ALMOST SQUARE TRIANGULAR NUMBERS

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1. INTRODUCTION

While undergoing the study of Square Triangular Numbers (STN), it was observed that there are certain triangular numbers (TN) which, although not squares, are "very close" to squares. If we restrict this closeness to just unity, we obtain what we shall call "Almost Square Triangular Numbers" (ASTN). More precisely, an ASTN is a TN that differs from a perfect square exactly by unity.

The very description of ASTN leads to their two types: first, those TN that exceed a perfect square by one; second, those that fall short of a perfect square by one.

The purpose of this paper is to account for all the ASTN of both types by linking them with STN.

2. SOME PRELIMINARIES

2.1 (Def.) α-ASTN

A TN x will be called an ASTN of the type α (α-ASTN) iff $x-1$ is a perfect square.

The first ten α-ASTN are:

10, 325, 11026, 374545, 12723490, 432224101, 14682895930, 498786237505, 16944049179226, and 575598885856165.

2.2 (Def.) β-ASTN

A TN y will be called an ASTN of the type β (β-ASTN) iff $y+1$ is a perfect square.

The first ten β-ASTN are:

3, 15, 120, 528, 4095, 17955, 139128, 609960, 4726276, and 20720703.

We will need the following notations:

$$\alpha_n = \text{ the } n^{\text{th}}\ \alpha\text{-ASTN}, \quad \beta_n = \text{ the } n^{\text{th}}\ \beta\text{-ASTN}, \quad t_n = \text{ the } n^{\text{th}} \text{ STN},$$
$$U_n = \sqrt{t_n}, \quad a_n = (\alpha_n - 1)^{1/2}, \quad b_n = (\beta_n + 1)^{1/2}.$$

We will also need the results (in addition to the well-known fact that x is a triangular number iff $8x+1$ is a perfect square) from our earlier works:

$$U_n = 6U_{n-1} - U_{n-2} \qquad \text{(from [1]);} \tag{2.1}$$

$$U_{n-1}U_{n+1} + 1 = U_n^2 \qquad \text{(from [2]).} \tag{2.2}$$

3. THE α-ASTN

Our first result paves the way for constructing an α-ASTN using a given STN, thus guaranteeing the infinitude of the set of all α-ASTN.

Lemma 3.1: If x is an STN, then $9x+1$ is an α-ASTN.

Proof: Note that $8(9x+1)+1=(3\sqrt{8x+1})^2$. Since x is a TN, $8x+1$ must be a perfect square, thus making $9x+1$ a TN. Moreover, x itself is a perfect square, say z^2, so $9x+1=(3z)^2+1$, which means that $9x+1$ is an α-ASTN. □

That this construction indeed exhausts all the α-ASTN is confirmed by the following result.

Lemma 3.2: If x is an α-ASTN, then $(x-1)/9$ must be an STN.

Proof: In order that the lemma may make any sense, we must ensure that $x-1$ is indeed a multiple of 9. For this, we note that whenever x is an α-ASTN, $x-1$ is a perfect square. As a result, $8x \equiv 8, 7, 4, 1 \pmod 9$. On the other hand, whenever x is a TN, $8x+1$ is a perfect square. Thus, $8x \equiv 8, 0, 3, 6 \pmod 9$. Therefore, $x \equiv 1 \pmod 9$. Let $(x-1)/9 = z$. Clearly, z is a perfect square. Also, $8z+1=(8x+1)/9$ is a perfect square. This means that z is a TN and, hence, an STN. □

Our next result establishes a direct link between α_n and t_n. In what follows, n will always denote an arbitrary natural number.

Theorem 3.1: $\alpha_n = 9t_n + 1$.

Proof: First, note that $\alpha_1 = 10 = 9t_1 + 1$. Assume the assertion is true for $n=k$, so that $\alpha_k = 9t_k + 1$. If possible, let $\alpha_{k+1} \neq 9t_{k+1}+1$. But $(\alpha_{k+1}-1)/9$ is an STN (by Lemma 3.2), so let $(\alpha_{k+1}-1)/9 = t_m$ for some m. We have $\alpha_{k+1} > \alpha_n$ so that $t_m > t_k$. This means $m > k$. But m cannot be equal to $k+1$ (by our assumption), so $m > k+1$. Also, $9t_{k+1}+1$ is an α-ASTN (by Lemma 3.1), so let $9t_{k+1}+1 = \alpha_p$ for some p. We have $t_k < t_{k+1} < t_m$. This leads to $\alpha_k < \alpha_p < \alpha_{k+1}$, an absurdity. Hence, by mathematical induction, $\alpha_n = 9t_n + 1$. □

4. THE β-ASTN

As in the case of the α-ASTN, our first attempt would be toward constructing a β-ASTN from STN. But here, unlike the case of α-ASTN, we need two consecutive STN. First, we will need the following auxiliary results.

Lemma 4.1: $4U_nU_{n+1}+1=(U_{n+1}-U_n)^2$.

Proof: We have

$$\begin{aligned} U_n^2 &= U_{n-1}U_{n+1}+1 && \text{[by (2.2)]} \\ &= 6U_nU_{n+1} - U_{n+1}^2 + 1 && \text{[by (2.1)].} \end{aligned}$$

Hence, $4U_nU_{n+1}+1=(U_{n+1}-U_n)^2$. □

Lemma 4.2: $8U_nU_{n+1}+1=(U_{n+1}+U_n)^2$.

Proof: Proceed as in Lemma 4.1. □

Lemma 4.3: $U_{n+1} = 3U_n + \sqrt{8U_n^2+1}$.

Proof: While proving Lemma 4.1, we found that $U_n^2 = 6U_{n+1}U_n - U_{n+1}^2 + 1$. This yields $(U_{n+1} - 3U_n)^2 = 8U_n^2 + 1$. But $U_{n+1} - 3U_n = 3U_n - U_{n-1} > 0$. Hence,

$$U_{n+1} - 3U_n = \sqrt{8U_n^2 + 1}. \ \square$$

Theorem 4.1: $(U_{n+1} - 2U_n)^2 - 1$ is a β-ASTN.

Proof: Let

$$x = (U_{n+1} - 2U_n)^2 - 1 = \left\{U_n + \sqrt{8U_n^2 + 1}\right\}^2 - 1 \quad \text{(by Lemma 4.3)}.$$

Thus, $8x + 1 = \left\{8U_n + \sqrt{8U_n^2 + 1}\right\}^2$, a perfect square. As a result, x is a TN and, consequently, $(U_{n+1} - 2U_n)^2 - 1$ is a β-ASTN. $\square$

Theorem 4.1 guarantees the infinitude of the set of all β-ASTN, but it does not guarantee that this construction accounts for all the β-ASTN. In fact, it cannot do so because there do exist β-ASTN that cannot be obtained by the application of this theorem, e.g., the very first β-ASTN viz. 3 cannot be expressed as $(U_{n+1} - 2U_n)^2 - 1$ for any n.

In fact, there are infinitely many such exceptions viz. 3, 120, 4095, ... (i.e., all the odd-indexed β-ASTN). Of course, all the even-indexed β-ASTN are taken care of by the above theorem.

Theorem 4.2: $(U_{n+1} - 4U_n)^2 - 1$ is a β-ASTN.

Proof: Let

$$y = (U_{n+1} - 4U_n)^2 - 1 = \left\{\sqrt{8U_n^2 + 1} - U_n\right\}^2 - 1 \quad \text{(by Lemma 4.3)}.$$

Hence, $8y + 1 = \left\{8U_n - \sqrt{8U_n^2 + 1}\right\}^2$, so that y is a TN. This means that $(U_{n+1} - 4U_n)^2 - 1$ is a β-ASTN. $\square$

It appears that Theorems 4.1 and 4.2 jointly account for all the β-ASTN. The same is confirmed by the following theorem.

Theorem 4.3: $b_{2n} = U_{n+1} - 2U_n$ and $b_{2n-1} = U_{n+1} - 4U_n$.

Before attacking the proof of Theorem 4.3 (our main theorem), we must prove the following three lemmas.

Lemma 4.4: If $b^2 - 1$ is a β-ASTN, then either $\{(R+b)/7\}^2$ or $\{(R-b)/7\}^2$ must be an STN, where $R = (8b^2 - 7)^{1/2}$.

Proof: For this lemma to make any sense, we have to ensure that either $(R+b)/7$ or $(R-b)/7$ must be an integer. To this end, we argue that whenever $b^2 - 1$ is a β-ASTN, $b^2 - 1$ is a TN, so $8(b^2 - 1) + 1 = R^2$ must be a perfect square. Thus, R is an integer. Also $(R-b)(R+b) = 7(b^2 - 1)$. This ensures that $(R-b)/7$ or $(R+b)/7$ is an integer.

<u>**Case 1**</u>. Let $(R+b)/7$ be an integer, say x. Then $8x^2 + 1 = \{(8b+R)/7\}^2$, a perfect square. Hence, x^2 must be a TN. This means that $\{(R+b)/7\}^2$ is an STN.

Case 2. Let $(R-b)/7$ be an integer, say y. Then $8y^2+1=\{(8b-R)/7\}^2$, a perfect square. Hence, y^2 must be a TN. This means that $\{(R-b)/7\}^2$ is an STN. Now, we claim that $(R+b)/7$ and $(R-b)/7$ cannot both be integers at the same time. For, if the contrary is true, then $\{(R+b)/7\}\{(R-b)/7\}=(b^2-1)/7$, which means that b^2-1 is a multiple of 7. Also, $\{(R+b)/7\}-\{(R-b)/7\}=2b/7$, which would mean b is a multiple of 7. This leads to a contradiction. □

Lemma 4.5: If b^2-1 is a β-ASTN and $R-b$ is a multiple of 7, then $b=U_{m+1}-2U_m$ for some m.

Proof: By Lemma 4.4, $\{(R-b)/7\}^2$ is an STN. Hence, $(R-b)/7=U_m$ for some m, so that $(b-U_m)^2=8U_m^2+1$. We claim that $b>U_m$, otherwise b will become $U_m-(8U_m^2+1)^{1/2}$ which is negative, an absurdity. Thus, $b-U_m=(8U_m^2+1)^{1/2}$, i.e., $b=U_{m+1}-2U_m$ (by Lemma 4.3). □

Lemma 4.6: If b^2-1 is a β-ASTN and $R+b$ is a multiple of 7, then $b=U_{k+1}-4U_k$ for some k.

Proof: As before, $(R+b)/7=U_k$ for some k, so that $b=-U_k+(8U_k^2+1)^{1/2}$. □

Proof of Theorem 4.3: Define the sequences $\langle x_r\rangle$ and $\langle y_r\rangle$, respectively, by $x_r=U_{r+1}-4U_r$ and $y_r=U_{r+1}-2U_r$. Clearly, for each r, $x_r<y_r$. Also,

$$x_{r+1}=U_{r+2}-4U_{r+1}=2U_{r+1}-U_r=y_r+(U_{r+1}+U_r)>y_r.$$

Thus, $x_r<y_r<x_{r+1}<y_{r+1}$. Hence, the sequence $\langle z_r\rangle$, defined by $z_{2r-1}=x_r$ and $z_{2r}=y_r$, is monotonically increasing. We claim that the sequence $\langle b_n\rangle$ is a subsequence of the sequence $\langle z_n\rangle$ because, for any n, either $(R_n+b)/7$ or $(R_n-b)/7$ is equal to U_k for some k [where $R_n=(8b_n^2-7)^{1/2}$]. Thus, $b_n=U_{k+1}-2U_k$ or $b_n=U_{k+1}-4U_k$. Also, by Theorems 4.1 and 4.2, for each r, $y_r=b_m$ and $x_r=b_k$ for some m and k. Hence, $\langle z_n\rangle$ and $\langle b_n\rangle$ are identical. □

We conclude by rewriting the statement of Theorem 4.3 in a more useful form, as follows.

Corollary: $\beta_{2n}=(U_{n+1}-2U_n)^2-1$ and $\beta_{2n-1}=(U_{n+1}-4U_n)^2-1$.

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