ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

Please send all material for ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. STANLEY RABINOWITZ; 12 VINE BROOK RD; WESTFORD, MA 01886-4212 USA. Correspondence may also be sent to the problem editor by electronic mail to Fibonacci@MathPro.com *on Internet. All correspondence will be acknowledged.*

Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original. A problem should not be submitted elsewhere while it is under consideration for publication in this column.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-778 ***Proposed by Eliot Jacobson, Ohio University, Athens, OH***

While paging through an old text in our library, I found a tattered and yellowed page, clearly out of place, as if it had been torn from a book, long forgotten. After months of tedious work, I have completed the translation of the scribbled markings on that page. In the margin was noted:

> *I have found a truly wondrous demonstration of the following theorem; unfortunately the margin of this page is too small to contain it.*

And then followed:

> **Fibonacci's Last Theorem:** The equation $x^n + y^n = z^n$ has no nontrivial solutions consisting entirely of Fibonacci numbers, for $n \geq 2$.

Can you supply the missing proof?

B-779 ***Proposed by Andrew Cusumano, Great Neck, NY***

Find integers a, b, c, and d (with $1 < a < b < c < d$) that make the following an identity:

$$F_n = F_{n-a} + 6F_{n-b} + F_{n-c} + F_{n-d}.$$

B-780 ***Proposed by Zdravko F. Starc, Vršac, Yugoslavia***

Prove that:

(a) $F_1 \cdot F_2 \cdot F_3 \cdot \cdots \cdot F_n \leq \exp(F_{n+2} - n - 1)$;

(b) $F_1 \cdot F_3 \cdot F_5 \cdot \cdots \cdot F_{2n-1} \le \exp(F_{2n} - n)$;

(c) $F_2 \cdot F_4 \cdot F_6 \cdot \cdots \cdot F_{2n} \le \exp(F_{2n+1} - n - 1)$.

B-781 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Let $F(j) = F_j$. Find a closed form for

$$\sum_{k=0}^{n} F\left(k - \left\lfloor \sqrt{k} \right\rfloor^2\right).$$

(The notation $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

B-782 ***Proposed by László Cseh, Stuttgart, Germany, & Imre Merény, Budapest, Hungary***

Express $(F_{n+h}^2 + F_n^2 + F_h^2)(F_{n+h+k}^2 + F_{n+k}^2 + F_k^2)$ as the sum of three squares.

B-783 ***Proposed by David Zeitlin, Minneapolis, MN***

Find a rational function $P(x, y)$ such that

$$P(F_n, F_{2n}) = \frac{105n^5 - 1365n^3 + 1764n}{25n^6 + 175n^4 - 5600n^2 + 5904}$$

for $n = 0, 1, 2, 3, 4, 5, 6$.

SOLUTIONS

Fun with Unit Fractions

B-745 ***Proposed by Richard André-Jeannin, Longwy, France (Vol. 31, no. 3, August 1993)***

Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{F_{2n}} = 1 + \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}F_{2n}F_{2n+1}}$.

Solution by Paul S. Bruckman, Everett, WA

Let $S = \sum_{n=1}^{\infty} 1/F_{2n}$, $D_n = F_{2n-1}F_{2n}F_{2n+1}$, and $T = \sum_{n=1}^{\infty} 1/D_n$. We want to show that $S = 1 + T$. Clearly, the sums defining S and T are absolutely convergent, which justifies the following manipulations:

$$\begin{aligned} S &= \sum_{n=1}^{\infty} \frac{F_{2n-1}F_{2n+1}}{D_n} = \sum_{n=1}^{\infty} \frac{F_{2n}^2 + 1}{D_n} = T + \sum_{n=1}^{\infty} \frac{F_{2n}}{F_{2n-1}F_{2n+1}} \\ &= T + \sum_{n=1}^{\infty} \frac{F_{2n+1} - F_{2n-1}}{F_{2n-1}F_{2n+1}} = T + \sum_{n=1}^{\infty} \left(\frac{1}{F_{2n-1}} - \frac{1}{F_{2n+1}} \right) \\ &= T + \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}} - \sum_{n=2}^{\infty} \frac{1}{F_{2n-1}} = T + \frac{1}{F_1} = T + 1. \end{aligned}$$

Also solved by Leonard A. G. Dresel, Piero Filipponi, Russell Jay Hendel, Norbert Jensen, Murray S. Klamkin, Joseph J. Kostal, Bob Prielipp, Almas Rumov, H.-J. Seiffert, J. Suck, A. N. 't Woord, David Zeitlin, and the proposer.

L_{3^n} Recurs

B-746 *Proposed by Seung-Jin Bang, Albany, CA (Vol. 31, no. 3, August 1993)*

Solve the recurrence equation $a_{n+1} = 4a_n^3 + 3a_n$, $n \geq 0$, with initial condition $a_0 = 1/2$.

Solution by Chris Long, Bridgewater, New Jersey

I claim that $a_n = \frac{1}{2} L_{3^n}$. Indeed, using the Binet form and the fact that $\alpha\beta = -1$, it follows that $L_n^3 = L_{3n} - 3L_n$. Thus, $L_{3n} = L_n^3 + 3L_n$, which implies that $L_{3n}/2 = 4 \cdot L_n / 2^3 + 3 \cdot L_n / 2$. The result follows, since $a_0 = 1/2 = L_1/2$.

The proposer stated that this problem was inspired by Problem 1809 in Crux Mathematicorum 19 (1993):16, proposed by David Doster.

Also solved by Paul S. Bruckman, Leonard A. G. Dresel, Piero Filipponi, F. J. Flanigan, Norbert Jensen, Hans Kappus, Murray S. Klamkin, Juan Pla, Bob Prielipp, Almas Rumov, H.-J. Seiffert, J. Suck, David Zeitlin, and the proposer.

Great Sums from Partial Sums

B-747 *Proposed by Piero Filipponi, Fond. U. Bordoni, Rome, Italy (Vol. 31, no. 3, August 1993)*

Let

$$S_1 = \sum_{n=2}^{\infty} \frac{1}{(-1)^n L_{2n-1} - 1} \quad \text{and} \quad S_2 = \sum_{n=2}^{\infty} \frac{1}{(-1)^n L_{2n-1} + 1}.$$

Prove that $S_1 / S_2 = \sqrt{5}$.

Solution by Hans Kappus, Rodersdorf, Switzerland

Consider the partial sums

$$s_1(n) = \sum_{k=2}^{n+1} \frac{1}{(-1)^k L_{2k-1} - 1} \quad \text{and} \quad s_2(n) = \sum_{k=2}^{n+1} \frac{1}{(-1)^k L_{2k-1} + 1}.$$

We shall prove that

$$s_1(n) = F_n / L_{n+1} \quad \text{and} \quad s_2(n) = F_n / (5F_{n+1}). \tag{1}$$

From (1), it follows easily that

$$S_1 = \lim_{n\to\infty} s_1(n) = \frac{1}{\sqrt{5}\alpha} = \frac{5-\sqrt{5}}{10} \quad \text{and} \quad S_2 = \lim_{n\to\infty} s_2(n) = \frac{1}{5\alpha} = \frac{\sqrt{5}-1}{10}.$$

Hence, $S_1 / S_2 = \sqrt{5}$.

Proof of (1): In the known relations (see [1], p. 177)

$$L_{k+m} + (-1)^m L_{k-m} = L_m L_k \quad \text{and} \quad L_{k+m} - (-1)^m L_{k-m} = F_m F_k,$$

we put $m = k-1$. We then have

$$s_1(n)=\sum_{k=2}^{n+1}\frac{(-1)^k}{L_{k-1}L_k}=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{L_kL_{k+1}} \quad\text{and}\quad s_2(n)=\sum_{k=2}^{n+1}\frac{(-1)^k}{F_{k-1}F_k}=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{F_kF_{k+1}}.$$

The latter expressions are special cases of sums considered in Problem B-697 (see [2]), and it is readily seen that their closed forms are just those given by (1).

References

1. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications.* Chichester: Ellis Horwood ltd., 1989.
2. Richard André-Jeannin. Problem B-697. *The Fibonacci Quarterly* **30.3** (1992):280.

Also solved by Paul S. Bruckman, Charles K. Cook, Leonard A. G. Dresel, Russell Jay Hendel, Norbert Jensen, Bob Prielipp, H.-J. Seiffert, J. Suck, and the proposer.

A Recurrence for F_{kn}

B-748 ***Proposed by Herta T. Freitag, Roanoke, VA (Vol. 31, no. 4, November 1993)***

Let $u_k = F_{kn}/F_n$ for some fixed positive integer n. Find a recurrence satisfied by the sequence $\langle u_k\rangle$.

Solution by Tony Shannon, University of Technology, Sydney, Australia

We have

$$u_{k+1}=\frac{\alpha^{kn}\alpha^n-\beta^{kn}\beta^n}{\alpha^n-\beta^n}=\alpha^n u_k+\beta^n u_k+\frac{\alpha^n\beta^{kn}-\beta^n\alpha^{kn}}{\alpha^n-\beta^n}=L_n u_k-(-1)^n u_{k-1}.$$

Haukkanen noted that for any function f(n), the sequence $\langle u_k\rangle$, given by $u_k = F_{kn}f(n)$, satisfies the recurrence $u_{k+2}=L_nu_{k+1}-(-1)^nu_k$. Libis expressed the recurrence neatly as $u_{k+2}=u_2u_{k+1}-(-1)^nu_k$. Kostal found the recurrence $u_k=L_{(k-1)n}+(-1)^nu_{k-2}$. Ballieu found the recurrence $u_k=\alpha^nu_{k-1}+(\beta^n)^{k-1}$. Somer reported that Lehmer found that the recurrence $u_{k+2}=L_nu_{k+1}-(-1)^nu_k$ is satisfied by the more general sequence defined by $u_k=W_{kn}/W_n$, where n is a fixed positive integer, $W_0=0$, $W_1=1$, and $W_{i+2}=\sqrt{R}\,W_{i+1}-QW_i$, where R and Q are relatively prime integers. See page 437 in D. H. Lehmer, "An Extended Theory of Lucas' Functions," Annals of Mathematics, Series 2, 31 (1930):419-448. The proposer stated that this problem was inspired by David Englund.

Also solved by Michel Ballieu, Paul S. Bruckman, Leonard A. G. Dresel, Steve Edwards, C. Georghiou, Pentti Haukkanen (two solutions), Russell Jay Hendel, Norbert Jensen, Joseph J. Kostal, Harris Kwong, Carl Libis, Bob Prielipp, H.-J. Seiffert, Lawrence Somer, J. Suck, David C. Terr, A. N. 't Woord, David Zeitlin, and the proposer.

No Remainder

B-749 ***Proposed by Richard André-Jeannin, Longwy, France (Vol. 31, no. 4, November 1993)***

For n a positive integer, define the polynomial $P_n(x)$ by $P_n(x)=x^{n+2}-x^{n+1}-F_nx-F_{n-1}$. Find the quotient and remainder when $P_n(x)$ is divided by x^2-x-1.

Solution by H. K. Krishnapriyan, Drake University, Des Moines, IA, and by Joseph J. Kostal, Chicago, IL (independently)

Direct multiplication confirms that

$$P_n(x) = (F_{n-1} + F_{n-2}x + F_{n-3}x^2 + \cdots + F_0x^{n-1} + x^n)(x^2 - x - 1).$$

Thus, the quotient is $\sum_{k=0}^{n} F_{n-k-1}x^k$ and the remainder is 0.

Beasley found the analog for Lucas numbers: $x^{n+2} + x^{n+1} - 2x^n - L_nx - L_{n-1}$ *is divisible by* $x^2 - x - 1$. *Redmond found that if r and s are distinct roots of* $x^2 - ax - b = 0$ *and* $u_n = \frac{r^n - s^n}{r-s}$ *then* $x^{n+2} - ax^{n+1} + bu_nx - b^2u_{n-1}$ *is divisible by* $x^2 - ax + b$. *Suck showed that if* f_n *satisfies the recurrence* $a_0f_n + a_1f_{n+1} + \cdots + a_rf_{n+r} = 0$ *then, for* $n \ge r - 1$,

$$\sum_{i=0}^{r-1}\sum_{j=0}^{i} a_jf_{m+n-i+j}x^i + \sum_{i=1}^{r}\sum_{j=i}^{r} a_jf_{m-i+j}x^{n+i}$$

is divisible by $a_0 + a_1x + \cdots + a_rx^r$. *The given problem is the special case* $f_n = F_n$, $r = 2$, $a_0 = a_1 = -1$, $a_2 = 1$, *and* $m = -1$. *These solvers found the quotient in each case as well. Zeitlin found that* $x^{n+1} - x^{n-1} - F_nx - F_{n-1}$ *is also divisible by* $x^2 - x - 1$.

Also solved by Charles Ashbacher, Brian D. Beasley, Paul S. Bruckman, Leonard A. G. Dresel, Steve Edwards, F. J. Flanigan, Herta Freitag, C. Georghiou, Russell Jay Hendel, Norbert Jensen, Hans Kappus, Harris Kwong, Carl Libis, Bob Prielipp, Don Redmond, H.-J. Seiffert, Tony Shannon, J. Suck, A. N. 't Woord, David Zeitlin, and the proposer.

A Linear Transformation that Shifts

B-750 ***Proposed by Seung-Jin Bang, Albany, CA (Vol. 31, no. 4, November 1993)***

Find a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(F_n, L_n) = (F_{n+1}, L_{n+1})$.

Solution by Leonard A. G. Dresel, Reading, England

Adding the identities $F_n = F_{n+1} - F_{n-1}$ and $L_n = F_{n+1} + F_{n-1}$, we obtain $F_n + L_n = 2F_{n+1}$. Similarly, adding the identities $L_n = L_{n+1} - L_{n-1}$ and $5F_n = L_{n+1} + L_{n-1}$, we obtain $5F_n + L_n = 2L_{n+1}$.

Hence, the required transformation is

$$T(x, y) = \left(\frac{x+y}{2}, \frac{5x+y}{2}\right).$$

This can also be written as

$$\begin{pmatrix} F_{n+1} \\ L_{n+1} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 5/2 & 1/2 \end{pmatrix}\begin{pmatrix} F_n \\ L_n \end{pmatrix}.$$

Redmond generalized to the sequences defined by $u_n = (r^n - s^n)/(r - s)$ and $v_n = r^n + s^n$, *where r and s are the distinct roots of* $x^2 - ax + b = 0$. *In this case, he found that if*

$$T(x, y) = \left(\frac{1}{2}xv_k + \frac{1}{2}yu_k, \frac{1}{2}x(r-s)^2u_k + \frac{1}{2}yv_k\right)$$

then $T(u_n, v_n) = (u_{n+k}, v_{n+k})$. *The given problem is the special case where* $a = 1, b = -1$, and $k = 1$. *Several solvers found a transformation such as* $T(x, y) = (y - xF_{n-1} / F_n, 5x - yL_{n-1} / L_n)$ *which, for a given fixed n, is a linear transformation. However, these solutions are not as elegant as the featured solution in which the linear transformation found is independent of n.*

Also solved by Charles Ashbacher, Michel Ballieu, Paul S. Bruckman, Charles K. Cook, Steve Edwards, F. J. Flanigan, C. Georghiou, Russell Jay Hendel, Norbert Jensen, Hans Kappus, Joseph J. Kostal, H. K. Krishnapriyan, Harris Kwong, Stanley Wu-Wei Liu, Bob Prielipp, Don Redmond, H.-J. Seiffert, Lawrence Somer, J. Suck, David C. Terr, A. N. 't Woord, and the proposer.

Divisibility by 25

B-751 ***Proposed by Jayantibhai M. Patel, Bhavan's R. A. Col. Sci., Gujarat State, India (Vol. 31, no. 4, November 1993)***

Prove that $6L_{n+3}L_{3n+4} + 7$ and $6L_nL_{3n+5} - 7$ are divisible by 25.

Solution by Russell Jay Hendel, University of Louisville, Louisville, KY

This and similar problems can always be solved swiftly using periodicity properties.

Looking at L_n (mod 25), namely, 2, 1, 3, 4, 7, 11, 18, 4, 22, 1, -2, -1, ... shows that the function L_n modulo 25 has period 20. It immediately follows that the functions L_{n+3}, L_{3n+4}, L_n, and L_{3n+5} all have period 20 modulo 25. To prove the given assertions, it therefore suffices to check that the given two functions when calculated modulo 25 on $n = 0, 1, 2, \ldots, 19$ all equal 0.

As an example, if $n = 2$ then, modulo 25, we find $L_{n+3} \equiv 11$ and $L_{3n+4} = L_{10} \equiv 23$, so

$$6L_{n+3}L_{3n+4} + 7 \equiv 6(11)(-2) + 7 \equiv 0.$$

The editor found the explicit representations

$$6L_{n+3}L_{3n+4} + 7 = 25[7 + 69(-1)^n F_n^2 + 87F_n^4 + 15(-1)^n F_{2n} + 39F_n^3 L_n]$$

and

$$6L_nL_{3n+5} - 7 = 25[5 + 33(-1)^n F_n^2 + 33F_n^4 + 3(-1)^n F_{2n} + 3F_{4n}],$$

directly showing that these expressions are divisible by 25. He wonders if explicit representations can be found for all similar divisibility problems. (See, for example, the solution to Problem B-741 in the previous issue.)

Also solved by Charles Ashbacher, Paul S. Bruckman, Leonard A. G. Dresel, C. Georghiou, Norbert Jensen, H.-J. Seiffert, J. Suck, David Zeitlin, and the proposer.

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