INVERSE TRIGONOMETRIC AND HYPERBOLIC SUMMATION FORMULAS INVOLVING GENERALIZED FIBONACCI NUMBERS

R. S. Melham and A. G. Shannon
University of Technology, Sydney, 2007, Australia
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1. INTRODUCTION

Define the sequences $\{U_n\}_{n=0}^{\infty}$ and $\{V_n\}_{n=0}^{\infty}$ for all integers n by

$$U_n = pU_{n-1} + U_{n-2},\ U_0 = 0,\ U_1 = 1,\ n \geq 2, \tag{1.1}$$

$$V_n = pV_{n-1} + V_{n-2},\ V_0 = 2,\ V_1 = p,\ n \geq 2. \tag{1.2}$$

Of course, these sequences can be extended to negative subscripts by the use of (1.1) and (1.2). The Binet forms for U_n and V_n are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{1.3}$$

and

$$V_n = \alpha^n + \beta^n, \tag{1.4}$$

where

$$\alpha = \frac{p + \sqrt{p^2+4}}{2}, \quad \beta = \frac{p - \sqrt{p^2+4}}{2}. \tag{1.5}$$

Certain specializations of the parameter p produce sequences that are of interest here and Table 1 summarizes these.

TABLE 1

p	1	2	$2x$
U_n	F_n	P_n	$P_n(x)$
V_n	L_n	Q_n	$Q_n(x)$

Here $\{F_n\}$ and $\{L_n\}$ are the Fibonacci and Lucas sequences, respectively. The sequences $\{P_n\}$ and $\{Q_n\}$ are the Pell and Pell-Lucas numbers, respectively, and appear, for example, in [3], [5], [7], [11], and [17]. The sequences $\{P_n(x)\}$ and $\{Q_n(x)\}$ are the Pell and Pell-Lucas polynomials, respectively, and have been studied, for example, in [12], [14], [15], and [16].

Hoggatt and Ruggles [9] produced some summation identities for Fibonacci and Lucas numbers involving the arctan function. Their results are of the same type as the striking result of D. H. Lehmer,

$$\sum_{i=1}^{\infty} \tan^{-1}\left(\frac{1}{F_{2i+1}}\right) = \frac{\pi}{4}, \tag{1.6}$$

to which reference is made in their above-mentioned article [9]. Mahon and Horadam [13] produced identities for Pell and Pell-Lucas polynomials leading to summation formulas for Pell and Pell-Lucas numbers similar to (1.6). For example,

$$\sum_{i=0}^{\infty} \tan^{-1}\left(\frac{2}{P_{2i+1}}\right) = \frac{\pi}{2}. \tag{1.7}$$

Here, we produce similar results involving the arctan function and terms from the sequences $\{U_n\}$ and $\{V_n\}$. Some of our results are equivalent to those obtained in [13] but most are new. We also obtain results involving the arctanh function, all of which we believe are new.

2. PRELIMINARY RESULTS

We make consistent use of the following results which appear in [1] and [6]:

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{ if } xy < 1, \tag{2.1}$$

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right), \text{ if } xy > -1, \tag{2.2}$$

$$\tanh^{-1} x + \tanh^{-1} y = \tanh^{-1}\left(\frac{x+y}{1+xy}\right), \tag{2.3}$$

$$\tanh^{-1} x - \tanh^{-1} y = \tanh^{-1}\left(\frac{x-y}{1-xy}\right), \tag{2.4}$$

$$\tanh^{-1} x = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right), \ |x| < 1, \tag{2.5}$$

$$\coth^{-1} x = \frac{1}{2}\log_e\left(\frac{x+1}{x-1}\right), \ |x| > 1, \tag{2.6}$$

$$\tan^{-1} x = \cot^{-1}\left(\frac{1}{x}\right), \tag{2.7}$$

$$\tanh^{-1} x = \coth^{-1}\left(\frac{1}{x}\right). \tag{2.8}$$

We note from (2.7) and (2.8) that all results obtained involving arctan (arctanh) can be expressed equivalently using arccot (arccoth).

If k and n are integers, and writing

$$\Delta = (\alpha - \beta)^2 = p^2 + 4, \tag{2.9}$$

we also have the following:

$$U_n^2 - U_{n+k}U_{n-k} = (-1)^{n+k}U_k^2, \tag{2.10}$$

$$V_{n+k}V_{n-k} - V_n^2 = \Delta(-1)^{n+k}U_k^2, \tag{2.11}$$

$$U_{n+k} - U_{n-k} = \begin{cases} U_kV_n, & k \text{ even}, \\ U_nV_k, & k \text{ odd}, \end{cases} \tag{2.12}$$

$$U_{n+k} + U_{n-k} = \begin{cases} U_nV_k, & k \text{ even}, \\ U_kV_n, & k \text{ odd}, \end{cases} \tag{2.13}$$

$$V_{n+k} - V_{n-k} = \begin{cases} \Delta U_kU_n, & k \text{ even}, \\ V_kV_n, & k \text{ odd}, \end{cases} \tag{2.14}$$

$$V_{n+k} + V_{n-k} = \begin{cases} V_kV_n, & k \text{ even}, \\ \Delta U_kU_n, & k \text{ odd}. \end{cases} \tag{2.15}$$

$$U_{n+2} + U_n = V_{n+1}, \tag{2.16}$$

$$U_nU_{n+2} + (-1)^n = U_{n+1}^2. \tag{2.17}$$

Identities (2.12)-(2.15) occur as (56)-(63) in [2], and the remainder can be proved using Binet forms. Indeed, (2.10) and (2.11) resemble the famous Catalan identity for Fibonacci numbers,

$$F_n^2 - F_{n+k}F_{n-k} = (-1)^{n-k}F_k^2. \tag{2.18}$$

We assume throughout that the parameter p is real and $|p| \geq 1$. If $p \geq 1$, then $\{U_n\}_{n=2}^{\infty}$ and $\{V_n\}_{n=1}^{\infty}$ are increasing sequences. If $p \leq -1$, then $\{|U_n|\}_{n=2}^{\infty}$ and $\{|V_n|\}_{n=1}^{\infty}$ are increasing sequences and, if $n > 0$, then

$$\begin{cases} U_n < 0, & n \text{ even}, \\ U_n > 0, & n \text{ odd}, \\ V_n < 0, & n \text{ odd}, \\ V_n > 0, & n \text{ even}. \end{cases} \tag{2.19}$$

Furthermore, if $|p| \geq 1$, then using Binet forms it is seen that

$$\lim_{n\to\infty} \frac{U_{n+m}}{U_n} = \lim_{n\to\infty} \frac{V_{n+m}}{V_n} = \begin{cases} \delta^m, & m \text{ even or } p \geq 1, \\ -\delta^m, & m \text{ odd and } p \leq -1, \end{cases} \tag{2.20}$$

where

$$\delta = \frac{|p| + \sqrt{p^2+4}}{2}. \tag{2.21}$$

3. MAIN RESULTS

Theorem 1: If n is an integer, then

$$\tan^{-1} U_{n+2} - \tan^{-1} U_n = \tan^{-1}\left(\frac{p}{U_{n+1}}\right), \quad n \text{ even}, \tag{3.1}$$

$$\tan^{-1}\left(\frac{1}{U_n}\right) + \tan^{-1}\left(\frac{1}{U_{n+2}}\right) = \tan^{-1}\left(\frac{V_{n+1}}{U_{n+1}^2}\right), \quad n \text{ odd}, \ n \neq -1. \tag{3.2}$$

Proof:

$$\tan^{-1} U_{n+2} - \tan^{-1} U_n = \tan^{-1}\left(\frac{U_{n+2} - U_n}{1 + U_n U_{n+2}}\right) = \tan^{-1}\left(\frac{p}{U_{n+1}}\right),$$

where we have used (2.2), (1.1), and (2.17).

To prove (3.2), proceed similarly using (2.1), (2.16), and (2.17). □

Now, in (3.1), replacing n by 0, 2, ..., $2n-2$, we obtain a sum which telescopes to yield

$$\sum_{i=1}^{n} \tan^{-1}\left(\frac{p}{U_{2i-1}}\right) = \tan^{-1} U_{2n}. \tag{3.3}$$

Similarly, in (3.2), replacing n by 1, 3, ..., $2n-1$ yields

$$\sum_{i=1}^{n} (-1)^{i-1} \tan^{-1}\left(\frac{V_{2i}}{U_{2i}^2}\right) = \frac{\pi}{4} + (-1)^{n-1} \tan^{-1}\left(\frac{1}{U_{2n+1}}\right). \tag{3.4}$$

The corresponding limiting sums are

$$\sum_{i=1}^{\infty} \tan^{-1}\left(\frac{p}{U_{2i-1}}\right) = \begin{cases} \frac{\pi}{2}, & p \geq 1, \\ \frac{-\pi}{2}, & p \leq -1, \end{cases} \tag{3.5}$$

$$\sum_{i=1}^{\infty} (-1)^{i-1} \tan^{-1}\left(\frac{V_{2i}}{U_{2i}^2}\right) = \frac{\pi}{4}. \tag{3.6}$$

We note here that (3.3) and (3.4) were essentially obtained by Mahon and Horadam [13], (3.3) in a slightly different form. When $p = 1$, (3.5) reduces essentially to Lehmer's result (1.6) stated earlier.

Theorem 2: For positive integers k and n,

$$\tan^{-1}\left(\frac{U_n}{U_{n+k}}\right) - \tan^{-1}\left(\frac{U_{n-k}}{U_n}\right) = \begin{cases} \tan^{-1}\left(\frac{(-1)^n U_k^2}{V_k U_n^2}\right), & k \text{ even}, \\ \tan^{-1}\left(\frac{(-1)^{n-1} U_k}{U_{2n}}\right), & k \text{ odd}. \end{cases} \tag{3.7}$$

Proof: Use (2.2), (2.10), and (2.13). □

Now, in (3.7), replacing n by k, $2k$, ..., nk to form a telescoping sum yields

$$\sum_{i=1}^{n}\tan^{-1}\left(\frac{U_k^2}{V_k U_{ik}^2}\right)=\tan^{-1}\left(\frac{U_{nk}}{U_{(n+1)k}}\right),\quad k \text{ even} \tag{3.8}$$

$$\sum_{i=1}^{n}\tan^{-1}\left(\frac{(-1)^{i-1}U_k}{U_{2ik}}\right)=\tan^{-1}\left(\frac{U_{nk}}{U_{(n+1)k}}\right),\quad k \text{ odd.} \tag{3.9}$$

Using (2.20), the limiting sums are, respectively,

$$\sum_{i=1}^{\infty}\tan^{-1}\left(\frac{U_k^2}{V_k U_{ik}^2}\right)=\tan^{-1}(\delta^{-k}),\quad k \text{ even,} \tag{3.10}$$

$$\sum_{i=1}^{\infty}\tan^{-1}\left(\frac{(-1)^{i-1}U_k}{U_{2ik}}\right)=\begin{cases}\tan^{-1}(\delta^{-k}), & k \text{ odd, } p\ge 1,\\ -\tan^{-1}(\delta^{-k}), & k \text{ odd, } p\le -1.\end{cases} \tag{3.11}$$

Theorem 3: For positive integers k and n,

$$\tan^{-1}\left(\frac{V_{n-k}}{V_n}\right)-\tan^{-1}\left(\frac{V_n}{V_{n+k}}\right)=\begin{cases}\tan^{-1}\left(\dfrac{\Delta(-1)^n U_k^2}{V_k V_n^2}\right), & k \text{ even,}\\ \tan^{-1}\left(\dfrac{(-1)^{n-1}U_k}{U_{2n}}\right), & k \text{ odd.}\end{cases} \tag{3.12}$$

Proof: Use (2.2), (2.11), and (2.15). □

Again in (3.12), replacing n by k, $2k$, ..., nk yields

$$\sum_{i=1}^{n}\tan^{-1}\left(\frac{\Delta U_k^2}{V_k V_{ik}^2}\right)=\tan^{-1}\left(\frac{2}{V_k}\right)-\tan^{-1}\left(\frac{V_{nk}}{V_{(n+1)k}}\right),\quad k \text{ even,} \tag{3.13}$$

$$\sum_{i=1}^{n}\tan^{-1}\left(\frac{(-1)^{i-1}U_k}{U_{2ik}}\right)=\tan^{-1}\left(\frac{2}{V_k}\right)-\tan^{-1}\left(\frac{V_{nk}}{V_{(n+1)k}}\right),\quad k \text{ odd.} \tag{3.14}$$

Since the left sides of (3.9) and (3.14) are the same, we can write

$$\tan^{-1}\left(\frac{U_{nk}}{U_{(n+1)k}}\right)+\tan^{-1}\left(\frac{V_{nk}}{V_{(n+1)k}}\right)=\tan^{-1}\left(\frac{2}{V_k}\right),\quad k \text{ odd,} \tag{3.15}$$

and taking limits using (2.20) gives

$$\tan^{-1}(\delta^{-k})=\frac{1}{2}\tan^{-1}\left(\frac{2}{|V_k|}\right),\quad k \text{ odd.} \tag{3.16}$$

The limiting sum arising from (3.13) is

$$\sum_{i=1}^{\infty}\tan^{-1}\left(\frac{\Delta U_k^2}{V_k V_{ik}^2}\right)=\tan^{-1}\left(\frac{2}{V_k}\right)-\tan^{-1}(\delta^{-k}),\ k \text{ even}. \tag{3.17}$$

Theorem 4: If $n>2$, then

$$\tanh^{-1}\left(\frac{1}{U_n}\right)+\tanh^{-1}\left(\frac{1}{U_{n+2}}\right)=\tanh^{-1}\left(\frac{V_{n+1}}{U_{n+1}^2}\right),\ n \text{ even}, \tag{3.18}$$

$$\tanh^{-1}\left(\frac{1}{U_n}\right)-\tanh^{-1}\left(\frac{1}{U_{n+2}}\right)=\tanh^{-1}\left(\frac{p}{U_{n+1}}\right),\ n \text{ odd}. \tag{3.19}$$

Proof: To prove (3.18) use (2.3), (2.16), and (2.17); (3.19) is proved similarly. □

These results lead, respectively, to

$$\sum_{i=1}^{n}(-1)^{i-1}\tanh^{-1}\left(\frac{V_{2i+3}}{U_{2i+3}^2}\right)=\tanh^{-1}\left(\frac{1}{U_4}\right)+(-1)^{n-1}\tanh^{-1}\left(\frac{1}{U_{2n+4}}\right), \tag{3.20}$$

$$\sum_{i=1}^{n}\tanh^{-1}\left(\frac{p}{U_{2i+2}}\right)=\tanh^{-1}\left(\frac{1}{U_3}\right)-\tanh^{-1}\left(\frac{1}{U_{2n+3}}\right). \tag{3.21}$$

Note that in Theorem 4 our assumption that $n>2$, together with our earlier assumption that $|p|\geq 1$, is necessary to ensure that the arctanh function is defined. The corresponding limiting sums are

$$\sum_{i=1}^{\infty}(-1)^{i-1}\tanh^{-1}\left(\frac{V_{2i+3}}{U_{2i+3}^2}\right)=\tanh^{-1}\left(\frac{1}{U_4}\right), \tag{3.22}$$

$$\sum_{i=1}^{\infty}\tanh^{-1}\left(\frac{p}{U_{2i+2}}\right)=\tanh^{-1}\left(\frac{1}{U_3}\right). \tag{3.23}$$

We refrain from giving proofs for the theorems that follow, since the proofs are similar to those already given.

Theorem 5: Let $n\geq k$ be positive integers where $(k,n)\neq(1,1)$ if $p=\pm 1$. Then

$$\tanh^{-1}\left(\frac{U_n}{U_{n+k}}\right)-\tanh^{-1}\left(\frac{U_{n-k}}{U_n}\right)=\begin{cases}\tanh^{-1}\left(\dfrac{(-1)^n U_k}{U_{2n}}\right), & k \text{ even},\\[2ex] \tanh^{-1}\left(\dfrac{(-1)^{n-1}U_k^2}{V_k U_n^2}\right), & k \text{ odd}.\end{cases} \tag{3.24}$$

This leads to

$$\sum_{i=1}^{n}\tanh^{-1}\left(\frac{U_k}{U_{2ik}}\right)=\tanh^{-1}\left(\frac{U_{nk}}{U_{(n+1)k}}\right),\ k \text{ even}, \tag{3.25}$$

$$\sum_{i=1}^{n} \tanh^{-1}\left(\frac{(-1)^{i-1}U_k^2}{V_k U_{ik}^2}\right) = \tanh^{-1}\left(\frac{U_{nk}}{U_{(n+1)k}}\right), \quad k \text{ odd.} \tag{3.26}$$

The corresponding limiting sums are

$$\sum_{i=1}^{\infty} \tanh^{-1}\left(\frac{U_k}{U_{2ik}}\right) = \tanh^{-1}(\delta^{-k}), \quad k \text{ even,} \tag{3.27}$$

$$\sum_{i=1}^{\infty} \tanh^{-1}\left(\frac{(-1)^{i-1}U_k^2}{V_k U_{ik}^2}\right) = \begin{cases} \tanh^{-1}(\delta^{-k}), & k \text{ odd, } p \geq 1, \\ -\tanh^{-1}(\delta^{-k}), & k \text{ odd, } p \leq -1. \end{cases} \tag{3.28}$$

Theorem 6: Let $n \geq k$ be positive integers where $(k, n) \neq (1, 1)$ if $1 \leq |p| \leq 2$. Then

$$\tanh^{-1}\left(\frac{V_{n-k}}{V_n}\right) - \tanh^{-1}\left(\frac{V_n}{V_{n+k}}\right) = \begin{cases} \tanh^{-1}\left(\frac{(-1)^n U_k}{U_{2n}}\right), & k \text{ even,} \\ \tanh^{-1}\left(\frac{\Delta(-1)^{n-1}U_k^2}{V_k V_n^2}\right), & k \text{ odd.} \end{cases} \tag{3.29}$$

The resulting sums are

$$\sum_{i=1}^{n} \tanh^{-1}\left(\frac{U_k}{U_{2ik}}\right) = \tanh^{-1}\left(\frac{2}{V_k}\right) - \tanh^{-1}\left(\frac{V_{nk}}{V_{(n+1)k}}\right), \quad k \text{ even,} \tag{3.30}$$

$$\sum_{i=1}^{n} \tanh^{-1}\left(\frac{\Delta(-1)^{i-1}U_k^2}{V_k V_{ik}^2}\right) = \tanh^{-1}\left(\frac{2}{V_k}\right) - \tanh^{-1}\left(\frac{V_{nk}}{V_{(n+1)k}}\right), \quad k \text{ odd.} \tag{3.31}$$

Since the left sides of (3.25) and (3.30) are the same, we can write

$$\tanh^{-1}\left(\frac{U_{nk}}{U_{(n+1)k}}\right) + \tanh^{-1}\left(\frac{V_{nk}}{V_{(n+1)k}}\right) = \tanh^{-1}\left(\frac{2}{V_k}\right), \quad k \text{ even,} \tag{3.32}$$

and taking limits yields

$$\tanh^{-1}(\delta^{-k}) = \frac{1}{2}\tanh^{-1}\left(\frac{2}{V_k}\right), \quad k \text{ even.} \tag{3.33}$$

This should be compared with (3.16). The limiting sum arising from (3.31) is

$$\sum_{i=1}^{\infty} \tanh^{-1}\left(\frac{\Delta(-1)^{i-1}U_k^2}{V_k V_{ik}^2}\right) = \begin{cases} \tanh^{-1}\left(\frac{2}{V_k}\right) - \tanh^{-1}(\delta^{-k}), & k \text{ odd, } p \geq 1, \\ \tanh^{-1}\left(\frac{2}{V_k}\right) + \tanh^{-1}(\delta^{-k}), & k \text{ odd, } p \leq -1. \end{cases} \tag{3.34}$$

At this point we remark that Mahon and Horadam [13] obtained results similar to our Theorems 2 and 3 and derived summation formulas from them. However, in our notation, they considered only the case k odd.

4. APPLICATIONS

We now use some of our results to obtain identities for the Fibonacci and Lucas numbers. From (3.22) and (3.23), we have

$$\sum_{i=1}^{\infty}(-1)^{i-1}\tanh^{-1}\left(\frac{L_{2i+3}}{F_{2i+3}^2}\right)=\frac{1}{2}\log_e 2, \tag{4.1}$$

$$\sum_{i=1}^{\infty}\tanh^{-1}\left(\frac{1}{F_{2i+2}}\right)=\frac{1}{2}\log_e 3. \tag{4.2}$$

In terms of infinite products, these become, respectively,

$$\prod_{i=1}^{\infty}\frac{F_{2i+3}^2+(-1)^{i-1}L_{2i+3}}{F_{2i+3}^2+(-1)^{i}L_{2i+3}}=2, \tag{4.3}$$

$$\prod_{i=1}^{\infty}\frac{F_{2i+2}+1}{F_{2i+2}-1}=3. \tag{4.4}$$

In (3.28), keeping in mind the constraints in the statement of Theorem 5 and taking $k=3$, we obtain

$$\sum_{i=1}^{\infty}(-1)^{i-1}\tanh^{-1}\left(\frac{1}{F_{3i}^2}\right)=\frac{1}{2}\log_e\phi, \tag{4.5}$$

or

$$\prod_{i=1}^{\infty}\frac{F_{3i}^2+(-1)^{i-1}}{F_{3i}^2+(-1)^{i}}=\phi, \tag{4.6}$$

where $\phi=\frac{1+\sqrt{5}}{2}$ is the Golden Ratio.

Finally, (3.34) yields, after simplifying the right side,

$$\sum_{i=1}^{\infty}(-1)^{i-1}\tanh^{-1}\left(\frac{5}{L_{3i}^2}\right)=\frac{1}{2}\log_e(3(\phi-1)), \tag{4.7}$$

or

$$\prod_{i=1}^{\infty}\frac{L_{3i}^2+(-1)^{i-1}5}{L_{3i}^2+(-1)^{i}5}=3(\phi-1). \tag{4.8}$$

Of course, many other examples can be given by varying the parameter k.

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