PATH-COUNTING AND FIBONACCI NUMBERS

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1. INTRODUCTION

Consider the set of points (i, j) given by nonnegative integers i and j. This lattice may be viewed as an unbounded rectangle with boundary consisting of points $(i, 0)$ on a horizontal x-axis and points $(0, j)$ on a vertical y-axis. There are many systematic ways to draw paths from this boundary into the interior of the lattice. Enumerations of such paths yield arrays associated with Fibonacci numbers and other recurrence sequences. Such enumerations also apply to various classes of compositions of nonnegative integers. In order to investigate such enumerations, we begin with some notation:

$R = \{(i, j) : i \text{ and } j \text{ are nonnegative integers}\}$,
$R^+ = \{(i, j) : i \text{ and } j \text{ are positive integers}\}$,
$R^0 = R - R^+$.

Suppose G is a circuit-free graph on R, directed so that for each (i, j) in R^+ every path to (i, j) is rooted in a vertex in R^0. Each edge entering (i, j) has a tail (x, y); let

$$E(i, j) = \{(x_{i,j}(k),\ y_{i,j}(k)) : k = 1, 2, \ldots, n(i, j)\}$$

be the set of tails. Suppose now that a number $R(i, j)$ is assigned to each (i, j) in R^0, and for each (i, j) in R^+ define inductively

$$R(i, j) = \sum_{k=1}^{n(i,j)} R(p_k), \tag{1}$$

where the points p_k are the vertices in $E(i, j)$. The numbers $R(i, j)$ comprise a rectangular array:

$$\begin{array}{ccccc} \vdots & & & & \\ R(0,2) & R(1,2) & R(2,2) & R(3,2) & \cdots \\ R(0,1) & R(1,1) & R(2,1) & R(3,1) & \cdots \\ R(0,0) & R(1,0) & R(2,0) & R(3,0) & \cdots \end{array}$$

which can be expressed in triangular form:

$$\begin{array}{ccccccc} & & & R(0,0) & & & \\ & & R(1,0) & & R(0,1) & & \\ & R(2,0) & & R(1,1) & & R(0,2) & \\ \cdots & & & & & & \end{array}$$

or

$$\begin{array}{ccccccc} & & & T(0,0) & & & \\ & & T(1,0) & & T(1,1) & & \\ & T(2,0) & & T(2,1) & & T(2,2) & \\ \cdots. & & & & & & \end{array}$$

Explicitly,

$$T(i, j) := R(i - j, j) \text{ for } 0 \le j \le i. \tag{2}$$

Henceforth, except for Examples 3C and 3D, we posit that for all (i, j) in R^0, the number $R(i, j)$ is the out-degree of (i, j), satisfying

$$R(i, 0) = 1 \quad \text{and} \quad R(0, j) \in \{0, 1\}. \tag{3}$$

Then, for (i, j) in R^+, the number of paths to (i, j) is $R(i, j)$, hence $T(i+j, j)$. The initial values $R(i, j)$ for (i, j) in R^0 imply initial values $T(i, 0)$ and $T(i, i)$ for $i \geq 0$; these values occupy the outermost wedge of the triangular array T. We call $\{R(i, j)\}$ the *path-counting rectangle of* G, and $\{T(i, j)\}$ the *path-counting triangle of* G for the given initial values. For reasons of notational convenience hereafter, define

$$R(i, j) = 0 \text{ if } i < 0 \text{ or } j < 0; \tag{4}$$

$$T(i, j) = 0 \text{ if } i < 0 \text{ or } j < 0 \text{ or } j > i. \tag{5}$$

2. INTEGER STRINGS AND COMPOSITIONS

In this section we restrict attention to path-counting under these conditions:

(i) $T(i, 0) = 1$ for $i \geq 0$;

(ii) for (i, j) in R^+, each (x, y) in $E(i, j)$ has the form $(i-1, j+q)$, where q is an element of a prescribed set Q of nonnegative integers.

By (1) and (2),

$$R(i, j) = \sum_{k=1}^{n} R(i-1, j+q_k), \tag{6}$$

$$T(i, j) = \sum_{k=1}^{n} T(i-q_k-1, j+q_k). \tag{7}$$

Theorem 1: Let Q be a nonempty set of nonnegative integers, and let i and j be positive integers. If $0 \in Q$, then the number of strings $(s_1, s_2, \ldots, s_m)$ of nonnegative integers s_k satisfying the three conditions,

(a) $s_k - s_{k-1} \in Q$ for $k = 2, 3, \ldots, m$,

(b) $s_m = j$,

(c) $m = i+1$,

is given as in (6) by $R(i, j)$ or, equivalently, by $T(i+j, j)$. If $0 \notin Q$, then the number of strings $(s_1, s_2, \ldots, s_m)$ of nonnegative integers s_k satisfying ***(a)***, ***(b)***, and

(c)' $m \leq i+1$

is given as in (6) by $R(i, j)$ or, equivalently, by $T(i+j, j)$.

Proof:

Case 1: $0 \in Q$. The paths counted by $R(i, j)$ consist of edges $(k-1, j_k)$-to-(k, j_{k+1}), where $j_{k+1} - j_k \in Q$ for $k = 1, 2, \ldots, i$, and $j_{i+1} = j$. Let $s_k = j_{k+1} - j_k$ for $k = 1, 2, \ldots, i+1$. Then $(s_1, s_2, \ldots, s_m)$ is a string of the sort described. Conversely, for $m = i+1$, each such string yields a path with initial point $(0, j_1)$ for some $j_1 \geq 0$ and terminal point (i, j), where $j = s_{i+1}$. This one-to-one correspondence between the paths and strings establishes that the number of strings is $R(i, j)$.

Case 2: $0 \notin Q$. Here, the initial point of a path can be of the form $(i_0, 0)$, where $0 \le i_0 \le i-1$. The one-to-one correspondence holds, but the length of a string can be $\le i+1$. □

Corollary 1A: Suppose $T(i,i)=0$ for $i \ge 1$. If $0 \in Q$, then $R(i, j)$, hence also $T(i+j, j)$, is the number of compositions of j consisting of i parts in the set Q. If $0 \notin Q$, then $R(i, j)$, hence also $T(i+j, j)$, is the number of compositions of j consisting of at most i parts, all in the set Q.

Proof: If $0 \in Q$, the i differences $j_{k+1} - j_k$ in the proof of Theorem 1 lie in Q and have sum j. Thus, there is a one-to-one correspondence between the paths counted by $R(i, j)$ and the compositions. If $0 \notin Q$, the same argument applies, except that the root of a path to (i, j) may be a point $(h, 0)$ for $0 \le h \le i-1$, and the corresponding number of parts is $i-h$. □

The following two corollaries have similar, omitted, proofs.

Corollary 1B: Suppose $h \ge 1$, $T(i,i)=1$ for $i \le h$, and $T(i,i)=0$ for $i > h$. If $0 \in Q$, then $R(i, j)$, hence also $T(i+j, j)$, is the number of compositions of the numbers $j, j-1, j-2, \ldots, j-h$ consisting of i parts in the set Q. If $0 \notin Q$, then $R(i, j)$, hence also $T(i+j, j)$, is the number of compositions of the numbers $j, j-1, j-2, \ldots, j-h$ consisting of at most i parts, all in the set Q.

Corollary 1C: Suppose $T(i,i)=1$ for all $i \ge 0$. If $0 \in Q$, then $R(i, j)$, hence also $T(i+j, j)$, is the number of compositions of the numbers $0, 1, 2, \ldots, j$ consisting of i parts in the set Q. If $0 \notin Q$, then $R(i, j)$, hence also $T(i+j, j)$, is the number of compositions of the numbers $0, 1, 2, \ldots, j$ consisting of at most i parts, all in the set Q.

Theorem 2: Suppose $n \ge 2$ and Q is a set of n nonnegative integers q_k. Suppose also that $q_1 < q_2 < \cdots < q_n$. Let S_i be the sum of numbers in row i of array $T(i, j)$. Then (S_i) is a linear recurrence sequence of order $q_n + 1$.

Proof:

$$S_i = \sum_{j=0}^{i} T(i,j) = T(i,0) + T(i,i) + \sum_{j=1}^{i-1} \sum_{k=1}^{n} T(i-q_k-1, j-q_k)$$
$$= T(i,0) + T(i,i) + \sum_{k=1}^{n} \sum_{j=1}^{i-1} T(i-q_k-1, j-q_k),$$

so that, by (5),

$$S_i = T(i,0) + T(i,i) + \sum_{k=1}^{n} \sum_{j=1}^{i-q_k-1} T(i-q_k-1, j)$$
$$= 1 + T(i,i) + \begin{cases} -T(i-q_k-1, 0) + \sum_{k=1}^{n} \sum_{j=0}^{i-q_k-1} T(i-q_k-1, j) & \text{if } q_1 = 0, \\ \sum_{k=1}^{n} \sum_{j=0}^{i-q_k-1} T(i-q_k-1, j) & \text{if } q_1 > 0, \end{cases}$$
$$= \begin{cases} T(i,i) + \sum_{k=1}^{n} S_{i-q_k-1} & \text{if } q_i = 0, \\ 1 + T(i,i) + \sum_{k=1}^{n} S_{i-q_k-1} & \text{if } q_1 > 0. \end{cases} \quad \square$$

The proof of Theorem 2 shows that, if $q_1 = 0$ and $T(i,i) = 0$ for all i greater than some i_0, then the linear recurrence is homogeneous for $i > i_0$. This is illustrated by Example 1C.

We turn now to applications of Theorems 1 and 2, in the form of Examples 1A-E, with particular interest in the appearance of Fibonacci and Lucas numbers in row sums or the central column.

Example 1A: A011973 in Sloane [5]

Initial values	$T(i,0) = 1$ for $i \geq 0$, $T(i,i) = 0$ for $i \geq 1$
Q	$\{0,1\}$
Recurrence	$T(i,j) = T(i-1,j) + T(i-2,j-1)$ for $1 \leq j \leq i-1$
Row sums	$1, 1, 2, 3, 5, 8, \ldots$ (Fibonacci numbers)

This is essentially the triangular array of coefficients of the Fibonacci polynomials [1], having rows (1), (1), (1, 1), (1, 2), (1, 3, 1), ... The two arrays have identical nonzero entries. Note that the southeast diagonals of nonzero entries form Pascal triangle: $T(i,j) = C(i-j,j)$.

1
1 0
1 1 0
1 2 0 0
1 3 1 0 0
1 4 3 0 0 0
1 5 6 1 0 0 0
1 6 10 4 0 0 0 0

For example, $T(6,2) = 6$ counts the compositions of 2 into 4 parts, each a 0 or 1, and it also counts strings of length 5, starting with 0 and ending in 2, with gaps of size 0 or 1:

compositions	0011	0101	0110	1001	1010	1100
strings	00012	00112	00122	01112	01122	01222

Example 1B: A005794 in Sloane [5]

Initial values	$T(i,0) = 1$ for $i \geq 0$; $T(i,i) = 1$ for $0 \leq i \leq 3$, else $T(i,i) = 0$
Q	$\{0,1\}$
Recurrence	$T(i,j) = T(i-1,j) + T(i-2,j-1)$ for $1 \leq j \leq i-1$
Row sums	$1, 2, 4, 7, 11, 18, 29, 47, \ldots$ (Lucas numbers)
Central column	$1, 2, 4, 8, 15, 26, 42, 64, \ldots$ (Cake numbers, A000125 in Sloane [5])

1
1 1
1 2 1
1 3 2 1
1 4 4 2 0
1 5 7 4 1 0
1 6 11 8 3 0 0
1 7 16 15 7 1 0 0
1 8 22 26 15 4 0 0 0

For example, $T(7,4) = 7$ counts the compositions of 1, 2, 3 into 3 parts, each a 0 or 1, and it also counts strings of length 4 starting with 0, 1, 2, or 3 and ending in 4, with gaps of size 0 or 1:

compositions	001	010	100	011	101	110	111
strings	3334	3344	3444	2234	2334	2344	1234

Regarding row sums, for $n \geq 2$, the number of strings $(s_1, s_2, \ldots, s_m)$ having gap sizes 0 or 1 and $m + s_m = n+1$ is the n^{th} Lucas number; e.g., for $n = 4$, the $L_4 = 11$ strings are as follows:

$$00000;\ 0001, 0111, 1111;\ 012, 112, 122, 222;\ 33, 23.$$

Example 1C: A052509 in Sloane [5]

Initial values	$T(i,0) = T(i,i) = 1$ for $i \geq 0$
Q	$\{0,1\}$
Recurrence	$T(i,j) = T(i-1,j) + T(i-2,j-1)$ for $1 \leq j \leq i-1$
Row sums	$1, 2, 4, 7, 12, \ldots$ (Fibonacci numbers minus 1)
Central column	$1, 1, 2, 4, 8, 16, \ldots$ (powers of 2)

1
1 1
1 2 1
1 3 2 1
1 4 4 2 1
1 5 7 4 2 1
1 6 11 8 4 2 1
1 7 16 15 8 4 2 1
1 8 22 26 16 8 4 2 1

For example, $T(5,2) = 7$ counts the compositions of 0, 1, 2 into 3 parts, each a 0 or 1, and it also counts strings of length 4 ending in 2 with gaps of size 0 or 1:

compositions	000	001	010	100	011	101	110
strings	2222	1112	1122	1222	0012	0112	0122

By Theorem 2, $S_i = 1 + S_{i-1} + S_{i-2}$ for $i \geq 2$. As a first step in an induction argument, we have $S_0 = F_3 - 1$ and $S_1 = F_4 - 1$. The hypothesis that $S_k = F_{k+3} - 1$ for all $k \leq i-1$ yields $S_i = 1 + F_{i+2} - 1 + F_{i+1} - 1 = F_{i+3} - 1$.

Example 1D: A055215 in Sloane [5]

Initial values	$T(i,0) = T(i,i) = 1$ for $i \geq 0$
Q	$\{1,2\}$
Recurrence	$T(i,j) = T(i-2,j-1) + T(i-3,j-2)$ for $1 \leq j \leq i-1$
Central column	$1, 1, 2, 3, 5, 8, \ldots$ (Fibonacci numbers)

1
1 1
1 1 1
1 1 2 1
1 1 2 2 1
1 1 2 3 2 1
1 1 2 3 4 2 1
1 1 2 3 5 4 2 1
1 1 2 3 5 7 4 2 1
1 1 2 3 5 8 8 4 2 1
1 1 2 3 5 8 12 8 4 2 1

$T(8, 5) = 7$ counts the compositions of the integers 1, 2, 3, 4, 5 into 3 parts, each a 1 or 2, and it also counts strings of length 4 ending in 5 with gaps of sizes 1 or 2:

compositions	111	112	121	211	122	212	221
strings	2345	1235	1245	1345	0135	0235	0245

In accord with Theorem 2, the sequence (S_i) of row sums satisfies the recurrence $S_i = S_{i-2} + S_{i-3} + 2$.

Example 1E: A055216 in Sloane [5]

Initial values	$T(i,0) = T(i,i) = 1$ for $i \geq 0$
Q	$\{0,1,2\}$
Recurrence	$T(i,j) = T(i-1,j) + T(i-2,j-1) + T(i-3,j-2)$ for $1 \leq j \leq i-1$
Central column	A027914 in Sloane [4]

1
1 1
1 2 1
1 3 3 1
1 4 6 3 1
1 5 10 8 3 1
1 6 15 17 9 3 1
1 7 21 31 23 9 3 1
1 8 28 51 50 26 9 3 1
1 9 36 78 96 66 27 9 3 1

$T(5, 3) = 8$ counts the compositions into 2 parts, each a 0, 1, or 2, of nonnegative integers ≤ 3, and it also counts strings of length 3 ending in 3 with gaps of size 0, 1, or 2:

compositions	00	01	10	11	02	20	12	21
strings	333	223	233	123	133	113	023	013

The array in Example 1E has interesting connections with the array of coefficients of $(1+x+x^2)^n$ considered by Hoggatt and Bicknell [3]. That array, $U(i, j)$ consists of trinomial coefficients. Written in left-justified form as in Comtet [2], we have

1										
1	1	1								
1	2	3	2	1						
1	3	6	7	6	3	1				
1	4	10	16	19	16	10	4	1		
1	5	15	30	45	51	45	30	15	5	1

For example, the partial row-sums, $\sum_{j=1}^{n} U(i, j)$, beginning with 1, 2, 6, 17, 50, form the central column of the preceding array.

Example 2A: A055800 in Sloane [5]

Initial values	$T(i,0) = 1$ for $i \geq 0$; $T(i,i) = 0$ for $i \geq 1$
Q	$\{1,3,5,7,9,\ldots\}$
Recurrence	$T(i,j) = \sum_{k=1}^{\infty} T(i-2k, j-2k+1)$ for $1 \leq j \leq i-1$
Row sums	$S_i = 2^{\lfloor i/2 \rfloor}$ (powers of 2)
Central column	$1,1,1,2,3,5,8,\ldots$ (Fibonacci numbers)

1
1 0
1 1 0
1 1 0 0
1 1 1 1 0
1 1 1 1 0 0
1 1 1 2 2 1 0
1 1 1 2 2 1 0 0
1 1 1 2 3 4 3 1 0
1 1 1 2 3 4 3 1 0 0
1 1 1 2 3 5 7 7 4 1 0

For example, $T(10, 5) = 5$ counts the compositions of 5 into parts in the set $\{1, 3, 5\}$, and it also counts strings ending in 5 with gaps of size 1, 3, or 5:

compositions	11111	113	131	311	5
strings	012345	0125	0145	0345	05

Example 2A points toward a more general result.

Theorem 3: The number of compositions of the positive integers $\leq n$ into odd parts is F_n.

Proof: By Corollary 1A, the number of compositions of $0, 1, 2, \ldots, n$ into odd parts is $T(2n, n)$. Therefore, it suffices to prove that $T(2n, n) = F_n$. We shall prove somewhat more: that the first $n+1$ terms of row $2n$ are $1, F_1, F_2, \ldots, F_{n-2}, F_{n-1}, F_n$ for $n \geq 1$. Assume for arbitrary $n \geq 2$ that this has been established for all $m \leq n-1$. Then, for row $2n$, we have $T(2n, 0) = 1$ and for $1 \leq j \leq n$,

$$T(2n, j) = \sum_{k=1}^{\infty} T(2n-2k, j-2k+1) = \sum_{h=0}^{\lfloor \frac{j-1}{2} \rfloor} F_{j-2h-1} = F_j. \quad \square$$

Example 2B: **A055801 in Sloane [5]**

Initial values	$T(i, 0) = T(i, i) = 1$ for $i \geq 0$
Q	$\{1, 3, 5, 7, 9, \ldots\}$
Recurrence	$T(i, j) = \sum_{k=1}^{\infty} T(i-2k, j-2k+1)$ for $1 \leq j \leq i-1$
Central column	$1, 1, 1, 2, 3, 5, 8, \ldots$ (Fibonacci numbers)

1
1 1
1 1 1
1 1 1 1
1 1 1 2 1
1 1 1 2 2 1
1 1 1 2 3 3 1
1 1 1 2 3 4 3 1
1 1 1 2 3 5 6 4 1
1 1 1 2 3 5 7 7 4 1
1 1 1 2 3 5 8 11 10 5 1

For example, $T(9, 6) = 7$ counts the compositions of numbers ≤ 6 using up to 3 parts, each an odd number, and it also counts strings of length ≤ 4 ending in 6 with no even gap sizes:

compositions	111	113	131	311	33	15	51
strings	3456	1236	1256	1456	036	016	056

3. ARRAYS BASED ON RECTANGULAR SETS $E(i, j)$

In Section 2, the tail-set $E(i, j)$ as defined in Section 1 is of the form $\{(i-1, j+q)\}$. That is to say, all edges into vertex (i, j) emanate from a single column of array $\{R(i, j)\}$. In Section 3, we consider paths for which $E(i, j)$ is a rectangle of more than one column.

***Example 3A:* A055807 in Sloane [5]**

Initial values	$R(i,0) = 1$ for $i \geq 0$, $R(0,j) = 0$ for $j \geq 1$
Recurrence	$R(i,j) = \sum_{i'=0}^{i-1} \sum_{j'=0}^{j} R(i',j')$ for $i \geq 1$, $j \geq 1$; $T(i,j) = R(i-j,j)$
Row sums	$1, 1, 2, 5, 13, \ldots$ (odd-indexed Fibonacci numbers)

```
              1
            1   0
          1   1   0
        1   3   1   0
      1   7   4   1   0
    1  15  12   5   1   0
  1  31  32  18   6   1   0
1  63  80  56  25   7   1   0
```

The array obtained by reflecting this one about its central column appears as A050143 in Sloane [5]. In order to see that the row sums in Example 3A are odd-indexed Fibonacci numbers, we first record an identity having an easy omitted proof:

$$R(i, j) = 2R(i-1, j) + R(i, j-1) - R(i-1, j-1) \text{ for } i \geq 1, \ j \geq 2. \tag{8}$$

$$S_n = \sum_{j=0}^{n} R(n-j, j) = 2^{n-1} + 1 + \sum_{j=2}^{n-1} R(n-j, j)$$
$$= 2^{n-1} + 1 + \sum_{j=2}^{n-1} [2R(n-j-1, j) + r(n-j, j-1) - R(n-j-1, j-1)],$$

by (8), so that

$$\begin{aligned} S_n &= 2\left[2^{n-2} + 1 + \sum_{j=2}^{n-3} R(n-j-1, j)\right] + 1 + R(n-2, 1) \\ &\quad + \sum_{j=2}^{n-3} R(n-j-1, j) - \sum_{j=2}^{n-2} R(n-j-1, j-1) \\ &= (2S_{n-1} + 1) + (2^{n-2} - 1 + S_{n-1} - 2^{n-2} - 1) - \left[2^{n-3} - 1 + \sum_{j=2}^{n-4} R(n-j-2, j)\right] \\ &= 3S_{n-1} - S_{n-2}. \end{aligned} \tag{9}$$

Since both sequences (S_n) and (F_{2n-1}) are uniquely determined by initial values $S_0 = 1$ and $S_1 = 2$ together with the recurrence in (9), we have $S_n = F_{2n-1}$ for $n \geq 0$.

***Example 3B:* A055818 in Sloane [5]**

Initial values	$R(i,0) = R(0,i) = 1$ for $i \geq 0$
Recurrence	$R(i,j) = \sum_{i'=0}^{i-1} \sum_{j'=0}^{j} R(i',j')$ for $i \geq 1,\ j \geq 1$; $T(i,j) = R(i-j,j)$
Row sums	$1, 2, 4, 10, 26, \ldots$ (twice odd-indexed Fibonacci numbers)

$$\begin{array}{ccccccccccccccc} &&&&&&&1&&&&&&& \\ &&&&&&1&&1&&&&&& \\ &&&&&1&&2&&1&&&&& \\ &&&&1&&5&&3&&1&&&& \\ &&&1&&11&&9&&4&&1&&& \\ &&1&&23&&24&&14&&5&&1&& \\ &1&&47&&60&&43&&20&&6&&1& \\ 1&&95&&144&&122&&69&&27&&7&&1 \end{array}$$

The recurrences (8) hold for this array and can be used to prove that the row sums are given by $S_n = 2F_{2n-1}$ for $n \geq 1$.

Next, we break free of the initial values (3). When counting paths into the point (3, 0), for example, rather than counting only the edge (0, 0)-to-(3, 0) as a path, we can treat each of the following as paths:

(0, 0)-to-(3, 0),
(0, 0)-to-(2, 0)-to-(3, 0),
(0, 0)-to-(1, 0)-to-(3, 0),
(0, 0)-to-(1, 0)-to-(2, 0)-to-(3, 0).

More generally, for this sort of path, the number of paths entering $(i, 0)$ is 2^{i-1} for $i \geq 1$. Using as initial values $R(i, 0) = 2^{i-1}$, we count certain paths over rectangular tail-sets and obtain another array.

***Example 3C:* A049600 in Sloane [5]**

Initial values	$R(0,0) = 1$, $R(i,0) = 2^{i-1}$ and $R(0,i) = 0$ for $i \geq 1$
Recurrence	$R(i,j) = \sum_{i'=0}^{i-1} \sum_{j'=0}^{j} R(i',j')$ for $i \geq 1,\ j \geq 1$; $T(i,j) = R(i-j,j)$
Row sums	$1, 1, 3, 8, 21, 55, \ldots$ (even-indexed Fibonacci numbers)
Alternating row sums	$1, 1, 1, 2, 3, 5, 8, \ldots$ (Fibonacci numbers)

$$\begin{array}{ccccc} 1 &&&& \\ 1 & 0 &&& \\ 2 & 1 & 0 && \\ 4 & 3 & 1 & 0 & \\ 8 & 8 & 4 & 1 & 0 \end{array}$$

This array and its connections to compositions are considered in [4]. Again, the recurrences (8) prevail and can be used to prove that the row sums are given by $S_n = F_{2n}$ for $n \geq 1$, and that alternating rows sums defined by

$$A_n = T(n,0) - T(n,1) + T(n,2) - \cdots + (-1)^n T(n,n)$$

satisfy $A_n = F_n$ for $n \geq 1$.

Next, we consider rectangular tail-sets restricted to just two columns. On the x-axis the initial values $R(1,0)=1$ and $R(2,0)=1$, together with the two-column recurrence, determine the Fibonacci sequence for values of $R(i,0)$.

***Example 3D:* A055830 in Sloane [5]**

Initial values	$R(0,0)=1,\ R(i,0)=F_{i+1}$ for $i \geq 1,\ R(0,j)=0$ for $j \geq 1$
Recurrence	$R(i,j)=\sum_{i'=i-2}^{i-1} \sum_{j'=0}^{j} R(i',j')$ for $i \geq 1,\ j \geq 1;\ T(i,j)=R(i-j,j)$
1st diagonal	$1,1,1,2,3,5,8,\ldots$ (Fibonacci numbers)

1
1 0
2 1 0
3 3 1 0
5 7 4 1 0
8 15 12 5 1 0
13 30 31 18 6 1 0
21 58 73 54 25 7 1 0

In [4], this sort of array is discussed not only for 2-column tail-sets, but also for m-column tail-sets for $m > 2$.

4. A SYMMETRIC ARRAY

We consider one more array, this one given by one recurrence for points beneath the line $y = x$ and another, symmetric to the first, for the points above the line $y = x$.

***Example 4:* A038792 in Sloane [5]**

Initial values	$T(i,0)=T(i,i)=1$ for $i \geq 0$
Recurrence	$T(i,j)=T(i-1,j)+T(i-2,j-1)$ if $i < j/2$, else $T(i,j)=T(i,i-1)$
Central column	$1,2,5,13,34,\ldots$ (odd-indexed Fibonacci numbers)

1
1 1
1 2 1
1 3 3 1
1 4 5 4 1
1 5 8 8 5 1
1 6 12 13 12 6 1
1 7 17 21 21 17 7 1

Note that the recurrence can be written in symmetric form, as follows:

$$T(i,j) = \begin{cases} T(i-1,j) + T(i-2,j-1) & \text{if } 2i \leq j, \\ T(i-1,j-1) + T(i-2,j-1) & \text{if } 2i > j. \end{cases}$$

It is easy to prove that the central column of this array is the sequence of odd-indexed Fibonacci numbers, in conjunction with the fact that each column adjacent to the central one is the even-indexed Fibonacci sequence.

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