RISING DIAGONAL POLYNOMIALS ASSOCIATED WITH MORGAN-VOYCE POLYNOMIALS

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1. INTRODUCTION

Diagonal polynomials have been defined for Chebyshev, Fermat, Fibonacci, Lucas, Jacobsthal and other polynomials, and their properties have been studied (see, e.g., [9]. [5], and [7]). However, these are not applicable to the diagonal polynomials associated with the Morgan-Voyce polynomials (hereafter denoted as MVPs) $B_n(x), b_n(x), c_n(x)$, and $C_n(x)$, defined by:

$$B_n(x) = (x+2)B_{n-1}(x) - B_{n-2}(x) \quad (n \geq 2), \tag{1.1a}$$

with

$$B_0(x) = 1, \quad B_1(x) = x+2; \tag{1.1b}$$

$$b_n(x) = (x+2)b_{n-1}(x) - b_{n-2}(x) \quad (n \geq 2), \tag{1.2a}$$

with

$$b_0(x) = 1, \quad b_1(x) = x+1; \tag{1.2b}$$

$$c_n(x) = (x+2)c_{n-1}(x) - c_{n-2}(x) \quad (n \geq 2), \tag{1.3a}$$

with

$$c_0(x) = 1, \quad c_1(x) = x+3; \tag{1.3b}$$

$$C_n(x) = (x+2)C_{n-1}(x) - C_{n-2}(x) \quad (n \geq 2), \tag{1.4a}$$

with

$$C_0(x) = 2, \quad C_1(x) = x+2. \tag{1.4b}$$

Many interesting results have been proved regarding these MVPs (see [10], [11], [14], [12], [1], [2], [6], and [8]), and some of the important known results are listed in Section 2 for ready reference as well as for establishing the results regarding their associated diagonal polynomials.

2. SOME IMPORTANT PROPERTIES OF THE MORGAN-VOYCE POLYNOMIALS

Interrelations:

$$b_n(x) = B_n(x) - B_{n-1}(x) \quad (n \geq 1), \quad \text{from [10].} \tag{2.1}$$

$$xB_n(x) = b_{n+1}(x) - b_n(x), \quad \text{from [10].} \tag{2.2}$$

$$C_n(x) = B_n(x) - B_{n-2}(x) \quad (n \geq 2), \quad \text{from [14], [13].} \tag{2.3}$$

$$C_n(x) = b_n(x) + b_{n-1}(x) \quad (n \geq 2), \quad \text{from [14], [13].} \tag{2.4}$$

$$xc_n(x) = b_{n+1}(x) - b_{n-1}(x) \quad (n \geq 1), \quad \text{from [6].} \tag{2.5}$$

$$C_n(x) = c_n(x) - c_{n-1}(x) \quad (n \geq 1), \quad \text{from [6], [13].} \tag{2.6}$$

$$xc_n(x) = C_{n+1}(x) - C_n(x), \quad \text{from (2.4) and (2.5).} \tag{2.7}$$

$$c_n(x) = B_n(x) + B_{n-1}(x) \quad (n \geq 1), \quad \text{from [13].} \tag{2.8}$$

Closed-Form Expressions:

$$B_n(x) = \sum_{k=0}^{n} \binom{n+k+1}{n-k} x^k, \quad \text{from [11].} \tag{2.9}$$

$$b_n(x) = \sum_{k=0}^{n} \binom{n+k}{n-k} x^k, \quad \text{from [11].} \tag{2.10}$$

$$c_n(x) = \sum_{k=0}^{n} \frac{2n+1}{2k+1} \cdot \binom{n+k}{n-k} x^k, \quad \text{from (2.8) and (2.9).} \tag{2.11}$$

$$C_n(x) = 2 + \sum_{k=1}^{n} \frac{n}{k} \cdot \binom{n+k-1}{n-k} x^k, \quad \text{from (2.4) and (2.10).} \tag{2.12}$$

It should be noted that (2.12) has been derived earlier (see [2]).

Zeros:

$$B_n(x)\colon\ x_r = -4\sin^2\left\{\frac{r}{n+1}\cdot\frac{\pi}{2}\right\},\ r = 1, 2, \ldots, n, \quad \text{from [12].} \tag{2.13}$$

$$b_n(x)\colon\ x_r = -4\sin^2\left\{\frac{2r-1}{2n+1}\cdot\frac{\pi}{2}\right\},\ r = 1, 2, \ldots, n, \quad \text{from [12].} \tag{2.14}$$

$$c_n(x)\colon\ x_r = -4\sin^2\left\{\frac{2r}{2n+1}\cdot\frac{\pi}{2}\right\},\ r = 1, 2, \ldots, n, \quad \text{from [1].} \tag{2.15}$$

$$C_n(x)\colon\ x_r = -4\sin^2\left\{\frac{2r-1}{2n}\cdot\frac{\pi}{2}\right\},\ r = 1, 2, \ldots, n, \quad \text{from [14].} \tag{2.16}$$

Generating Functions:

$$B(x, t) = \sum_{0}^{\infty} B_n(x)t^n = [1-(xt+2t-t^2)]^{-1}, \quad \text{from (1.1a).} \tag{2.17}$$

$$b(x, t) = \sum_{0}^{\infty} b_n(x)t^n = (1-t)B(x, t), \quad \text{from (2.1) and (2.17).} \tag{2.18}$$

$$c(x, t) = \sum_{0}^{\infty} c_n(x)t^n = (1+t)B(x, t), \quad \text{from (2.8) and (2.17).} \tag{2.19}$$

$$C(x, t) = \sum_{0}^{\infty} C_n(x)t^n = 1+(1-t^2)B(x, t), \quad \text{from (2.3) and (2.17).} \tag{2.20}$$

Differential Equations:

$$B_n(x)\colon\ x(x+4)y'' + 3(x+2)y' - n(n+2)y = 0, \quad \text{from [12].} \tag{2.21}$$

$$b_n(x)\colon\ x(x+4)y'' + 2(x+1)y' - n(n+1)y = 0, \quad \text{from [12].} \tag{2.22}$$

$$c_n(x)\colon\ x(x+4)y'' + 2(x+3)y' - n(n+1)y = 0, \quad \text{from [13].} \tag{2.23}$$

$$C_n(x)\colon\ x(x+4)y'' + (x+2)y' - n^2y = 0, \quad \text{from [3].} \tag{2.24}$$

Orthogonality Property:

$B_n(x)$: Orthogonal over $(-4, 0)$ with respect to the weight function $\sqrt{-x(x+4)}$, from [11]. (2.25)

$b_n(x)$: Orthogonal over $(-4, 0)$ with respect to the weight function $\sqrt{-(x+4)/x}$, from [11]. (2.26)

$c_n(x)$: Orthogonal over $(-4, 0)$ with respect to the weight function $\sqrt{-x/(x+4)}$, from [13]. (2.27)

$C_n(x)$: Orthogonal over $(-4, 0)$ with respect to the weight function $1/\sqrt{-x(x+4)}$, from [2]. (2.28)

Simson Formulas:

$$B_{n+1}(x)B_{n-1}(x) - B_n^2(x) = -1, \quad \text{from [11].} \tag{2.29}$$

$$b_{n+1}(x)b_{n-1}(x) - b_n^2(x) = x, \quad \text{from [12].} \tag{2.30}$$

$$c_{n+1}(x)c_{n-1}(x) - c_n^2(x) = -(x+4), \quad \text{from [13].} \tag{2.31}$$

$$C_{n+1}(x)C_{n-1}(x) - C_n^2(x) = x(x+4), \quad \text{from [13].} \tag{2.32}$$

3. RISING DIAGONAL POLYNOMIALS

In order to define the diagonal polynomials associated with the Morgan-Voyce polynomials in a manner similar to the diagonal polynomials defined for Chebyshev, Fermat, Fibonacci, and other polynomials (see [9], [5], [7]), we first need to express the MVPs $B_n(x)$, $b_n(x)$, $c_n(x)$, and $C_n(x)$ in descending powers of x. By letting $i = n-k$ in (2.9), (2.10), (2.11), and (2.12), we get the following expressions for the MVPs:

$$B_n(x) = \sum_{i=0}^{n} \binom{2n+1-i}{i} x^{n-i}; \tag{3.1}$$

$$b_n(x) = \sum_{i=0}^{n} \binom{2n-i}{i} x^{n-i}; \tag{3.2}$$

$$c_n(x) = \sum_{i=0}^{n} \frac{2n+1}{2n+1-2i} \cdot \binom{2n-i}{i} x^{n-i}; \tag{3.3}$$

$$C_n(x) = x^n + \sum_{i=1}^{n-1} \frac{n}{n-i} \cdot \binom{2n-1-i}{i} x^{n-i} + 2. \tag{3.4}$$

We now rearrange $C_n(x)$ into a form that will help in formulating a closed-form expression for the corresponding rising diagonal polynomial. It can be shown that

$$\frac{n}{n-i} \cdot \binom{2n-1-i}{i} = \frac{2n}{i} \cdot \binom{2n-1-i}{i-1}.$$

Hence, (3.4) can be rewritten as

$$C_n(x) = x^n + \sum_{i=1}^{n-1} \frac{2n}{i} \cdot \binom{2n-1-i}{i-1} x^{n-i} + 2,$$

or

$$C_n(x) = x^n + \sum_{i=1}^{n} \frac{2n}{i} \cdot \binom{2n-1-i}{i-1} x^{n-i}. \tag{3.5}$$

Let us first consider the rising diagonal polynomial $R_n(x)$ associated with the MVP $B_n(x)$. We see from (3.1) that

$$R_0(x) = 1,\ R_1(x) = x,\ R_2(x) = x^2 + 2,\ R_3(x) = x^3 + 4x, \ldots,$$

$$R_n(x) = x^n + \binom{2n-2}{1} x^{n-2} + \binom{2n-5}{2} x^{n-4} + \binom{2n-8}{3} x^{n-6} + \cdots.$$

The above may be rewritten as

$$R_n(x) = \binom{2n+1}{0} x^n + \binom{2n-2}{1} x^{n-2} + \binom{2n-5}{2} x^{n-4} + \cdots + \binom{2n+1-3\left[\frac{n}{2}\right]}{\left[\frac{n}{2}\right]} x^{n-2\left[\frac{n}{2}\right]}.$$

Hence,

$$R_n(x) = \sum_{i=0}^{[n/2]} \binom{2n+1-3i}{i} x^{n-2i}. \tag{3.6}$$

Similarly, starting with (3.2), (3.3), and (3.5), we may derive the following polynomial expressions for the rising diagonal polynomials $r_n(x)$, $\rho_n(x)$, and $\mathrm{P}_n(x)$ associated, respectively, with the MVPs $b_n(x)$, $c_n(x)$, and $C_n(x)$:

$$r_n(x) = \sum_{i=0}^{[n/2]} \binom{2n-3i}{i} x^{n-2i}; \tag{3.7}$$

$$\rho_n(x) = \sum_{i=0}^{[n/2]} \frac{2n+1-2i}{2n+1-4i} \cdot \binom{2n-3i}{i} x^{n-2i}; \tag{3.8}$$

$$\mathrm{P}_n(x) = x^n + \sum_{i=1}^{[n/2]} \frac{2(n-i)}{i} \cdot \binom{2n-1-3i}{i-1} x^{n-2i}. \tag{3.9}$$

It is readily seen that all the four sets of diagonal polynomials are even for even values of n and odd for odd values of n. Table 1 lists the diagonal polynomials up to $n = 8$.

4. SOME INTERRELATIONS AMONG $R_n(x)$, $r_n(x)$, $\rho_n(x)$ AND $\mathrm{P}_n(x)$

Consider the expression $R_n(x) - R_{n-2}(x)$. Then, from (3.6), we have

$$\begin{aligned} R_n(x) - R_{n-2}(x) &= \sum_{i=0}^{[n/2]} \binom{2n+1-3i}{i} x^{n-2i} - \sum_{i=0}^{[n/2]-1} \binom{2n-3-3i}{i} x^{n-2-2i} \\ &= x^n + \sum_{i=1}^{[n/2]} \binom{2n+1-3i}{i} x^{n-2i} - \sum_{i=1}^{[n/2]} \binom{2n-3i}{i-1} x^{n-2i} \\ &= x^n + \sum_{i=1}^{[n/2]} \frac{2n-4i+1}{i} \cdot \frac{(2n-3i)\ldots(2n-4i+2)}{(i-1)!} x^{n-2i} \end{aligned}$$

$$= x^n + \sum_{i=1}^{[n/2]} \binom{2n-3i}{i} x^{n-2i} = \sum_{i=0}^{[n/2]} \binom{2n-3i}{i} x^{n-2i}$$

$$= r_n(x), \text{ using (3.7).}$$

Hence, we have the result that

$$r_n(x) = R_n(x) - R_{n-2}(x) \quad (n \geq 2). \tag{4.1}$$

It is interesting to compare this result with the corresponding one relating the respective MVPs, namely,

$$b_n(x) = B_n(x) - B_{n-1}(x) \quad (n \geq 1).$$

We now prove that

$$xR_n(x) = r_{n+1}(x) - r_{n-1}(x) \quad (n \geq 1), \tag{4.2}$$

a result which corresponds to (2.2) with respect to the original MVPs $B_n(x)$ and $b_n(x)$. First, consider $r_{2n+1}(x) - r_{2n-1}(x)$. Then, from (3.7),

$$\begin{aligned} r_{2n+1}(x) - r_{2n-1}(x) &= \sum_{i=0}^{n} \binom{4n+2-3i}{i} x^{2n+1-2i} - \sum_{i=0}^{n-1} \binom{4n-2-3i}{i} x^{2n-1-2i} \\ &= x^{2n+1} + x\sum_{i=1}^{n} \binom{4n+2-3i}{i} x^{2n-2i} - x\sum_{i=1}^{n} \binom{4n+1-3i}{i-1} x^{2n-2i} \\ &= x^{2n+1} + x\sum_{i=1}^{n} \binom{4n+1-3i}{i} x^{2n-2i} \\ &= x\sum_{i=0}^{n} \binom{4n+1-3i}{i} x^{2n-2i} \\ &= xR_{2n}(x), \text{ using (3.6).} \end{aligned}$$

Similarly, we can show that

$$r_{2n+2}(x) - r_{2n}(x) = xR_{2n+1}(x).$$

Hence, the result (4.2).

Again, from (3.7), we have

$$\begin{aligned} r_{2n+1}(x) + r_{2n-1}(x) &= x^{2n+1} + x\sum_{i=1}^{n} \binom{4n+2-3i}{i} x^{2n-2i} + x\sum_{i=1}^{n} \binom{4n+1-3i}{i-1} x^{2n-2i} \\ &= x^{2n+1} + \sum_{i=1}^{n} \frac{2(2n+1-2i)}{i} \cdot \binom{4n+1-3i}{i-1} x^{2n+1-2i} \\ &= \mathrm{P}_{2n+1}(x), \text{ using (3.9).} \end{aligned} \tag{4.3a}$$

Similarly,

$$r_{2n+2}(x) + r_{2n}(x) = \mathrm{P}_{2n+2}(x). \tag{4.3b}$$

Combining (4.3a) and (4.3b), we get

$$\mathrm{P}_n(x) = r_n(x) + r_{n-2}(x) \quad (n \geq 2), \tag{4.4}$$

a result to be compared with (2.4). Using (4.1), the above relation may be rewritten as

$$\mathrm{P}_n(x) = R_n(x) - R_{n-4}(x) \quad (n \geq 4), \tag{4.5}$$

the corresponding result for the MVPs being (2.3).

Again starting with $R_n(x) + R_{n-2}(x)$ and using (3.6), we can show that

$$\rho_n(x) = R_n(x) + R_{n-2}(x) \quad (n \geq 2), \tag{4.6}$$

which should be compared with relation (2.8) for the corresponding MVPs. Now, using (4.6), we have

$$\rho_n(x) - \rho_{n-2}(x) = R_n(x) - R_{n-4}(x).$$

Hence, from (4.5), we get

$$\mathrm{P}_n(x) = \rho_n(x) - \rho_{n-2}(x) \quad (n \geq 2), \tag{4.7}$$

the corresponding relation for the MVPs being (2.6). Further, using (4.4), we have

$$\begin{aligned} \mathrm{P}_{n+1}(x) - \mathrm{P}_{n-1}(x) &= \{r_{n+1}(x) - r_{n-1}(x)\} + \{r_{n-1}(x) - r_{n-3}(x)\} \\ &= xR_n(x) + xR_{n-2}(x), \text{ using (4.2)}, \\ &= x\rho_n(x), \text{ using (4.6)}. \end{aligned}$$

Hence,

$$x\rho_n(x) = \mathrm{P}_{n+1}(x) - \mathrm{P}_{n-1}(x) \quad (n \geq 1), \tag{4.8}$$

a relation corresponding to (2.7) for the original MVPs.

We may derive a number of such interrelationships among the diagonal polynomials $R_n(x)$, $r_n(x)$, $\rho_n(x)$, and $\mathrm{P}_n(x)$ corresponding to those of the MVPs $B_n(x)$, $b_n(x)$, $c_n(x)$, and $C_n(x)$. We will only list the following:

$$\sum_{i=0}^{n} r_i(x) = R_n(x) + R_{n-1}(x); \tag{4.9}$$

$$x\sum_{i=0}^{n} R_i(x) = r_{n+1}(x) + r_n(x) - 1; \tag{4.10}$$

$$\sum_{i=0}^{n} \mathrm{P}_i(x) = \rho_n(x) + \rho_{n-1}(x) + 1; \tag{4.11}$$

$$x\sum_{i=0}^{n} \rho_i(x) = \mathrm{P}_{n+1}(x) + \mathrm{P}_n(x) - 2. \tag{4.12}$$

5. RECURRENCE RELATIONS AND GENERATING FUNCTIONS

From relation (4.2), we have

$$\begin{aligned} xR_n(x) &= r_{n+1}(x) - r_{n-1}(x) \quad (n \geq 1) \\ &= \{R_{n+1}(x) - R_{n-1}(x)\} - \{R_{n-1}(x) - R_{n-3}(x)\} \quad (n \geq 3), \text{ using (4.1)}. \end{aligned}$$

Hence,

$$R_{n+1}(x) = xR_n(x) + 2R_{n-1}(x) - R_{n-3}(x) \quad (n \geq 3).$$

Therefore, $R_n(x)$ satisfies the recurrence relation

$$R_n(x) = xR_{n-1}(x) + 2R_{n-2}(x) - R_{n-4}(x) \quad (n \geq 4), \tag{5.1a}$$

with

$$R_0(x) = 1,\ R_1(x) = x,\ R_2(x) = x^2 + 2,\ R_3(x) = x^3 + 4x. \tag{5.1b}$$

Similarly, we can deduce that $r_n(x)$, $\rho_n(x)$, and $\mathrm{P}_n(x)$ satisfy the following recurrence relations:

$$r_n(x) = xr_{n-1}(x) + 2r_{n-2}(x) - r_{n-4}(x) \quad (n \geq 4), \tag{5.2a}$$

with

$$r_0(x) = 1,\ r_1(x) = x,\ r_2(x) = x^2 + 1,\ r_3(x) = x^3 + 3x; \tag{5.2b}$$

$$\rho_n(x) = x\rho_{n-1}(x) + 2\rho_{n-2}(x) - \rho_{n-4}(x) \quad (n \geq 4), \tag{5.3a}$$

with

$$\rho_0(x) = 1,\ \rho_1(x) = x,\ \rho_2(x) = x^2 + 3,\ \rho_3(x) = x^3 + 5x; \tag{5.3b}$$

$$\mathrm{P}_n(x) = x\mathrm{P}_{n-1}(x) + 2\mathrm{P}_{n-2}(x) - \mathrm{P}_{n-4}(x) \quad (n \geq 4), \tag{5.4a}$$

with

$$\mathrm{P}_0(x) = 2,\ \mathrm{P}_1(x) = x,\ \mathrm{P}_2(x) = x^2 + 2,\ \mathrm{P}_3(x) = x^3 + 4x. \tag{5.4b}$$

It is interesting to compare the above recurrence relations with those of the corresponding MVPs $B_n(x)$, $b_n(x)$, $c_n(x)$, and $C_n(x)$ given by (1.1), (1.2), (1.3), and (1.4), respectively.

We shall now derive generating functions for these diagonal polynomials using the standard technique. Let $g_n(x)$ represent any one of the diagonal polynomials $R_n(x)$, $r_n(x)$, $\rho_n(x)$, or $\mathrm{P}_n(x)$, and let $G(x, t)$ be the corresponding generating function. Then, from [4], we have

$$\begin{aligned} &t^{-4}[G(x,t) - g_0(x) - g_1(x)t - g_2(x)t^2 - g_3(x)t^3] \\ &= xt^{-3}[G(x,t) - g_0(x) - g_1(x)t - g_2(x)t^2] \\ &\qquad + 2t^{-2}[G(x,t) - g_0(x) - g_1(x)t] - G(x,t). \end{aligned}$$

Hence,

$$\begin{aligned} (1 - xt - 2t^2 + t^4)G(x,t) &= g_0(x) + \{g_1(x) - xg_0(x)\}t \\ &+ \{g_2(x) - xg_1(x) - 2g_0(x)\}t^2 + \{g_3(x) - xg_2(x) - 2g_1(x)\}t^4. \end{aligned} \tag{5.5}$$

Therefore, $R(x, t)$, the generating function for the diagonal polynomial $R_n(x)$, is given by

$$\begin{aligned} (1 - xt - 2t^2 + t^4)R(x,t) &= 1 + (x - x)t + (x^2 + 2 - x^2 - 2)t^2 \\ &+ (x^3 + 4x - x^3 - 2x - 2x)t^4 = 1. \end{aligned}$$

Hence,

$$R(x,t) = \sum_0^\infty R_i(x)t^i = [1 - (xt + 2t^2 - t^4)]^{-1}. \tag{5.6}$$

Similarly, by substituting for $g_n(x)$ the diagonal polynomials $r_n(x)$, $\rho_n(x)$, and $\mathrm{P}_n(x)$ in (5.5), we can derive the following generating functions for these polynomials:

$$r(x,t) = \sum_0^\infty r_i(x)t^i = (1 - t^2)R(x,t); \tag{5.7}$$

$$\rho(x,t) = \sum_0^\infty \rho_i(x)t^i = (1 + t^2)R(x,t); \tag{5.8}$$

$$\mathrm{P}(x,t)=\sum_0^\infty \mathrm{P}_i(x)t^i = 1+(1-t^4)R(x,t). \tag{5.9}$$

It is interesting to compare the generating functions (5.6), (5.7), (5.8), and (5.9) of the diagonal polynomials with those of the corresponding MVPs $B_n(x)$, $b_n(x)$, $c_n(x)$, and $C_n(x)$, namely, those given by (2.17), (2.18), (2.19), and (2.20).

Using the generating function (5.6), we will now derive an interesting relation among the derivatives. From (5.6),

$$\frac{\partial R(x,t)}{\partial x} = tR^2(x,t)$$

and

$$\frac{\partial R(x,t)}{\partial t} = (x+4t-4t^3)R^2(x,t).$$

Hence,

$$(x+4t-4t^3)\frac{\partial R(x,t)}{\partial x} = t\frac{\partial R(x,t)}{\partial t}. \tag{5.10}$$

Thus, from (5.6),

$$xR_n'(x)+4R_{n-1}'(x)-4R_{n-3}'(x)=nR_n(x). \tag{5.11}$$

However, from (5.1), we have

$$R_{n+1}'(x)=xR_n'(x)+R_n(x)+2R_{n-1}'(x)-R_{n-3}'(x). \tag{5.12}$$

Substituting for $xR_n'(x)$ from (5.12) in (5.11) and rearranging the terms, we get

$$(n+1)R_n(x)=\{R_{n+1}'(x)-R_{n-1}'(x)\}+3\{R_{n-1}'(x)-R_{n-3}'(x)\}.$$

Using (4.1) in the above expression, we have the result

$$(n+1)R_n(x)=r_{n+1}'(x)+3r_{n-1}'(x). \tag{5.13}$$

Apart from the above result, it has not been possible to derive any other simple derivative relation for the rising diagonal polynomials.

6. CONCLUDING REMARKS

We have thus defined and obtained polynomial expressions for the four sets of diagonal polynomials associated with the four sets of Morgan-Voyce polynomials $B_n(x)$, $b_n(x)$, $c_n(x)$, and $C_n(x)$. We have also obtained a number of interesting properties of these diagonal polynomials, including the recurrence relations they satisfy. It appears that these diagonal polynomials have a number of other interesting properties.

We would like to mention one such interesting property regarding the location of the zeros of these diagonal polynomials. Using the network properties of two-element-kind electrical networks, it is possible to show that, for $n=1,2,\ldots,8$, the following results hold:

(a) The zeros of $R_n(x)$, $r_n(x)$, $\rho_n(x)$, and $\mathrm{P}_n(x)$ are all simple and lie on the imaginary axis, that is, all the zeros are purely imaginary.

(b) The zeros of $R_{n+1}(x)$ interlace on the imaginary axis with those of $R_n(x)$, $r_n(x)$, $\rho_n(x)$, and $P_n(x)$. Also, the zeros of $r_{n+1}(x)$ interlace on the imaginary axis with those of $R_n(x)$, $r_n(x)$, and $P_n(x)$, the zeros of $\rho_{n+1}(x)$ interlace on the imaginary axis with those of $R_n(x)$, $r_n(x)$, $\rho_n(x)$, and $P_n(x)$, and those of $P_{n+1}(x)$ interlace on the imaginary axis with those of $R_n(x)$, $r_n(x)$, $\rho_n(x)$, and $P_n(x)$.

(c) However, the zeros of $r_{n+1}(x)$ and those of $\rho_n(x)$ do not interlace, except for the case of $n=1$.

We conjecture that the above results are true for any value of n.

TABLE 1

Rising Diagonal Polynomials for n = 0, 1, 2, ..., 8

$R_0(x)=1$ $R_1(x)=x$ $R_2(x)=x^2+2$ $R_3(x)=x^3+4x$ $R_4(x)=x^4+6x^2+3$ $R_5(x)=x^5+8x^3+10x$ $R_6(x)=x^6+10x^4+21x^2+4$ $R_7(x)=x^7+12x^5+36x^3+20x$ $R_8(x)=x^8+14x^6+55x^4+56x^2+5$	$r_0(x)=1$ $r_1(x)=x$ $r_2(x)=x^2+1$ $r_3(x)=x^3+3x$ $r_4(x)=x^4+5x^2+1$ $r_5(x)=x^5+7x^3+6x$ $r_6(x)=x^6+9x^4+15x^2+1$ $r_7(x)=x^7+11x^5+28x^3+10x$ $r_8(x)=x^8+13x^6+45x^4+35x^2+1$
$\rho_0(x)=1$ $\rho_1(x)=x$ $\rho_2(x)=x^2+3$ $\rho_3(x)=x^3+5x$ $\rho_4(x)=x^4+7x^2+5$ $\rho_5(x)=x^5+9x^3+14x$ $\rho_6(x)=x^6+11x^4+27x^2+7$ $\rho_7(x)=x^7+13x^5+44x^3+30x$ $\rho_8(x)=x^8+15x^6+65x^4+77x^2+9$	$P_0(x)=2$ $P_1(x)=x$ $P_2(x)=x^2+2$ $P_3(x)=x^3+4x$ $P_4(x)=x^4+6x^2+2$ $P_5(x)=x^5+8x^3+9x$ $P_6(x)=x^6+10x^4+20x^2+2$ $P_7(x)=x^7+12x^5+35x^3+16x$ $P_8(x)=x^8+14x^6+54x^4+50x^2+2$

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