ADVANCED PROBLEMS AND SOLUTIONS

Edited by Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-521 ***Proposed by Paul S. Bruckman, Edmonds, WA***

Let ρ denote any zero of the Riemann Zeta Function $\zeta(z)$ lying in the strip $S = \{z \in C: 0 < \mathrm{Re}(z) < 1\}$. Prove the following:

(1) $\sum_{\rho \in S} \left(\rho - \tfrac{1}{2}\right)^{-1} = 0$;

(2) $\sum_{\rho \in S} \rho^{-1} = 1 + \tfrac{1}{2}\gamma - \tfrac{1}{2}\log 4\pi$, where γ is Euler's Constant.

H-522 ***Proposed by N. Gauthier, Royal Military College, Kingston, Ontario, Canada***

Let A and B be the following 2×2 matrices:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Show that, for $m \geq 1$,

$$\sum_{n=0}^{m-1} 2^n A^{2^n} (A^{2^n} + B^{2^n})^{-1} = c_{2^m} C_{2^m} - (A + B),$$

where

$$c_m = m / (F_{m+1} + F_{m-1} - 2) \text{ and } C_m = \begin{pmatrix} F_{m+1} - 1 & F_m \\ F_m & F_{m-1} - 1 \end{pmatrix};$$

F_m is the m^{th} Fibonacci number.

H-523 ***Proposed by Paul S. Bruckman, Edmonds, WA***

Let $Z(n)$ denote the "Fibonacci entry-point" of n, i.e., $Z(n)$ is the smallest positive integer m such that $n | F_m$. Given any odd prime p, let $q = \tfrac{1}{2}(p-1)$; for any integer s, define $g_p(s)$ as follows:

$$g_p(s) = \sum_{k=1}^{q} \frac{s^k}{k}.$$

Prove the following assertion:

$$Z(p^2) = Z(p) \text{ iff } g_p(1) \equiv g_p(5) \pmod{p}. \tag{*}$$

H-524 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Let p be a prime with $p \equiv 1$ or $9 \pmod{20}$. It is known that $a := (p-1)/Z(p)$ is an even integer, where $Z(p)$ denotes the entry-point in the Fibonacci sequence [1]. Let $q := (p-1)/2$. Show that

(1) $(-1)^{a/2} \equiv (-5)^{q/2} \pmod{p}$ if $p \equiv 1 \pmod{20}$,

(2) $(-1)^{a/2} \equiv -(-5)^{q/2} \pmod{p}$ if $p \equiv 9 \pmod{20}$.

Reference

1. P. S. Bruckman. "Problem H-515." *The Fibonacci Quarterly* **34.4** (1996):379.

H-525 ***Proposed by Paul S. Bruckman, Edmonds, WA***

Let p be any prime $\neq 2, 5$. Let

$$q = \frac{1}{2}(p-1), \quad e = \left(\frac{5}{p}\right), \quad r = \frac{1}{2}(p-e).$$

Let $Z(p)$ denote the entry-point of p in the Fibonacci sequence. Given that $2^{p-1} \equiv 1 \pmod{p}$ and $5^q \equiv e \pmod{p}$, let

$$A = \frac{1}{p}(2^{p-1}-1), \quad B = \frac{1}{p}(5^q - e), \quad C = \sum_{k=1}^{q} \frac{5^{k-1}}{2k-1}.$$

Prove that $Z(p^2) = Z(p)$ if and only if $eA - B \equiv C \pmod{p}$.

SOLUTIONS

Another Complex Problem

H-504 ***Proposed by Z. W. Trzaska, Warsaw, Poland***
(Vol. 33, no. 5, November 1995)

Given a sequence of polynomials in complex variable $z \in C$ defined recursively by

(i) $$R_{k+1}(z) = (3+z)R_k(z) - R_{k-1}(z), \quad k = 0, 1, 2, \ldots,$$

with $R_0(z) = 1$ and $R_1(z) = (1+z)R_0$.

Prove that

(ii) $$R_k(0) = F_{2k+1},$$

where F_ℓ, $\ell = 0, 1, 2, \ldots$, denotes the ℓ^{th} term of the Fibonacci sequence.

Solution by Paul S. Bruckman, Edmonds, WA

The correct expression for $R_k(0)$ is F_{2k-1}, not F_{2k+1}.

Proof: Let $R_k(0) = S_k$, $k = 0, 1, \ldots$. The given recurrence reduces to the following one with constant coefficients, by setting $z = 0$:

$$S_{k+2} - 3S_{k+1} + S_k = 0, \quad k = 0, 1, \ldots; \tag{1}$$

also

$$S_0 = S_1 = 1. \tag{2}$$

The characteristic equation of this recurrence is

$$z^2 - 3z + 1 = 0, \tag{3}$$

which has the roots α^2 and β^2. Therefore, $S_k = AF_{2k} + BL_{2k}$, for appropriate constants A and B. Setting $k = 0$ and $k = 1$ yields $S_0 = 1 = 2B$ and $S_1 = 1 = A + 3B$, whence $A = -\frac{1}{2}$ and $B = \frac{1}{2}$. Then

$$S_k = \tfrac{1}{2}(L_{2k} - F_{2k}) = \tfrac{1}{2}(F_{2k+1} + F_{2k-1} - F_{2k})$$

or

$$R_k(0) = S_k = F_{2k-1}. \text{ Q.E.D.} \tag{4}$$

Also solved by L. A. G. Dresel, A. Dujella, J. Koštál, and the proposer.

Sum Formulae!

H-505 ***Proposed by Juan Pla, Paris, France***
(Vol. 33, no. 5, November 1995)

Edouard Lucas once noted: "On ne connaît pas de formule simple pour la somme des cubes du binôme" [No simple formula is known for the sum of the cubes of the binomial coefficients] (see Edouard Lucas, *Théorie des Nombres*, Paris, 1891, p. 133, as reprinted by Jacques Gabay, Paris, 1991).

The following problem is designed to find *closed*, if not quite "simple," formulas for the sum of the cubes of all the coefficients of the binomial $(1+x)^n$.

1) Prove that

$$\sum_{p=0}^{p=n}\binom{n}{p}^3 = \frac{2^n}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\{1+cos\varphi+cos\theta+cos(\varphi+\theta)\}^n d\theta\, d\varphi.$$

2) Prove that

$$\sum_{p=0}^{p=n}\binom{n}{p}^3 = \frac{8^n}{\pi^2}\int_0^{\pi}\int_0^{\pi}\{\cos\varphi\cos\theta\cos(\varphi+\theta)\}^n d\theta\, d\varphi.$$

Solution by Paul S. Bruckman, Edmonds, WA

Given $n = 0, 1, 2, \ldots$, define

$$S_n = \sum_{k=0}^{n}\binom{n}{k}^3, \tag{1}$$

$$A_n = \frac{2^n}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}[1+\cos x+\cos y+\cos(x+y)]^n dx\, dy, \tag{2}$$

$$B_n = \frac{8^n}{\pi^2}\int_0^{\pi}\int_0^{\pi}[\cos u\cdot\cos v\cdot\cos(u+v)]^n du\, dv. \tag{3}$$

Note that

$$1+\cos x+\cos y+\cos(x+y) = (1+\cos x)(1+\cos y) - \sin x\cdot\sin y$$

$$= 4\cos^2\frac{x}{2}\cdot\cos^2\frac{y}{2} - 4\sin\frac{x}{2}\cos\frac{x}{2}\sin\frac{y}{2}\cos\frac{y}{2}$$
$$= 4\cos\frac{x}{2}\cos\frac{y}{2}\left(\cos\frac{x}{2}\cos\frac{y}{2} - \sin\frac{x}{2}\sin\frac{y}{2}\right)$$
$$= 4\cos\frac{x}{2}\cos\frac{y}{2}\cos\left(\frac{x+y}{2}\right).$$

Therefore,

$$A_n = \frac{2^n}{4\pi^2}\cdot 4^n\int_0^{2\pi}\int_0^{2\pi}\left[\cos\frac{x}{2}\cdot\cos\frac{y}{2}\cdot\cos\left(\frac{x+y}{2}\right)\right]^n dx\,dy$$
$$= \frac{8^n}{4\pi^2}\cdot 4\int_0^{\pi}\int_0^{\pi}[\cos u\cdot\cos v\cdot\cos(u+v)]^n du\,dv;$$

thus,

$$A_n = B_n. \tag{4}$$

Now, it suffices to prove that $S_n = B_n$. Toward this end, we employ the following identity and integral (the latter valid for all integers m):

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) \text{ for all complex } z; \tag{5}$$

$$\frac{1}{\pi}\int_o^{\pi} e^{2itm}dt = \delta_{m:0} = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m \neq 0. \end{cases} \tag{6}$$

Note that

$$B_n = \frac{8^n}{\pi^2}\int_o^{\pi}(\cos v)^n C_n(v)\,dv, \text{ where } C_n(v) = \int_0^{\pi}[\cos u\cdot\cos(u+v)]^n du.$$

Then

$$C_n(v) = \int_0^{\pi} 4^{-n}[(e^{iu}+e^{-iu})(e^{i(u+v)}+e^{-i(u+v)})]^n du$$
$$= 4^{-n}\int_0^{\pi}\sum_{a=0}^{n}\sum_{b=0}^{n}\binom{n}{a}\binom{n}{b}e^{iu(n-2a)+i(u+v)(n-2b)}du$$
$$= 4^{-n}\sum_{a=0}^{n}\sum_{b=0}^{n}\binom{n}{a}\binom{n}{b}e^{iv(n-2b)}\int_0^{\pi}e^{2iu(n-a-b)}du.$$

Thus, using (6),

$$C_n(v) = 4^{-n}\sum_{a=0}^{n}\sum_{b=0}^{n}\binom{n}{a}\binom{n}{b}e^{iv(n-2b)}\cdot\pi\delta_{n-a-b:0}$$

or

$$C_n(v) = \pi\cdot 4^{-n}\sum_{b=0}^{n}\binom{n}{b}^2 e^{iv(n-2b)}. \tag{7}$$

Then

$$B_n = \frac{2^n}{\pi}\int_0^{\pi}2^{-n}(e^{iv}+e^{-iv})^n\sum_{b=0}^{n}\binom{n}{b}^2 e^{iv(n-2b)}dv$$

$$=\frac{1}{\pi}\int_0^{\pi}\sum_{b=0}^{n}\sum_{c=0}^{n}\binom{n}{b}^2\binom{n}{c}e^{iv(n-2b)+iv(n-2c)}dv$$

$$=\frac{1}{\pi}\sum_{b=0}^{n}\sum_{c=0}^{n}\binom{n}{b}^2\binom{n}{c}\int_0^{\pi}e^{2iv(n-b-c)}dv$$

$$=\sum_{b=0}^{n}\sum_{c=0}^{n}\binom{n}{b}^2\binom{n}{c}\delta_{n-b-c:0}=\sum_{b=0}^{n}\binom{n}{b}^3=S_n.\text{ Q.E.D.}$$

Also solved by the proposer.

Sum Figuring

H-506 ***Proposed by Paul S. Bruckman, Edmonds, WA***
(Vol. 34, no. 1, February 1996)

Let

$$A=\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{5n+1}+\frac{1}{5n+4}\right)\quad\text{and}\quad B=\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{5n+2}+\frac{1}{5n+3}\right).$$

Evaluate A and B, showing that $A=\alpha B$.

Solution by C. Georghiou, University of Patras, Patras, Greece

Since, for $|x|<1$,

$$\frac{1}{1+x^5}=1-x^5+x^{10}-x^{15}+\cdots=\sum_{n=0}^{\infty}(-1)^n x^{5n}.$$

we let, for $-1<x<1$,

$$A(x)=\int_0^x\frac{1+u^3}{1+u^5}du=\sum_{n=0}^{\infty}(-1)^n\left(\frac{x^{5n+1}}{5n+1}+\frac{x^{5n+4}}{5n+4}\right)$$

and

$$B(x)=\int_0^x\frac{u+u^2}{1+u^5}du=\sum_{n=0}^{\infty}(-1)^n\left(\frac{x^{5n+2}}{5n+2}+\frac{x^{5n+3}}{5n+3}\right).$$

By Abel's Limit Theorem, we have $A=A(1)$ and $B=B(1)$. But

$$\frac{1+x^3}{1+x^5}=\frac{x^2-x+1}{x^4-x^3+x^2-x+1}=\frac{1}{\alpha-\beta}\left(\frac{2\cos(\pi/5)}{x^2+2\cos(2\pi/5)x+1}+\frac{2\cos(2\pi/5)}{x^2-2\cos(\pi/5)x+1}\right)$$

and

$$\frac{x+x^2}{1+x^5}=\frac{x}{x^4-x^3+x^2-x+1}=\frac{1}{\alpha-\beta}\left(\frac{1}{x^2-2\cos(\pi/5)x+1}-\frac{1}{x^2+2\cos(2\pi/5)x+1}\right).$$

Now it is easy to verify that, for $0<\gamma<\pi$,

$$\int_0^x\frac{du}{u^2\pm 2u\cos\gamma+1}=\frac{1}{\sin\gamma}\tan^{-1}\frac{x\sin\gamma}{1\pm x\cos\gamma}$$

and, therefore,

$$\int_0^1 \frac{dx}{x^2+2x\cos(2\pi/5)+1} = \frac{1}{\sin(2\pi/5)}\tan^{-1}\frac{\sin(2\pi/5)}{1+\cos(2\pi/5)} = \frac{1}{\sin(2\pi/5)}\frac{\pi}{5}$$

and

$$\int_0^1 \frac{dx}{x^2-2x\cos(\pi/5)+1} = \frac{1}{\sin(\pi/5)}\tan^{-1}\frac{\sin(\pi/5)}{1-\cos(\pi/5)} = \frac{1}{\sin(\pi/5)}\frac{2\pi}{5}.$$

Finally, we find

$$A = \frac{\pi/5}{\sin(\pi/5)} = \frac{2\pi}{5\sqrt{3-\alpha}} \quad \text{and} \quad B = \frac{\pi/5}{\sin(2\pi/5)} = \frac{2\pi}{5\alpha\sqrt{3-\alpha}},$$

where we used the fact that $\alpha = 2\cos(\pi/5)$ and $\beta = -2\cos(2\pi/5)$.

Also solved by K. Davenport, H. Kappus, H.-J. Seiffert, D. Terr, and the proposer.

Triple Threat

H-507 (Corrected) ***Proposed by Mohammad K. Azarian, Univ. of Evansville, Evansville, IN (Vol. 34, no. 1, February 1996)***

Prove that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{n}\sum_{k=1}^{m}(-1)^i 2^{-(k+1)(i+j)}\left(\frac{n(i+1)(i+2)\cdots(i+n-1)}{j!(n-j)!}\right)(F_k)^{i+j} = m.$$

Solution by H.-J. Seiffert, Berlin, Germany

Let $x_1, \dots, x_m \in (-1, 1)$. Then

$$\begin{aligned} S &= \sum_{i=0}^{\infty}\sum_{j=0}^{n}\sum_{k=1}^{m}(-1)^i\left(\frac{n(i+1)(i+2)\cdots(i+n-1)}{j!(n-j)!}\right)x_k^{i+j} \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{n}\sum_{k=1}^{m}\binom{-n}{i}\binom{n}{j}x_k^{i+j} = \sum_{k=1}^{m}\sum_{i=0}^{\infty}\binom{-n}{i}x_k^i\sum_{j=0}^{n}\binom{n}{j}x_k^j \\ &= \sum_{k=1}^{m}\sum_{i=0}^{\infty}\binom{-n}{i}x_k^i(1+x_k)^n = \sum_{k=1}^{m}(1+x_k)^n\sum_{i=0}^{\infty}\binom{-n}{i}x_k^i \\ &= \sum_{k=1}^{m}(1+x_k)^n(1+x_k)^{-n}, \end{aligned}$$

or $S = m$. Since $0 < F_k < \alpha^k < 2^k < 2^{k+1}$, $k \ge 1$, we may take $x_k = 2^{-k-1}F_k$, $k = 1, \dots, m$. From the above, it follows that the sum in question has the value m; we note the mistake in the proposal.

Also solved by P. Bruckman.

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