SOME INFORMATION ABOUT THE BINOMIAL TRANSFORM

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A few days ago I saw the paper [4]. I think I can make some additional remarks that might not be totally useless for the Fibonacci Community!

Let (a_n) be a given sequence and $s_n = \sum_{k=0}^{n} \binom{n}{k} a_k$. Denoting the respective (ordinary) generating functions by $A(x)$ and $S(x)$, the paper in question mainly deals with the consequences of the formula

$$S(x) = \frac{1}{1-x} A\left(\frac{x}{1-x}\right).$$

(1)

Knuth [7] has introduced the *binomial transform* by

$$\hat{a}_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k,$$

and it is clear that this is the situation from above. But Philippe Flajolet and the present writer agreed about ten years ago that there are just *exponential generating functions* hidden! They have a convolution formula

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k},$$

and upon choosing the b_k's to be equal to 1, we have the old situation. So, denoting the exponential generating functions by $\overline{A}(x)$ and $\overline{S}(x)$, we have the even simpler formula $\overline{S}(x) = e^x \overline{A}(x)$. This can readily be inverted as $\overline{A}(x) = e^{-x} \overline{S}(x)$, whence

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} s_k.$$

These facts about exponential generating functions are of course folklore; one particular reference is [3].

Flajolet & Richmond [2], Schmid [8], and Kirschenhofer & Prodinger [6] all made heavy use of (1). Schmid observed (among other writers) that an exponential generating function will be transformed into an ordinary generating function by the *Borel transform*.

Now the *generalization*

$$s_n = \sum_{k=0}^{n} \binom{n}{k} b^{n-k} c^k a_k \quad \text{or} \quad S(x) = \frac{1}{1-bx} A\left(\frac{cx}{1-bx}\right)$$

translates into

$$\overline{S}(x) = e^{bx} \overline{A}(cx).$$

Since

$$\overline{A}(x) = e^{-\frac{b}{c}x} \overline{S}\left(\frac{x}{c}\right),$$

we find the inversion formula

$$a_n = c^{-n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b^{n-k} s_k.$$

The discussion in Theorem 2 becomes quite transparent, considering exponential generating functions. It is asked whenever we have

$$F_{pn+r} = \sum_{k=0}^{n} \binom{n}{k} t^{n-k} s^k F_{qk+r},$$

where F_n denote Fibonacci numbers. The exponential generating function of the Fibonacci numbers F_n is

$$\frac{1}{\sqrt{5}} (e^{\alpha x} - e^{\beta x}),$$

with the usual $\alpha = (1+\sqrt{5})/2$ and $\beta = -1/\alpha = (1-\sqrt{5})/2$. More generally, the sequence F_{pn+r} leads to

$$\frac{1}{\sqrt{5}} \left(\alpha^r e^{\alpha^p x} - \beta^r e^{\beta^p x} \right) = e^{tx} \frac{1}{\sqrt{5}} \left(\alpha^r e^{\alpha^q sx} - \beta^r e^{\beta^q sx} \right),$$

from which we deduce the two equations,

$$\alpha^p = t + \alpha^q s \quad \text{and} \quad \beta^p = t + \beta^q s.$$

Subtracting them, we see that

$$s = \frac{\alpha^p - \beta^p}{\alpha^q - \beta^q} = \frac{F_p}{F_q}.$$

Further,

$$t = \alpha^p - \alpha^q \frac{\alpha^p - \beta^p}{\alpha^q - \beta^q} = (-1)^p \frac{\alpha^{q-p} - \beta^{q-p}}{\alpha^q - \beta^q} = (-1)^p \frac{F_{q-p}}{F_q}.$$

To justify this *equating of coefficients*, we note that the functions $e^{\lambda x}$ are linearly independent; and the other possibility of grouping terms from the left and the right side would lead to the impossible equation $\alpha^r = -\beta^r$.

In [4] there is also the modification: What are the coefficients of

$$T(x) = A\left(\frac{cx}{1-bx} \right)?$$

That means: What is the effect of deleting the first factor? We can answer this much more generally by considering (with an arbitrary complex parameter d),

$$T(x) = \frac{1}{(1-bx)^d} A\left(\frac{cx}{1-bx} \right).$$

In this derivation, we will use the concept of *residues*, interesting *per se*.

We are using the substitution $w = \frac{cx}{1-bx}$ or $x = \frac{w}{c+bw}$. Therefore, $1 - bx = \frac{c}{c+bw}$ and $dx = \frac{c}{(c+bw)^2} dw$; thus,

$$t_n := [x^n]T(x) = \frac{1}{2\pi i} \oint \frac{dx}{x^{n+1}} T(x)$$

$$= \frac{1}{2\pi i} \oint \frac{dx}{x^{n+1}} \frac{(c+bw)^d}{c^d} A(w)$$

$$= \frac{1}{2\pi i} \oint \frac{cdw}{(c+bw)^2} \frac{(c+bw)^{n+1}}{w^{n+1}} \frac{(c+bw)^d}{c^d} A(w)$$

$$= c^{1-d}[w^n](c+bw)^{n+d-1} A(w)$$

$$= \sum_{k=0}^{n} \binom{n+d-1}{n-k} b^{n-k} c^k a_k.$$

Since

$$A(w) = \left(\frac{c}{c+bw}\right)^d T\left(\frac{w}{c+bw}\right),$$

we find in a similar way the inversion formula

$$a_n = c^{-n} \sum_{k=0}^{n} \binom{n+d-1}{n-k} (-1)^{n-k} b^{n-k} t_k.$$

The formula (1) is also useful to deal with Knuth's sum [5, eq. (7.6)]

$$u_n = \sum_{k=0}^{n} \binom{n}{k} \left(-\frac{1}{2}\right)^k \binom{2k}{k}.$$

Since

$$f(x) := \sum_{k \geq 0} \left(-\frac{1}{2}\right)^k \binom{2k}{k} x^k = \sum_{k \geq 0} \left(-\frac{x}{2}\right)^k \binom{2k}{k} = \frac{1}{\sqrt{1+2x}},$$

the generating function of the sequence u_n turns out to be

$$\frac{1}{1-x} \frac{1}{\sqrt{1+2\left(\frac{x}{1-x}\right)}} = \frac{1}{\sqrt{1-x^2}} = \sum_{n \geq 0} x^{2n} \binom{2n}{n} 4^{-n}.$$

From this, we see that $u_n = 2^{-n}\binom{n}{n/2}$ if n is even, and $u_n = 0$ otherwise.

I communicated this idea to Knuth, and he reported that Herbert Wilf came to this (or a similar) approach independently.

Formula (1) also has a *combinatorial interpretation*. If, for example, $A(x)$ enumerates certain words, so that a_n is the number of words of length n with a certain property, and we perform the operation "fill-in a new letter where and as often as you want," then the new "language" has the generating function $S(x)$. For further details on such *combinatorial constructions*, we refer the reader to [1].

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GENERALIZED PASCAL TRIANGLES AND PYRAMIDS:
THEIR FRACTALS, GRAPHS, AND APPLICATIONS

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This monograph was first published in Russia in 1990 and consists of seven chapters, a list of 406 references, an appendix with another 126 references, many illustrations and specific examples. Fundamental results in the book are formulated as theorems and algorithms or as equations and formulas. For more details on the contents of the book, see *The Fibonacci Quarterly* 31.1 (1993):52.

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