrightarrow{\bar{\partial}} \mathcal{A}^{0,2}(V) \xrightarrow{\bar{\partial}} \dots$$

The terms of this complex are the spaces of smooth V -valued differential forms of type $(0, k)$. In local coordinates, such a differential form has the form

$$\sum s_{i_1, \dots, i_k}(z) d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_k},$$

where s_{i_1, \dots, i_k} is a smooth section of V . Obviously, $H^k(B, V) = 0$ for $k > \dim B$. We will use only two properties of the cohomology groups.

First, $H^0(V)$ is the space of holomorphic sections of V . This follows directly from the definition.

Second, let K be the *canonical* line bundle over B , in other words the highest exterior power of the cotangent vector bundle. Let V^\vee be the dual vector bundle of V . Then $H^k(B, V)$ and $H^{\dim B - k}(B, K \otimes V^\vee)$ are dual vector spaces. This property is called the *Serre duality*.

These definitions, which we gave in the case of a smooth complex manifold B , can be generalized to any projective algebraic variety. In particular, the Serre duality holds on a stable curve C if we replace the canonical line bundle K by $\mathcal{L}|_C$.

3.1.4 K^0 , p_* , and $p_!$

Let X be a complex manifold. Consider the free abelian group generated by the vector bundles over X . To every short exact sequence

$$0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow 0$$

we assign the relation $V_1 - V_2 + V_3 = 0$ in this group. The *Grothendieck group* $K^0(X)$ is the factor of the free group by all such relations.

We would obtain the same group if the vector bundles were replaced by coherent sheaves, because every coherent sheaf has a finite resolution by vector bundles.

The Chern character determines a group morphism $\text{ch}: K^0(X) \rightarrow H^*(X, \mathbb{Q})$ from the Grothendieck group to the cohomology group of X .

Let $p: X \rightarrow Y$ be a proper morphism of complex manifolds (that is, the inverse images of compact sets are compact). Then p induces a morphism

$$p_*: H^*(X, \mathbb{Q}) \rightarrow H^*(Y, \mathbb{Q})$$

of cohomology groups (the fiberwise integration). It is defined by $\langle p_*\alpha, C \rangle = \langle \alpha, p^{-1}(C) \rangle$ for any cycle $C \subset Y$ in general position.

On the other hand, p also determines a morphism $p_!: K^0(X) \rightarrow K^0(Y)$, that we now describe.

Denote by X_y the fiber over a point $y \in Y$. Then we can consider the cohomology spaces $H^k(X_y, V)$ (where V actually stands for the restriction of V to X_y). For each k , the vector spaces $H^k(X_y, V)$ can be glued together into a sheaf over Y . This sheaf is denoted by $R^k p_*(V)$. If the dimension of all cohomology groups $H^k(X_y, V)$ do not depend on the point y , then the sheaves $R^k p_*(V)$ are vector bundles and their fibers are actually equal to $H^k(p, V)$. We will only apply the GRR formula in this situation. In the general case, the construction of the sheaves $R^k p_*(V)$ is more complicated and the fibers of $R^k p_*(V)$ do not necessarily coincide with $H^k(X_y, V)$ at every point (see [19], Chapter III, Section 8).

The morphism $p_!$ of Grothendieck groups is now defined by

$$p_!(V) = R^0 p_*(V) - R^1 p_*(V) + \cdots .$$

Now we have the following diagram of morphisms.

$$\begin{array}{ccc}
 K^0(X) & \xrightarrow{\text{ch}} & H^*(X, \mathbb{Q}) \\
 p_! \downarrow & & \downarrow p_* \\
 K^0(Y) & \xrightarrow{\text{ch}} & H^*(Y, \mathbb{Q})
 \end{array}
 \quad \begin{array}{l}
 \text{td}(p) \\
 \curvearrowright \\
 \leftarrow
 \end{array}$$

The question is: is it commutative? The answer, given by the GRR formula, is that it is not, but can be made to commute if we add a multiplicative factor $\text{td}(p)$.

3.1.5 The Grothendieck–Riemann–Roch formula

Let $p: X \rightarrow Y$ be a morphism of complex manifolds with compact fibers. Let V be a vector bundle over X . Denote by

$$\text{td}(p) = \frac{\text{td}(T^\vee X)}{\text{td}(p^*(T^\vee Y))}.$$

Theorem 3.11 (GRR, see [13], Chapter 9). *We have*

$$\text{ch}(p_! \mathcal{F}) = p_* [\text{ch}(\mathcal{F}) \text{td}(p)].$$

Exercise 3.12. Apply the GRR formula to the situation $\mathcal{F} \rightarrow X \rightarrow \text{point}$, where X is a compact Riemann surface and \mathcal{F} is a sheaf.

Exercise 3.13. Apply the GRR formula to the situation $\mathcal{F} \rightarrow X \rightarrow X$, where X is a smooth complex manifold and the map p is the identity.

Exercise 3.14. Suppose $D \subset X$ is a divisor. Apply the GRR formula to the situation $\mathcal{O}_D \rightarrow D \hookrightarrow X$.

Exercise 3.15. Suppose $x \in X$ is a point. Apply the GRR formula to the situation $\mathcal{O}_x \rightarrow x \hookrightarrow X$.

3.1.6 The Koszul resolution

The Koszul resolution (see [7], Chapter 17) is a tool that we will need to apply the GRR to the universal curve, so let's introduce it here.

Let $p: V \rightarrow X$ be a vector bundle over a smooth complex manifold X . Then X is embedded in the total space of V by the zero section, $X \subset V$. On the total space V , consider the sheaf \mathcal{O}_X supported on X . Its sections over an open set $U \subset V$ are the holomorphic functions on $U \cap X$. Thus the fiber of \mathcal{O}_X over a point of X is \mathbb{C} , while its fiber over a point outside X is 0.

It is always unpleasant to work with sheaves that are not vector bundles, therefore we would like to construct a *resolution* of \mathcal{O}_X , in other words, an exact sequence of sheaves that contains \mathcal{O}_X , but whose all other terms are vector bundles.

Example 3.16. Suppose V is a line bundle. Then over the total space of V there are two sheaves: the sheaf \mathcal{O}_V of holomorphic functions and the sheaf $\mathcal{O}_V(X)$ of holomorphic functions vanishing on X . The short exact sequence

$$0 \rightarrow \mathcal{O}_V(-X) \rightarrow \mathcal{O}_V \rightarrow \mathcal{O}_X \rightarrow 0$$

is a resolution of \mathcal{O}_X .

Our aim is to generalize this example. Denote by \mathbf{V} the pull-back of the vector bundle V to the total space of V :

$$\begin{array}{ccc} \mathbf{V} & \longrightarrow & V \\ \downarrow & & \downarrow \\ V & \longrightarrow & X \end{array}$$

Thus \mathbf{V} is a vector bundle over the total space of V .

Denote by $\mathcal{A}^k(\mathbf{V})$ the sheaf of skew-symmetric k -forms on \mathbf{V} . In particular, $\mathcal{A}^1(\mathbf{V})$ is the sheaf of sections of \mathbf{V}^\vee , while $\mathcal{A}^0(\mathbf{V}) = \mathcal{O}_V$.

Let x be a point in X , $v \in V_x$ a point in the fiber V_x over x , and α be a skew-symmetric k -form on V_x . Then the form $i_v\alpha$ obtained by substituting v as the first entry of α is a skew-symmetric $(k-1)$ -form on V_x . Thus we obtain a natural sheaf morphism $d: \mathcal{A}^k(\mathbf{V}) \rightarrow \mathcal{A}^{k-1}(\mathbf{V})$ by substituting v into α .

Denote by p the rank of V .

Theorem 3.17. *The following sequence of sheaves is exact.*

$$0 \rightarrow \mathcal{A}^p(\mathbf{V}) \rightarrow \mathcal{A}^{p-1}(\mathbf{V}) \rightarrow \cdots \rightarrow \mathcal{A}^1(\mathbf{V}) \rightarrow \mathcal{O}_V \rightarrow \mathcal{O}_X \rightarrow 0.$$

This exact sequence is called the *Koszul resolution* of the sheaf \mathcal{O}_X .

Proof of Theorem 3.17 in the case of a rank 2 vector bundle. For simplicity we restrict ourselves to the case $p = 2$, since it is the only case that we will need.

Choose an open chart in X and a trivialization of the vector bundle V over this chart. Let $t = (t_1, t_2, \dots)$ be the local coordinates on the chart and (x, y) the coordinates in the fibers of V . The maps d of the Koszul resolution can be explicitly written out as

$$0 \longmapsto 0 \cdot dx \wedge dy, \tag{1}$$

$$f(t; x, y) dx \wedge dy \longmapsto -yf(t; x, y) dx + xf(t; x, y) dy, \tag{2}$$

$$g_x(t; x, y) dx + g_y(t; x, y) dy \longmapsto xg_x(t; x, y) + yg_y(t; x, y), \tag{3}$$

$$h(t; x, y) \longmapsto h(t; 0, 0), \quad (4)$$

$$h(t; 0, 0) \longmapsto 0. \quad (5)$$

We must check that the image of each map coincides with the kernel of the next map.

The image of the map (1) and the kernel the map (2) vanish.

The kernel of the map (3) is given by the condition $xg_x + yg_y = 0$. In particular, this implies that g_x is divisible by y , while g_y is divisible by x . Hence an element of the kernel lies in the image of the first map, since we can take $f = -g_x/y = g_y/x$. Conversely, if $g_x = -yf$ and $g_y = xf$, then $xg_x + yg_y = 0$, hence the image of the map (2) is included in the kernel of the map (3).

The kernel of map (4) and the image of the map (3) are composed of maps that vanish at $x = y = 0$.

Finally, the map (4) is surjective and the map (5) takes everything to zero. \square

Corollary 3.18. *Let $D \subset X$ be a smooth codimension 1 subvariety. We have*

$$\frac{1}{\text{td}(\mathcal{O}_D)} = \frac{D}{1 - e^{-D}}.$$

Proof. This is immediately obtained from the exact sequence

$$0 \rightarrow \mathcal{O}(-D) \rightarrow \mathcal{O} \rightarrow \mathcal{O}_D \rightarrow 0$$

by the multiplicative property of the Todd class.

Corollary 3.19. *Let $\Delta \subset X$ be a smooth codimension 2 subvariety. Denote by ν_1 and ν_2 the Chern roots of its normal vector bundle. We have*

$$\frac{1}{\text{td}(\mathcal{O}_\Delta)} = 1 + \Delta \sum_{k \geq 1} \frac{B_{2k}}{(2k)!} \frac{\nu_1^{2k-1} + \nu_2^{2k-1}}{\nu_1 + \nu_2},$$

where B_{2k} are the Bernoulli numbers (see Remark 3.7).

Proof. It is enough to consider a tubular neighborhood of Δ , which is isomorphic to a tubular neighborhood of Δ in its normal bundle N . (As we mentioned, the characteristic classes are topological invariants of vector bundles. Therefore the characteristic classes of a sheaf supported on Δ depend only on its resolution in the neighborhood of Δ in $\bar{\mathcal{C}}_{g,n}$, which is topologically the same as the neighborhood of Δ in the total space of N .)

The Koszul resolution gives us the following exact sequence of sheaves:

$$0 \rightarrow \bigwedge^2 N^\vee \rightarrow N^\vee \rightarrow \mathcal{O}_N \rightarrow \mathcal{O}_\Delta \rightarrow 0,$$

where \mathcal{O}_N is the sheaf of holomorphic functions on the total space of the vector bundle N . Hence we have

$$\begin{aligned} \frac{1}{\text{td}(\mathcal{O}_\Delta)} &= \frac{\text{td}(N^\vee)}{\text{td}(\bigwedge^2 N^\vee)} = \frac{\nu_1}{1 - e^{-\nu_1}} \cdot \frac{\nu_2}{1 - e^{-\nu_2}} \cdot \frac{1 - e^{-(\nu_1 + \nu_2)}}{\nu_1 + \nu_2} \\ &= \frac{\nu_1 \nu_2}{\nu_1 + \nu_2} \cdot \frac{1 - e^{-(\nu_1 + \nu_2)}}{(1 - e^{-\nu_1})(1 - e^{-\nu_2})} \\ &= \frac{\nu_1 \nu_2}{\nu_1 + \nu_2} \cdot \frac{1 - e^{-\nu_1} + e^{-\nu_1} - e^{-(\nu_1 + \nu_2)}}{(1 - e^{-\nu_1})(1 - e^{-\nu_2})} \\ &= \frac{\nu_1 \nu_2}{\nu_1 + \nu_2} \cdot \left[\frac{1}{1 - e^{-\nu_2}} + \frac{e^{-\nu_1}}{1 - e^{-\nu_1}} \right] \\ &= \frac{\nu_1 \nu_2}{\nu_1 + \nu_2} \cdot \left[\frac{1}{1 - e^{-\nu_2}} + \frac{1}{1 - e^{-\nu_1}} - 1 \right] \\ &= 1 + \Delta \cdot \sum_{k \geq 1} \frac{B_{2k}}{(2k)!} \frac{\nu_1^{2k-1} + \nu_2^{2k-1}}{\nu_1 + \nu_2}. \end{aligned}$$

Exercise 3.20. Prove the Koszul resolution theorem for vector bundles of any rank. Hint: proceed by induction on n by decomposing each differential form α as $\alpha = \alpha' - dx_n \wedge \alpha''$.

Exercise 3.21. Let $p: X \hookrightarrow Y$ be an embedding of smooth manifolds and $N \rightarrow X$ the normal vector bundle to X in Y . Let n be the rank of N .

- a) Show that $p_*(\alpha) = c_n(N) \cdot \alpha$.
- b) Apply GRR to the situation $\mathcal{O}_X \rightarrow X \hookrightarrow Y$.
- c) Express both sides of the equality obtained in (b) in terms of Chern roots of N and check that they do coincide.

3.2 Applying GRR to the universal curve

The aim of this section is to apply the GRR formula to the case where the morphism of complex manifolds is the universal curve $p: \overline{\mathcal{C}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$, while the vector bundle over $\overline{\mathcal{C}}_{g,n}$ is the trivial line bundle \mathcal{O} . We follow Mumford’s paper [31].

A careful reader may wonder whether the GRR formula is applicable to orbifolds. Such worries are well-founded, because in general it is not. As an example, consider the line bundle \mathcal{L}_1 over $\overline{\mathcal{M}}_{1,1}$. As we saw in Section 2.2.2, the holomorphic sections of \mathcal{L}_1^k are the modular forms of weight k . It is well-known (see, for instance, [32], VII, 3, Theorem 4) that modular forms are homogeneous polynomials in E_4 and E_6 , which allows one to find the dimension of $H^0(\mathcal{L}_1^k)$ and to work out the necessary modifications of the Riemann–Roch formula in this example.

However, the GRR formula applies without changes to a morphism between two *orbifolds* if every *fiber* of the morphism is a compact *manifold*, *i.e.*, the orbifold structure of the fibers is trivial. This is, of course, true for the universal curve, because its fibers are stable curves. If the fibers have a nontrivial orbifold structure, the GRR formula must be modified to take into account the stabilizers of different points of the fibers.

Taking a look at the GRR formula, we see that we must compute two things: $\text{ch}(p_! \mathcal{O})$ and $\text{td}(p)$. Here “compute” means express either via the ψ -, κ -, δ -, and λ -classes (on the moduli space) or via the classes D_i , K , Δ and ν_1 , ν_2 (on the universal curve).

Proposition 3.22. *We have $p_! \mathcal{O} = \mathbb{C} - \Lambda^\vee$, where Λ is the Hodge bundle (Definition 2.11) and \mathbb{C} the trivial line bundle over $\overline{\mathcal{M}}_{g,n}$.*

Proof. Let C be a stable curve. Obviously, $H^0(C, \mathcal{O}) = \mathbb{C}$. By Serre’s duality, the space $H^1(C, \mathcal{O})$ is dual to $H^0(C, \mathcal{L} \otimes \mathcal{O}^\vee) = H^0(C, \mathcal{L}) = \Lambda$. (In Section 3.1.3 we only described the Serre duality over smooth manifolds, but it can be extended to stable curves by using the line bundle \mathcal{L} instead of the cotangent line bundle.) Thus $R^1(p, \mathcal{O})$ is the dual of the Hodge bundle. \square

Corollary 3.23. *We have*

$$\text{ch}(p_! \mathcal{O}) = 1 - \text{ch}(\Lambda^\vee).$$

Thus the left-hand side of the GRR formula involves λ -classes. Once we have computed the right-hand side we will be able to express the λ -classes via other classes.

3.2.1 An exact sequence involving p

Computing the first ingredient of the GRR formula was easy, but the computation of $\text{td}(p)$ requires more work. If all the fibers of the universal curve were smooth, there would have been an exact sequence relating the vector bundles $T^\vee \overline{\mathcal{M}}_{g,n}$, $T^\vee \overline{\mathcal{C}}_{g,n}$ and \mathcal{L} . The Todd class $\text{td}(p)$ would then be equal to the Todd class of \mathcal{L} . In reality, however, some fibers of p are singular and the singularity locus is Δ . Therefore we will get a more complicated expression for $\text{td}(p)$, involving \mathcal{L} and the sheaf \mathcal{O}_Δ . We will then compute use the expression for the Todd class of \mathcal{O}_Δ obtained in Corollary 3.19.

Denote by $\mathcal{L}^{\log} = \mathcal{L}(D_1 + \cdots + D_n)$ the relative dualizing sheaf twisted by the divisors D_i .

Proposition 3.24. *The sequence*

$$0 \longrightarrow p^*(T^\vee \overline{\mathcal{M}}_{g,n}) \xrightarrow{(dp)^\vee} T^\vee \overline{\mathcal{C}}_{g,n} \longrightarrow \mathcal{L}^{\log} \longrightarrow \mathcal{L}^{\log} \otimes \left(\mathcal{O}_\Delta \oplus \bigoplus_{i=1}^n \mathcal{O}_{D_i} \right) \longrightarrow 0$$

is an exact sequence of sheaves over $\overline{\mathcal{C}}_{g,n}$.

Proof. The maps in this exact sequence are (i) the adjoint map of the differential of p , (ii) the restriction of a 1-form on the tangent space to $\overline{\mathcal{C}}_{g,n}$ to the tangent space of a fiber, and (iii) restricting a section of \mathcal{L} with allowed simple poles at the marked points to $\Delta \cup D_1 \cup \cdots \cup D_n$.

First consider the map p at the neighborhood of a point $z \in \overline{\mathcal{C}}_{g,n}$ outside of Δ and the D_i . Let $t = p(z) \in \overline{\mathcal{M}}_{g,n}$ be its image in the moduli space. The cotangent space to $\overline{\mathcal{M}}_{g,n}$ at t is injected in the cotangent space to $\overline{\mathcal{C}}_{g,n}$ at z by dp^\vee . The cokernel is the cotangent space to the fiber, *i.e.*, \mathcal{L} .

Second, consider a point $z \in D_i$. The preceding paragraph would apply without changes if we replaced \mathcal{L}^{\log} by \mathcal{L} in our sequence of sheaves. However, because we have \mathcal{L}^{\log} , we get a cokernel equal to the restriction of \mathcal{L}^{\log} to D_i .

Finally, take a point $z \in \Delta$ and let $t = p(z) \in \overline{\mathcal{M}}_{g,n}$ be its image in the moduli space. Consider the model local picture where p has the form $p: (X, Y) \mapsto T = XY$, where z is the point $X = Y = 0$ and t is the point $T = 0$. According to Section 1.4.4, in the general case the local picture is the direct product of our model local picture with a trivial map $p: B \rightarrow B$.

In the model local picture, the vector bundle $T^\vee \overline{\mathcal{M}}_{g,n}$ is the line bundle generated by dT . The vector bundle $T^\vee \overline{\mathcal{C}}_{g,n}$ is generated by dX and dY . Finally, the line bundle \mathcal{L} is generated by the sections $\frac{dX}{X}$ and $\frac{dY}{Y}$ modulo the

relation $\frac{dX}{X} + \frac{dY}{Y} = 0$. The maps of the sequence can be explicitly written out as follows:

$$f(T)dT \mapsto Y f(XY) dX + X f(XY) dY, \quad (\text{i})$$

$$\begin{aligned} g_X(X, Y) dX + g_Y(X, Y) dY &\mapsto (Xg_X(X, Y) - Yg_Y(X, Y))\frac{dX}{X} \\ &= (-Xg_X(X, Y) + Yg_Y(X, Y))\frac{dY}{Y}, \end{aligned} \quad (\text{ii})$$

$$h(X, Y)\frac{dX}{X} \mapsto h(0, 0). \quad (\text{iii})$$

These maps are very close to the maps of the Koszul resolution from Theorem 3.17 and the proof of the exactness is literally the same. \square

Remark 3.25. It seems that we have complicated our exact sequence unnecessarily by writing \mathcal{L}^{\log} instead of \mathcal{L} . The purpose of this is to simplify the subsequent computations. The author guarantees that we gain more than we lose by using \mathcal{L}^{\log} .

Corollary 3.26. *We have*

$$\text{td}(p) = \frac{\text{td}(\mathcal{L}^{\log})}{\text{td}(\mathcal{O}_\Delta) \prod_{i=1}^n \text{td}(\mathcal{O}_{D_i})}.$$

Proof. Using Proposition 3.24, we obtain

$$\text{td}(p) = \frac{\text{td}(\mathcal{L}^{\log})}{\text{td}(\mathcal{O}_\Delta \otimes \mathcal{L}^{\log}) \prod_{i=1}^n \text{td}(\mathcal{O}_{D_i} \otimes \mathcal{L}^{\log})},$$

because the Todd class is multiplicative. On the other hand, we have

$$\text{td}(\mathcal{O}_\Delta \otimes \mathcal{L}^{\log}) = \text{td}(\mathcal{O}_\Delta), \quad \text{td}(\mathcal{O}_{D_i} \otimes \mathcal{L}^{\log}) = \text{td}(\mathcal{O}_{D_i}).$$

Indeed, as we know, the restriction of the line bundle \mathcal{L}^{\log} to D_i is trivial, while its restriction to Δ is rationally trivial. Hence all characteristic classes of \mathcal{L}^{\log} on Δ and D_i are the same as for a trivial line bundle. \square

3.2.2 Computing $\text{td}(p)$

Now we can put together the results of our computations to obtain the final formula for $\text{td}(p)$.

According to Corollary 3.26 we have

$$\text{td}(p) = \frac{\text{td}(\mathcal{L}^{\log})}{\text{td}(\mathcal{O}_\Delta) \prod_{i=1}^n \text{td}(\mathcal{O}_{D_i})}.$$

The todd class of \mathcal{L}^{\log} is immediately evaluated to be

$$\text{td}(\mathcal{L}^{\log}) = \frac{K}{e^K - 1}.$$

The expressions for $1/\text{td}(\mathcal{O}_{D_i})$ and $1/\text{td}(\mathcal{O}_\Delta)$ were given in Corollaries 3.18 and 3.19. Putting everything together we get

$$\text{td}(p) = \frac{K}{e^K - 1} \prod_{i=1}^n \frac{D_i}{1 - e^{-D_i}} \left(1 + \Delta \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \frac{\nu_1^{2k-1} + \nu_2^{2k-1}}{\nu_1 + \nu_2} \right).$$

Now, taking into account that $D_i D_j = D_i \Delta = K D_i = K \Delta = 0$, we see that the product in the expression for $\text{td}(p)$ can be actually replaced by a sum:

$$\text{td}(p) = 1 + \left(\frac{K}{e^K - 1} - 1 \right) + \sum_{i=1}^n \left(\frac{D_i}{1 - e^{-D_i}} - 1 \right) + \left(\Delta \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \frac{\nu_1^{2k-1} + \nu_2^{2k-1}}{\nu_1 + \nu_2} \right).$$

Expanding the power series we get

$$\text{td}(p) = 1 - \frac{1}{2}(K - \sum D_i) + \sum_{k \geq 1} \frac{B_{2k}}{(2k)!} \left[K^{2k} - \sum_{i=1}^n D_i^{2k} + \Delta \cdot \frac{\nu_1^{2k-1} + \nu_2^{2k-1}}{\nu_1 + \nu_2} \right].$$

3.2.3 Assembling the GRR formula

Finally, according to the GRR formula, the push-forward p_* of $\text{td}(p)$ is equal to the class $1 - \text{ch}(\Lambda^\vee)$. For clarity, we separate this equality into homogeneous parts. Note that

$$\text{ch}_k(\Lambda^\vee) = \begin{cases} \text{ch}_k(\Lambda) & \text{if } k \text{ is even,} \\ -\text{ch}_k(\Lambda) & \text{if } k \text{ is odd.} \end{cases}$$

Thus, for instance, in degree 0 we have

$$1 - \text{ch}_0(\Lambda^\vee) = 1 - \text{ch}_0(\Lambda) = -\frac{1}{2}p_*(K - \sum D_i) = -\frac{1}{2}(2g - 2 + n - n) = 1 - g.$$

We obtain the following theorem.

Theorem 3.27. *We have*

$$\begin{aligned} \text{ch}_0(\Lambda) &= g, \\ \text{ch}_{2k}(\Lambda) &= 0, \\ \text{ch}_{2k-1}(\Lambda) &= \frac{B_{2k}}{(2k)!} \left[\kappa_{2k-1} - \sum_{i=1}^n \psi_i^{2k-1} + \delta_{2k-1}^\Lambda \right], \end{aligned}$$

where δ_{2k-1}^Λ is the push-forward p_* of the class $\Delta \cdot (\nu_1^{2k-1} + \nu_2^{2k-1}) / (\nu_1 + \nu_2)$.

A curious reader can check that

$$\nu_1^p + \nu_2^p = \sum_{l=0}^{[p/2]} (-1)^l \frac{p}{p-l} \binom{p-l}{l} (\nu_1 \nu_2)^l (\nu_1 + \nu_2)^{p-2l},$$

and hence the class δ_{2k-1}^Λ is expressed via the standard classes $\delta_{k,l}$ as

$$\delta_{2k-1}^\Lambda = \sum_{l=0}^{k-1} (-1)^l \frac{2k-1}{2k-1-l} \binom{2k-1-l}{l} \delta_{2k-2-2l,l}.$$

Theorem 3.27 expresses the Chern characters of the Hodge bundle via the ψ -, κ -, and δ -classes. To conclude this section we show how to obtain similar expressions for the classes $\lambda_i = c_i(\Lambda)$.

Exercise 3.28. Prove the equality

$$\begin{aligned} 1 + \lambda_1 + \lambda_2 + \cdots + \lambda_g &= \exp \left(\text{ch}_1 - \text{ch}_2 + \frac{\text{ch}_3}{2!} - \frac{\text{ch}_4}{3!} + \cdots \right) \\ &= \exp \left(\text{ch}_1 + \frac{\text{ch}_3}{2!} + \frac{\text{ch}_5}{4!} + \cdots \right), \end{aligned}$$

where $\text{ch}_k = \text{ch}_k(\Lambda)$.

Exercise 3.29. Deduce from the previous exercise that every class λ_i is a polynomial in the ψ -, κ -, and δ -classes.

Exercise 3.30. Show that $c(\Lambda)c(\Lambda^\vee) = 1$.

Exercise 3.31. Show that

$$\begin{aligned}\lambda_1 &= \frac{1}{12}(\kappa_1 - \sum_{i=1}^n \psi_i + \delta_{0,0}); \\ \lambda_2 &= \frac{1}{2}\lambda_1^2; \\ \lambda_3 &= \frac{1}{6}\lambda_1^3 - \frac{1}{360}(\kappa_3 - \sum_{i=1}^n \psi_i^3 + \delta_{2,0} - 3\delta_{0,1}).\end{aligned}$$

Exercise 3.32. Compute the integrals over $\overline{\mathcal{M}}_{1,1}$ of the classes λ_1 , κ_1 , ψ_1 , and $\delta_{0,0}$. Check the equality

$$\lambda_1 = \frac{1}{12}(\kappa_1 - \sum_{i=1}^n \psi_i + \delta_{0,0})$$

on $\overline{\mathcal{M}}_{1,1}$ numerically.

3.3 Eliminating κ - and δ -classes

In the previous section we expressed the λ -classes in terms of ψ -, κ -, and δ -classes.

Now we will reduce any integral over $\overline{\mathcal{M}}_{g,n}$ involving ψ -, κ -, and δ -classes to a combination of integrals involving only ψ -classes. This does not mean that the κ - and δ -classes can be expressed in terms of ψ -classes. Indeed, the integrals that we will obtain involve moduli spaces $\overline{\mathcal{M}}_{g',n'}$ with g' and n' not necessarily equal to g and n . To do that, we will use computations with attaching and forgetful maps.

Before going into our computations, recall the following basic property that we will use.

Let $f: X \rightarrow Y$ be a morphism of smooth compact manifolds. Then f induces two maps in cohomology: the pull-back $f^*: H^*(Y, \mathbb{Q}) \rightarrow H^*(X, \mathbb{Q})$ and the fiberwise integration $f_*: H^*(X, \mathbb{Q}) \rightarrow H^*(Y, \mathbb{Q})$. If the cohomology classes are represented by generic Poincaré dual cycles, then f^* and f_* are

the geometric preimage and image of cycles. Let $\alpha \in H^*(X, \mathbb{Q})$ and $\beta \in H^*(Y, \mathbb{Q})$. In general $f^*(f_*(\alpha)) \neq \alpha$ (and these classes have different degrees). Similarly, in general $f_*(f^*(\beta)) \neq \beta$ (and these classes also have different degrees). Instead, the following equality is true:

$$f_*(\alpha f^*(\beta)) = f_*(\alpha)\beta$$

and therefore

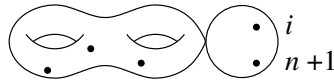
$$\int_X \alpha f^*(\beta) = \int_Y f_*(\alpha)\beta.$$

This property is called the *projection formula*.

3.3.1 Equivalence between $\overline{\mathcal{M}}_{g,n+1}$ and $\overline{\mathcal{C}}_{g,n}$

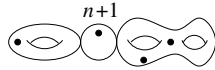
Recall that $\overline{\mathcal{M}}_{g,n+1}$ and $\overline{\mathcal{C}}_{g,n}$ are naturally isomorphic as families over $\overline{\mathcal{M}}_{g,n}$ (see Proposition 2.2). Let us make a small dictionary between the tautological classes defined on this family viewed as a moduli space and as a universal curve.

Consider the set of stable curves that contain a spherical component with exactly three special points: a node and the marked points number i and $n+1$.



The points encoding such curves form a divisor $\delta_{(i,n+1)}$ of $\overline{\mathcal{M}}_{g,n+1}$.

Further, consider the set of stable curves that contain a spherical component with exactly three special points: two nodes and the marked point number $n+1$.



The points encoding such curves form a codimension 2 subvariety $\delta_{(n+1)}$ of $\overline{\mathcal{M}}_{g,n+1}$.

Lemma 3.33. *Under the identification of $\overline{\mathcal{M}}_{g,n+1}$ with $\overline{\mathcal{C}}_{g,n}$, the divisor $\delta_{(i,n+1)}$ is identified with the divisor D_i , the subvariety $\delta_{(n+1)}$ is identified with Δ , while the class ψ_{n+1} is identified with K .*

Proof. The first two identifications follow immediately from the proof of Proposition 2.2: they correspond to cases (ii) and (iii).

The identification of ψ_{n+1} with K is more delicate. Recall that K is the first Chern class of $\mathcal{L}(D)$. In case (i) of Proposition 2.2, the fiber of \mathcal{L}_{n+1} over $\overline{\mathcal{M}}_{g,n+1}$ is naturally identified with the fiber of \mathcal{L} over $\overline{\mathcal{C}}_{g,n}$.

This identification fails in cases (ii) and (iii). However, case (iii) is not important, since it only occurs on a codimension 2 subvariety, hence does not influence the first Chern class. Case (ii), on the other hand, must be inspected more closely.

A holomorphic section of \mathcal{L} at the neighborhood of \hat{x}_i in the figure on page 22 is a holomorphic 1-form on the fibers of $\overline{\mathcal{C}}_{g,n}$. It is naturally extended by 0 on the genus 0 component containing x_i and x_{n+1} (cf. Exercise 2.22). Thus the sections of \mathcal{L}_{n+1} acquire an additional zero whenever we are in case (ii). Thus \mathcal{L}_{n+1} is identified not with \mathcal{L} but with $\mathcal{L}^{\log} = \mathcal{L}(D)$. \square

3.3.2 Eliminating κ -classes: the forgetful map

The contents of this and the next section are a reformulation of certain results of [2].

Let $p: \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$ be the forgetful map. Our aim is to compare the tautological classes on $\overline{\mathcal{M}}_{g,n+1}$ and the pull-backs of analogous tautological class from $\overline{\mathcal{M}}_{g,n}$. It turns out that the difference is easily expressed if we consider $\overline{\mathcal{M}}_{g,n+1}$ as the universal curve over $\overline{\mathcal{M}}_{g,n}$.

Theorem 3.34. *We have*

$$\psi_{n+1} = K, \tag{1}$$

$$\psi_i^d - p^* \psi_i^d = (-D_i)^{d-1} D_i, \tag{2}$$

$$\kappa_m - p^* \kappa_m = K^m, \tag{3}$$

$$\delta_{k,0} - p^* \delta_{k,0} = (-D)^k D + (\nu_1 + \nu_2)^{k+1} - \nu_1^{k+1} - \nu_2^{k+1}, \tag{4}$$

$$\delta_{k,l} - p^* \delta_{k,l} = (\nu_1 + \nu_2)^{k+1} \Delta^l \quad \text{for } l \neq 0. \tag{5}$$

Proof. We are going to prove Equalities (1), (2), and (3), leaving the last two equalities as an exercise¹.

(1) According to Lemma 3.33, we have $\psi_{n+1} = K$.

¹The last two equalities are not more complicated than then first three, but require some juggling between $\Delta \subset \overline{\mathcal{C}}_{g,n}$, $\Delta \subset \overline{\mathcal{C}}_{g,n+1}$, and $\overline{\mathcal{M}}_{g,n+1}$ identified with $\overline{\mathcal{C}}_{g,n}$, which makes the text of the proof rather ugly.

(2) According to Exercise 2.22, we have $\psi_i - p^*\psi_i = \delta_{(i,n+1)} \iff \psi_i - \delta_{(i,n+1)} = p^*\psi_i$. On the other hand, the line bundle \mathcal{L}_i restricted to $\delta_{(i,n+1)}$ is trivial, hence $\psi_i\delta_{(i,n+1)} = 0$. Thus

$$\begin{aligned} (\psi_i - \delta_{(i,n+1)})^d &= p^*\psi_i^d \iff \psi_i^d + (-\delta_{(i,n+1)})^d = p^*\psi_i^d \\ &\iff \psi_i^d - p^*\psi_i^d = -(-\delta_{(i,n+1)})^d. \end{aligned}$$

It remains to note that, according to Lemma 3.33, $\delta_{(i,n+1)}$ is identified with D_i .

(3) If $\tilde{p}: \bar{\mathcal{C}}_{g,n+1} \rightarrow \bar{\mathcal{C}}_{g,n}$ is the forgetful map for universal curves, we have $\tilde{p}^*K = K - D_{n+1}$. On the other hand, $KD_{n+1} = 0$. Hence

$$(K - D_{n+1})^{m+1} = \tilde{p}^*K^{m+1} \iff K^{m+1} - D_{n+1}^{m+1} = \tilde{p}^*K^{m+1}.$$

The push-forward of this equality to the moduli spaces gives

$$\kappa_m - p^*\kappa_m = \psi_{n+1}^m = K^m. \quad \square$$

Corollary 3.35. *Let Q be a polynomial in the variables $\kappa_m, \delta_{k,l}, \psi_1, \dots, \psi_n$. Let \tilde{Q} be the polynomial obtained from Q by the substitution $\kappa_i \mapsto \kappa_i - \psi_{n+1}^i$. Then we have*

$$\int_{\bar{\mathcal{M}}_{g,n}} \kappa_m Q = \int_{\bar{\mathcal{M}}_{g,n+1}} \psi_{n+1}^{m+1} \tilde{Q}.$$

Proof. By definition, $\kappa_m = p_*(K^{m+1})$. By the projection formula, we obtain

$$\int_{\bar{\mathcal{M}}_{g,n}} p_*(K^{m+1}) \cdot Q = \int_{\bar{\mathcal{C}}_{g,n} = \bar{\mathcal{M}}_{g,n+1}} K^{m+1} \cdot p^*Q.$$

According to Theorem 3.34, the pull-back p^* modifies each term of the polynomial Q . However most of these modifications play no role, because they vanish when we multiply them by K^{m+1} .

Indeed, the difference between ψ_i^d and $p^*\psi_i^d$ is a multiple of D_i , and we know that $D_iK = 0$. Similarly, the difference between $\delta_{k,l}$ and $p^*\delta_{k,l}$ is either a multiple of Δ (if $l \neq 0$), or a sum of a multiple of Δ and a multiple of D (if $l = 0$). But we know that $K\Delta = KD = 0$.

The only remaining terms are κ_i , and for these the difference is important. According to Theorem 3.34, we have $p^*\kappa_i = \kappa_i - K^i$. Recalling that K (on $\bar{\mathcal{C}}_{g,n}$) is the same as ψ_{n+1} (on $\bar{\mathcal{M}}_{g,n+1}$), we obtain that we must replace every κ_i in Q by $\kappa_i - \psi_{n+1}^i$. Thus we obtain exactly the polynomial \tilde{Q} and the assertion of the corollary. \square

Corollary 3.35 allows us to express an integral involving at least one κ -class as a combination of integrals with fewer κ -classes.

Exercise 3.36. Compute

$$\int_{\overline{\mathcal{M}}_{0,4}} \kappa_1, \quad \int_{\overline{\mathcal{M}}_{0,5}} \kappa_1^2, \quad \int_{\overline{\mathcal{M}}_{0,5}} \kappa_2, \quad \int_{\overline{\mathcal{M}}_{0,5}} \kappa_1 \psi_1, \quad \int_{\overline{\mathcal{M}}_{0,6}} \kappa_1 \kappa_2.$$

Exercise 3.37. Express the class $\kappa_1 \in H^2(\overline{\mathcal{M}}_{1,2})$ in terms of boundary divisors.

Exercise 3.38. Show that in $H^2(\overline{\mathcal{M}}_{0,n})$ the sum $\psi_i + \psi_j + \psi_k$ does not depend on i, j, k .

3.3.3 Eliminating δ -classes: the attaching map

Recall that $\Delta \subset \overline{\mathcal{C}}_{g,n}$ is the set of nodes of the singular fibers. Take a singular stable curve, choose a node, unglue the branches of the curve at the node and number the two preimages of the node (there are two ways of doing that). We obtain a new stable curve that has two new marked points. It can either be connected of genus $g - 1$, or composed of two connected components of genera g_1 and g_2 , $g_1 + g_2 = g$. Denote by $\overline{\mathcal{M}}_{\text{split}}$ the disjoint union

$$\overline{\mathcal{M}}_{\text{split}} = \bigsqcup_{\substack{I_1 \cup I_2 = \{1, \dots, n\} \\ g_1 + g_2 = g}} \overline{\mathcal{M}}_{g_1, I_1 \cup \{n+1\}} \times \overline{\mathcal{M}}_{g_2, I_2 \cup \{n+2\}} \bigsqcup \overline{\mathcal{M}}_{g-1, n+2}.$$

We have actually proved the following statement.

Lemma 3.39. $\overline{\mathcal{M}}_{\text{split}}$ is a 2-sheeted covering of Δ .

The same 2-sheeted covering was previously denoted by $\tilde{\Delta}$.

Denote by $j: \overline{\mathcal{M}}_{\text{split}} \rightarrow \overline{\mathcal{M}}_{g,n}$ the composition $\overline{\mathcal{M}}_{\text{split}} \xrightarrow{c} \Delta \xrightarrow{p} \overline{\mathcal{M}}_{g,n}$. The image of j is the boundary $\delta_{0,0} = \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ of the moduli space.

A tautological class on $\overline{\mathcal{M}}_{\text{split}}$ is defined by taking the same tautological class on every component of $\overline{\mathcal{M}}_{\text{split}}$.

Our aim is now to study the difference between a tautological class on $\overline{\mathcal{M}}_{\text{split}}$ and the pull-back of the analogous class from $\overline{\mathcal{M}}_{g,n}$.

Theorem 3.40. *We have*

$$\kappa_m - j^* \kappa_m = 0, \quad (1)$$

$$\psi_i^d - j^* \psi_i^d = 0, \quad (2)$$

$$\delta_{k,l} - j^* \delta_{k,l} = (\psi_{n+1} + \psi_{n+2})^{k+1} (\psi_{n+1} \psi_{n+2})^l. \quad (3)$$

Proof. Equalities (1) and (2) follow from the obvious fact that $\tilde{j}^* K = K$ and $\tilde{j}^* D_i = D_i$, where \tilde{j} is the map of universal curves associated with j .

For Equality (3) we only sketch the argument.

First consider the case $k = l = 0$. Rewrite the equality as

$$j^* \delta_{0,0} = \delta_{0,0} - (\psi_{n+1} + \psi_{n+2}).$$

The class $j^* \delta_{0,0}$ is almost the same as the self-intersection of the boundary $\delta_{0,0}$ of the moduli space. More precisely, different connected components of $\overline{\mathcal{M}}_{\text{split}}$ are degree 2 coverings of irreducible components of $\delta_{0,0}$, and we are interested in the pull-backs of the intersections of these components with $\delta_{0,0}$.

Some of the intersection points encode stable curves with two nodes. Such intersections are transversal and they give rise to the term $\delta_{0,0}$ in the right-hand side.

But when we try to intersect a component of $\delta_{0,0}$ with itself we obtain a nontransversal intersection. To determine the intersection in cohomology we need to know the first Chern class of the normal line bundle to $\delta_{0,0}$ in $\overline{\mathcal{M}}_{g,n}$. It turns out that this first Chern class equals $-j_*(\psi_{n+1} + \psi_{n+2})$.

This can be seen using a local model of the smoothening of a node. Let B be a base manifold and T_1, T_2 two line bundles over B . Let $T = T_1 \otimes T_2$. Then we have a natural map $(x, y) \mapsto t = xy$ between the total spaces of the bundles $T_1 \oplus T_2$ and T . In this local model the zero section of T corresponds to $\delta_{0,0}$, the total space of T represents the neighborhood of $\delta_{0,0}$ in $\overline{\mathcal{M}}_{g,n}$, and T is the normal line bundle to $\delta_{0,0}$ in $\overline{\mathcal{M}}_{g,n}$. The line bundles T_1 and T_2 are the tangent line bundles to the branches of the universal curve at the node. Thus we see that $c_1(T) = c_1(T_1) + c_1(T_2)$.

Thus the nontransversal intersection gives us the correction term $-(\psi_{n+1} + \psi_{n+2})$ in the right-hand side of (3).

The case of general k and l is similar, since $\delta_{k,l}$ is the intersection of $\delta_{0,0}$ with $(\nu_1 + \nu_2)^k (\nu_1 \nu_2)^l$, which is transformed into $(\psi_{n+1} + \psi_{n+2})^k (\psi_{n+1} \psi_{n+2})^l$ on $\overline{\mathcal{M}}_{\text{split}}$. \square

Corollary 3.41. *Let Q be a polynomial in variables $\kappa_m, \delta_{k,l}, \psi_1, \dots, \psi_n$. Let \tilde{Q} be the polynomial obtained from Q by the substitution*

$$\delta_{i,j} \mapsto \delta_{i,j} - (\psi_{n+1} + \psi_{n+2})^{i+1} (\psi_{n+1} \psi_{n+2})^j.$$

Then we have

$$\int_{\overline{\mathcal{M}}_{g,n}} \delta_{k,l} Q = \frac{1}{2} \int_{\overline{\mathcal{M}}_{\text{split}}} (\psi_{n+1} + \psi_{n+2})^k (\psi_{n+1} \psi_{n+2})^l \tilde{Q}.$$

Proof. By definition,

$$\delta_{k,l} = \frac{1}{2} j_* \left((\psi_{n+1} + \psi_{n+2})^k (\psi_{n+1} \psi_{n+2})^l \right),$$

where the factor $1/2$ appears because $\overline{\mathcal{M}}_{\text{split}}$ is a double covering of Δ . The projection formula implies that

$$\begin{aligned} \frac{1}{2} \int_{\overline{\mathcal{M}}_{g,n}} j_* \left((\psi_{n+1} + \psi_{n+2})^k (\psi_{n+1} \psi_{n+2})^l \right) \cdot Q \\ = \frac{1}{2} \int_{\overline{\mathcal{M}}_{\text{split}}} (\psi_{n+1} + \psi_{n+2})^k (\psi_{n+1} \psi_{n+2})^l \cdot j^* Q. \end{aligned}$$

According to Theorem 3.40, every term of Q remains unchanged after a pull-back by j , except for the terms $\delta_{k,l}$, that are transformed according to the rule

$$j^* \delta_{k,l} = \delta_{k,l} - (\psi_{n+1} + \psi_{n+2})^{k+1} (\psi_{n+1} \psi_{n+2})^l.$$

In other words, $j^* Q = \tilde{Q}$, so we obtain the claim of the corollary. \square

Exercise 3.42. Compute the integrals

$$\int_{\overline{\mathcal{M}}_{1,1}} \delta_{0,0}, \quad \int_{\overline{\mathcal{M}}_{1,2}} \delta_{0,0}^2, \quad \int_{\overline{\mathcal{M}}_{1,2}} \delta_{0,1}, \quad \int_{\overline{\mathcal{M}}_{1,2}} \delta_{0,0} \kappa_1.$$

Corollary 3.41 allows us to express an integral involving at least one δ -class as a combination of integrals with fewer δ -classes.

Applying Corollaries 3.35 and 3.41 several times allow us to reduce any integral involving ψ -, κ -, and δ -classes to a combination of integrals involving only ψ -classes. These integrals are the subject of the next section.

4 Around Witten's conjecture

Now our aim is to compute the integrals

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}.$$

4.1 The string and dilaton equations

Proposition 4.1. *For $2 - 2g - n < 0$ we have*

$$\begin{aligned} \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{d_1} \cdots \psi_n^{d_n} &= \sum_{i=1}^n \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_i^{d_i-1} \cdots \psi_n^{d_n}; \\ \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{d_1} \cdots \psi_n^{d_n} \psi_{n+1} &= (2g - 2 + n) \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}. \end{aligned}$$

These equations are called the *string* and the *dilaton* equations. We borrowed their proof from Witten's paper [35].

Proof. We will need to consider the line bundles \mathcal{L}_i both on $\overline{\mathcal{M}}_{g,n}$ and on $\overline{\mathcal{M}}_{g,n+1}$. We momentarily denote the former by \mathcal{L}'_i , $1 \leq i \leq n$, the latter retaining the notation \mathcal{L}_i , $1 \leq i \leq n+1$. Further, denote by ψ'_i the first Chern class of the line bundle \mathcal{L}'_i on $\overline{\mathcal{M}}_{g,n}$ and, by abuse of notation, its pull-back to $\overline{\mathcal{M}}_{g,n+1}$ by the map forgetting the $(n+1)$ st marked point. Let ψ_i be the first Chern class of \mathcal{L}_i on $\overline{\mathcal{M}}_{g,n+1}$.

According to Exercise 2.22, we have

$$\psi_i = \psi'_i + \delta_{(i,n+1)}. \quad (4)$$

Moreover, we have

$$\psi_i \cdot \delta_{(i,n+1)} = \psi_{n+1} \cdot \delta_{(i,n+1)} = 0 \quad (5)$$

(because the line bundles \mathcal{L}_i and \mathcal{L}_{n+1} are trivial over $\delta_{(i,n+1)}$) and

$$\delta_{(i,n+1)} \cdot \delta_{(j,n+1)} = 0 \quad \text{for } i \neq j \quad (6)$$

(because the divisors have an empty geometric intersection).

Let us first prove the string relation. We have

$$\psi_i^d - (\psi'_i)^d \stackrel{(1)}{=} \delta_{(i,n+1)} (\psi_i^{d-1} + \dots + (\psi'_i)^{d-1}) \stackrel{(2)}{=} \delta_{(i,n+1)} (\psi'_i)^{d-1},$$

where we set by convention $(\psi'_i)^{-1} = 0$. Thus

$$\psi_i^d = (\psi'_i)^d + \delta_{(i,n+1)} (\psi'_i)^{d-1}.$$

It follows that

$$\begin{aligned} & \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{d_1} \dots \psi_n^{d_n} \\ &= \int_{\overline{\mathcal{M}}_{g,n+1}} \left[(\psi'_1)^{d_1} + \delta_{(1,n+1)} (\psi'_1)^{d_1-1} \right] \dots \left[(\psi'_n)^{d_n} + \delta_{(n,n+1)} (\psi'_n)^{d_n-1} \right] \\ &\stackrel{(3)}{=} \int_{\overline{\mathcal{M}}_{g,n+1}} (\psi'_1)^{d_1} \dots (\psi'_n)^{d_n} + \sum_{i=1}^n \int_{\overline{\mathcal{M}}_{g,n+1}} (\psi'_1)^{d_1} \dots \delta_{(i,n+1)} (\psi'_i)^{d_i-1} \dots (\psi'_n)^{d_n}. \end{aligned}$$

The first integral is equal to 0, because the integrand is a pull-back from $\overline{\mathcal{M}}_{g,n}$. As for the integrals composing the sum, we integrate the class $\delta_{(i,n+1)}$ over the fibers of the projection $\overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$. This is equivalent to restricting the integral to the divisor $\delta_{(i,n+1)}$, which is naturally isomorphic to $\overline{\mathcal{M}}_{g,n}$. Finally, we obtain

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{d_1} \dots \psi_n^{d_n} = \sum_{i=1}^n \int_{\overline{\mathcal{M}}_{g,n}} (\psi'_1)^{d_1} \dots (\psi'_i)^{d_i-1} \dots (\psi'_n)^{d_n}.$$

This proves the string relation.

Now let us prove the dilaton relation. We have

$$\begin{aligned} & \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{d_1} \dots \psi_n^{d_n} \psi_{n+1} \\ &\stackrel{(1)}{=} \int_{\overline{\mathcal{M}}_{g,n+1}} (\psi'_1 + \delta_{(1,n+1)})^{d_1} \dots (\psi'_n + \delta_{(n,n+1)})^{d_n} \psi_{n+1} \\ &\stackrel{(2)}{=} \int_{\overline{\mathcal{M}}_{g,n+1}} (\psi'_1)^{d_1} \dots (\psi'_n)^{d_n} \psi_{n+1} \\ &= (2g - 2 + n) \int_{\overline{\mathcal{M}}_{g,n}} (\psi'_1)^{d_1} \dots (\psi'_n)^{d_n}. \end{aligned}$$

The last equality is obtained by integrating the factor ψ_{n+1} over the fibers of the projection $\overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$. This proves the dilaton relation. \square

Exercise 4.2. Show that the string and the dilaton equations, together with the initial conditions

$$\int_{\overline{\mathcal{M}}_{0,3}} 1 = 1 \quad \text{and} \quad \int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = \frac{1}{24}$$

allow one to compute all integrals

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}$$

for $g = 0, 1$.

Exercise 4.3. Compute the integrals

$$\int_{\overline{\mathcal{M}}_{1,2}} \psi_1^2, \quad \int_{\overline{\mathcal{M}}_{1,2}} \psi_1 \psi_2, \quad \int_{\overline{\mathcal{M}}_{1,3}} \psi_1^2 \psi_2, \quad \int_{\overline{\mathcal{M}}_{1,2}} \psi_1 \kappa_1, \quad \int_{\overline{\mathcal{M}}_{2,0}} \delta_{0,1} \kappa_1.$$

4.2 KdV and Virasoro

It looks like we are very close to our goal of computing all intersection numbers of ψ -classes and thus of all tautological classes. However the last step is actually quite hard; therefore we only formulate the theorems that make it possible to compute the remaining integrals, but do not give the proofs.

Introduce the following generating series:

$$F(t_0, t_1, \dots) = \sum_{\substack{g \geq 0, n \geq 1 \\ 2-2g-n < 0}} \sum_{d_1, \dots, d_n} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \frac{t_{d_1} \cdots t_{d_n}}{n!}.$$

The coefficients of this series encode all possible integrals involving ψ -classes.

Theorem 4.4. *The series F satisfies the following partial differential equation:*

$$\frac{\partial^2 F}{\partial t_0 \partial t_1} = \frac{1}{2} \left(\frac{\partial^2 F}{\partial t_0^2} \right)^2 + \frac{1}{12} \frac{\partial^4 F}{\partial t_0^4}.$$

This partial differential equation is one of the forms of the *Korteweg-de Vries* or *KdV* equation². The result of Theorem 4.4 was conjectured by E. Witten in [35] and proved by M. Kontsevich [24]. Today several other proofs exist, [22] being, probably the simplest one.

Let $G(p_1, p_3, p_5, \dots)$ be the power series obtained from F by the substitution $t_d = (2d - 1)!! p_{2d+1}$.

For $k \geq 1$, denote by a_k the operator of multiplication by p_k and by a_{-k} the operator $k \frac{\partial}{\partial p_k}$. Let $a_0 = 0$.

Further, for any integer k define an operator b_k by

$$b_k = \frac{1}{2} \sum_{i+j=k} a_i a_j \quad (k \neq 0), \quad b_0 = \frac{1}{8} + \sum_{i \geq 0} a_i a_{-i}.$$

Theorem 4.5. *For every integer $m \geq -1$ we have*

$$(a_{-(2m+3)} - b_{-2m})e^G = 0.$$

These equations are called the *Virasoro constraints*.

Example 4.6. The Virasoro constraints corresponding to $m = -1$ and $m = 0$ are equivalent to the string and the dilaton equations.

For completeness let us also write out the Virasoro constraints for $m \geq 1$ in terms of integrals of ψ -classes over moduli spaces. The concise expression of Theorem 4.5 translates, for $m \geq 1$, into the following identity:

$$\begin{aligned} & (2m + 3)!! \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{d_1} \cdots \psi_n^{d_n} \psi_{n+1}^{m+1} \\ &= \sum_{i=1}^n \frac{(2d_i + 2m + 1)!!}{(2d_i - 1)!!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_i^{d_i+m} \cdots \psi_n^{d_n} \\ &+ \frac{1}{2} \sum_{a+b=m-1} (2a + 1)!!(2b + 1)!! \int_{\overline{\mathcal{M}}_{g-1,n+2}} \psi_1^{d_1} \cdots \psi_n^{d_n} \psi_{n+1}^a \psi_{n+2}^b \\ &+ \frac{1}{2} \sum_{\substack{a+b=m-1 \\ I \sqcup J = \{1, \dots, n\} \\ g_1 + g_2 = g}} (2a + 1)!!(2b + 1)!! \int_{\overline{\mathcal{M}}_{g_1, I \cup \{n+1\}}} \prod_{i \in I} \psi_i^{d_i} \psi_{n+1}^a \int_{\overline{\mathcal{M}}_{g_2, J \cup \{n+2\}}} \prod_{i \in J} \psi_i^{d_i} \psi_{n+2}^b, \end{aligned}$$

²The usual form of the KdV equation is $\partial U / \partial t_1 = U \partial U / \partial t_0 + \frac{1}{12} \partial^3 U / \partial t_0^3$ obtained by differentiating our equation once with respect to t_0 and letting $U = \partial^2 F / \partial t_0^2$.

where $a, b \geq 0$.

It is straightforward to see that the Virasoro constraints together with the initial condition $F = t_0^3/6 + \dots$ determine the series F completely. Indeed, the above equality expresses an integral over some moduli space via integrals over other moduli spaces with smaller values of $2g - 2 + n$.

A slightly more ingenious argument shows that the KdV and the string equation, together with the initial condition $F = t_0^3/6 + \dots$, determine F uniquely. (For a proof see [35], Section 2a.) Thus Theorems 4.4 and 4.5 describe the same series F in two different ways. The equivalence between these two theorems was first established in [6] and [12] before either of them was proved.

A modern proof of both theorems can be found in [20]. We do not give it here; instead we will say a few words about the origins of Witten's conjecture in 2-dimensional quantum gravity.

In general relativity, the gravitational field is a quadratic form of signature $(1, 3)$ on the 4-dimensional space-time. Since the problem of constructing a quantum theory of gravity is extremely difficult, physicists started with a simpler model of 2-dimensional gravity. In this model, the space-time is a surface, while the gravitational field is a Riemannian metric on this surface.

The aim of a quantum theory of gravity is then to define and compute certain integrals over the space of all possible Riemannian metrics on all possible surfaces. The space of metrics is infinite-dimensional and does not carry a natural measure, therefore the definition of such integrals is problematic. Physicists found two ways to give a meaning to integrals over the space of metrics.

The first way is to replace Riemannian metrics by a discrete approximation, namely, surfaces obtained by glueing together very small equilateral triangles. In this method, integrals over the space of metrics are replaced by sums over triangulations. This leads to combinatorial problems of enumerating triangulations. These problems, although difficult, can be solved, and the KdV equation appeared in the works devoted to the enumeration of triangulations.

The second way to define infinite-dimensional integrals is to first perform the integration over the space of conformally equivalent metrics.

Two Riemannian metrics are *conformally equivalent* if one is obtained from the other by multiplication by a positive function. An equivalence class of conformally equivalent metrics is exactly the same thing as a Riemann surface. Indeed, one of the ways to introduce a complex structure on a

surface is to define a linear operator J acting on its tangent bundle such that $J^2 = -1$. The action of the operator is then interpreted as multiplication of the tangent vectors by i . In the case where the surface is endowed with a Riemannian metric, the operator J is simply the rotation by 90° , and this does not change if we multiply the metric by a positive function.

The space of positive functions is still infinite-dimensional, but it turns out that one can perform the integration over this space by a formal trick. (It would be more precise to say that the integral is *defined* in a meaningful way and the trick serves as a motivation.) After that, there remains an integral over the moduli space of Riemann surfaces. It so happens that this integral is

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n}.$$

What Witten's conjecture actually says is that we obtain the same answer using the two methods of defining infinite-dimensional integrals.

Solutions of exercises

Example .7. Recall the first equality of Example ??:

$$\lambda_1 = \frac{1}{12}(\kappa_1 - \sum \psi_i + \delta_{0,0}).$$

This equality is true for all g and n , but let us consider the case $g = n = 1$. We have (cf. Example ??)

$$\int_{\overline{\mathcal{M}}_{1,1}} \lambda_1 = \frac{1}{24}; \quad \int_{\overline{\mathcal{M}}_{1,1}} \kappa_1 = \int_{\overline{\mathcal{M}}_{1,2}} \psi_2^2 = \int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = \frac{1}{24}; \quad \int_{\overline{\mathcal{M}}_{1,1}} \delta_{0,0} = \frac{1}{2}.$$

Thus we have checked that the expression for λ_1 over $\overline{\mathcal{M}}_{1,1}$ is indeed correct: $1/24 = 1/12 \cdot (1/24 - 1/24 + 1/2)$.

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