ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Stanley Rabinowitz

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Each solution should be on a separate sheet (or sheets) and must be received within six months of publication of the problem. Solutions typed in the format used below will be given preference. Proposers of problems should normally include solutions. Although this Elementary Problem section does not insist on original problems, we do ask that proposers inform us of the history of the problem, if it is not original, and require that they do not submit the problem elsewhere while it is under consideration for publication herein.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \ F_0 = 0, \ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \ L_0 = 2, \ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

Nary a month goes by without my receiving a problem proposal from the inveterate problemist, Herta Taussig Freitag. So, as a tribute to Herta, and to reduce my backlog, all of the problems in this issue come from her. As usual, generalizations are always welcome.

B-772 ***Proposed by Herta T. Freitag, Roanoke, Virginia***

Prove that

$$\frac{L_n^2 + L_{n+a}^2}{F_n^2 + F_{n+a}^2}$$

is always an integer if a is odd. How should this problem be modified if a is even?

B-773 ***Proposed by Herta T. Freitag, Roanoke, Virginia***

Find the number of terms in the Zeckendorf representation of $\sum_{i=1}^{n} F_i^2$.

[The Zeckendorf representation of an integer expresses that integer as a sum of distinct non-consecutive Fibonacci numbers.]

B-774 ***Proposed by Herta T. Freitag, Roanoke, Virginia***

Let $\langle H_n \rangle$ be any sequence of integers such that $H_{n+2} = H_{n+1} + H_n$ for all n. Let p and m be positive integers such that $H_{n+p} \equiv H_n \pmod{m}$ for all integers n. Prove that the sum of any p consecutive terms of the sequence is divisible by m.

B-775 ***Proposed by Herta T. Freitag, Roanoke, Virginia***

Let $g = \alpha + 2$. Express g^{17} in the form $p\alpha + q$ where p and q are integers.

B-776 ***Proposed by Herta T. Freitag, Roanoke, Virginia***

Find all values of n for which $\sum_{k=1}^{n} kF_k$ is even.

B-777 ***Proposed by Herta T. Freitag, Roanoke, Virginia***

Find all integers a such that $n \equiv a \pmod 4$ if and only if $L_n \equiv a \pmod 5$.

SOLUTIONS

Fibonacci Fractions

B-739 ***Proposed by Ralph Thomas, University of Chicago, Dundee, Illinois (Vol. 31, no. 2, May 1993)***

Let $S = \left\{\frac{F_i}{F_j} \middle| i \geq 0, j > 0\right\}$. Is S dense in the set of nonnegative real numbers?

Solution by Margherita Barile, Universität Essen, Germany

The answer is no, since one has the following two facts:

(a) for all $i \geq 2$ and $j \geq i+2$, $\frac{F_i}{F_j} \leq \frac{F_i}{F_{i+2}} < \frac{1}{2}$;

(b) for all $i \geq 5$ and $0 < j \leq i+1$, $\frac{F_i}{F_j} \geq \frac{F_i}{F_{i+1}} > \frac{3}{5}$.

Hence, if $i \geq 5$, the fraction F_i / F_j cannot lie in the closed interval $[1/2, 3/5]$. This interval therefore only contains a finite number of elements of S and so S cannot be dense in that interval.

Both the claims (a) and (b) can be proved by induction in i. For claim (a), first note that

$$\frac{F_2}{F_4} = \frac{1}{3} < \frac{1}{2} \quad \text{and} \quad \frac{F_3}{F_5} = \frac{2}{5} < \frac{1}{2}.$$

Then take $i > 3$ and suppose the claim is true for $i-1$ and $i-2$. Then by induction,

$$\frac{F_i}{F_{i+2}} = \frac{F_{i-1} + F_{i-2}}{F_{i+1} + F_i} < \frac{\frac{1}{2}F_{i+1} + \frac{1}{2}F_i}{F_{i+1} + F_i} = \frac{1}{2}.$$

The proof of claim (b) proceeds similarly.

Several solvers stated that the set of limit points of S is $\{\alpha^p \mid p \in \mathbb{Z}\} \cup \{0\}$.

Also solved by Paul S. Bruckman, Russell Jay Hendel, H.-J. Seiffert, J. Suck, and the proposer.

Smarandache in Reverse

B-740 ***Proposed by Thomas Martin, Phoenix, Arizona (Vol. 31, no. 2, May 1993)***

Find all positive integers x such that 10 is the smallest integer, n, such that $n!$ is divisible by x.

Solution by Jane Friedman, University of San Diego, California

We are looking for all integers x such that $x|10!$ but $x \nmid n!$ for any $n < 10$. Let T be the set of all such integers. Since $10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7^1$, such integers must be of the form $2^a \cdot 3^b \cdot 5^c \cdot 7^d$ with a, b, c, and d nonnegative integers such that $a \le 8, b \le 4, c \le 2,$ and $d \le 1$. But $9! = 2^7 \cdot 3^4 \cdot 5^1 \cdot 7^1$, so we have the additional constraint that $a = 8$ or $c = 2$ or both. Thus,

$$T = \{x | x = 2^8 \cdot 3^b \cdot 5^c \cdot 7^d,\ 0 \le b \le 4, 0 \le c \le 2,\ 0 \le d \le 1\}$$
$$\cup \{x | x = 2^a \cdot 3^b \cdot 5^2 \cdot 7^d,\ 0 \le a \le 8,\ 0 \le b \le 4,\ 0 \le d \le 1\}.$$

There are 110 such integers, so I will not list them all explicitly.

*The proposer remarks that this problem is concerned with the inverse of the Smarandache Function S(n), which is defined to be the smallest integer such that S(n)! is divisible by n. For another problem about the Smarandache Function, see problem H-490 in this issue. For more information about the Smarandache Function, consult the "Smarandache Function Journal." Information about this journal can be obtained from its editor, Dr. R. Muller, at P.O. Box 10163, Glendale, AZ 85318-0163, U.S.A. For another solution to this problem, see "Elemente der Mathematik" **49** (1994):127.*

Also solved by Charles Ashbacher, Margherita Barile, Paul S. Bruckman, Pentti Haukkanen, Russell Jay Hendel, Joseph J. Kostal, H.-J. Seiffert, Sahib Singh, Lawrence Somer, J. Suck, Ralph Thomas, and the proposer.

Factor 54 Where Are You?

B-741 ***Proposed by Jayantibhai M. Patel, Bhavan's R. A. College of Science, Gujarat, India (Vol. 31, no. 2, May 1993)***

Prove that $S_n = F_{n+8}^4 + 331F_{n+4}^4 + F_n^4$ is always divisible by 54.

Solution by Piero Filipponi, Fond. U. Bordoni, Rome, Italy

Using the known identities

$$F_n^4 = \frac{L_{4n} - 4(-1)^n L_{2n} + 6}{25} \quad \text{and} \quad L_{n+k} + L_{n-k} = L_n L_k,\ k \text{ even}$$

(identities 81 and 17a in [1]), we get

$$S_n = \frac{(L_{16} + 331)L_{4n+16} - 4(-1)^n (L_8 + 331)L_{2n+8} + 1998}{25} = \frac{54}{25}[47L_{4n+16} - 28(-1)^n L_{2n+8} + 37].$$

Since 54 and 25 have no common factor, it follows that S_n is divisible by 54.

Dresel expressed S_n as $54[47F_{n+4}^4 + 32(-1)^n F_{n+4}^2 + 3]$, also elegantly showing that 54 is an explicit factor.

Reference

1. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section*. Chichester: Halsted, 1989.

Also solved by Paul S. Bruckman, Leonard A. G. Dresel, Russell Jay Hendel, Bob Prielipp, H.-J. Seiffert, Sahib Singh, J. Suck, Ralph Thomas, David Zeitlin, and the proposer.

Pell's Triggy Product

B-742 ***Proposed by Curtis Cooper & Robert E. Kennedy, Central Missouri State University, Warrensburg, Missouri***
(Vol. 31, no. 3, August 1993)

Pell numbers are defined by $P_0 = 0$, $P_1 = 1$, and $P_{n+1} = 2P_n + P_{n-1}$, for $n \geq 1$. Show that

$$P_{23} = 2^{11}\prod_{j=1}^{11}\left(3+\cos\frac{2j\pi}{23}\right).$$

Solution by Lou Shapiro, Silver Spring, Maryland

The Binet form [3] for the Pell numbers is $P_n = \frac{1}{2\sqrt{2}}(p^n - q^n)$ where $p = 1+\sqrt{2}$ and $q = 1-\sqrt{2}$. A form of the cyclotomic identity is

$$z^n - y^n = \prod_{j=0}^{n-1}(z - w^j y)$$

where $w = e^{2\pi i/n}$ is a primitive n^{th} root of unity. If n is odd, we have [1]

$$z^n - y^n = (z-y)\prod_{j=1}^{\frac{n-1}{2}}(z-w^j y)(z-w^{-j}y) = (z-y)\prod_{j=1}^{\frac{n-1}{2}}\left(z^2 - 2\cos\frac{2j\pi}{n}zy + y^2\right).$$

Now let $z = p = 1+\sqrt{2}$ and $y = q = 1-\sqrt{2}$, and note that $p - q = 2\sqrt{2}$, $p^2 + q^2 = 6$, and $pq = -1$. Thus, we have

$$p^n - q^n = 2\sqrt{2}\prod_{j=1}^{\frac{n-1}{2}}\left(6+2\cos\frac{2j\pi}{n}\right)$$

and, therefore,

$$P_n = 2^{(n-1)/2}\prod_{j=1}^{\frac{n-1}{2}}\left(3+\cos\frac{2j\pi}{n}\right).$$

Letting $n = 23$ finishes the problem.

A similar result can be obtained when n is even. Combining these two results gives the general formula

$$P_n = 2^{\lfloor n/2\rfloor}\prod_{j=1}^{\lfloor\frac{n-1}{2}\rfloor}\left(3+\cos\frac{2j\pi}{n}\right) \qquad (*)$$

which is true for all positive integers n.

Note that this same method gives an elementary proof of problems H-93 [2] and H-466 [4] which state that

$$F_n = \prod_{j=1}^{\lfloor\frac{n-1}{2}\rfloor}\left(3+2\cos\frac{2j\pi}{n}\right).$$

Suck refers to problem H-64 [5] where it is shown that $F_n = \prod_{j=1}^{n-1}(1-2i\cos\frac{j\pi}{n})$. *Zeitlin mentions that in his solution to H-64 [5], he showed that if* Z_n *satisfies the recurrence* $Z_{n+2} = dZ_{n+1} - cZ_n$, *with* $Z_0 = 0$ *and* $Z_1 = 1$, *then*

$$Z_n = c^{(n-1)/2}\prod_{j=1}^{n-1}\left(\frac{d}{\sqrt{c}} - 2\cos\frac{j\pi}{n}\right).$$

Hendel and Cook find recurrences for expressions similar to (∗) *where "3" is replaced by a fixed constant m. Seiffert shows that for all complex z,*

$$f_{2n-1}(z)=\prod_{j=1}^{n-1}\left(z^2+2+2\cos\frac{2j\pi}{2n-1}\right) \quad \text{and} \quad f_{2n}(z)=z\prod_{j=1}^{n-1}\left(z^2+2+2\cos\frac{j\pi}{n}\right)$$

where $\{f_n(x)\}$ *are the Fibonacci polynomials defined by* $f_{n+1}(x)=xf_n(x)+f_{n-1}(x)$ *with* $f_0(x)=0$ *and* $f_1(x)=1$.

References

1. I. S. Gradshteyn & I. M. Ryzhik. *Table of Integrals, Series, and Products* (corrected and enlarged edition), p. 34, formula 1.396.2. San Diego, Calif.: Academic Press, 1980.
2. Douglas Lind. "Problem H-93." *The Fibonacci Quarterly* **6.2** (1968):145-48.
3. Neville Robbins. *Beginning Number Theory*, p. 193. Dubuque, Iowa: Wm. C. Brown, 1993.
4. J. A. Sjogren. "Problem H-466: A Triggy Problem." *The Fibonacci Quarterly* **30.2** (1992): 188.
5. David Zeitlin. "Solution to Problem H-64." *The Fibonacci Quarterly* **5.1** (1967):74-75.

Also solved by Paul S. Bruckman, Leonard A. G. Dresel, Russell Jay Hendel, Norbert Jensen, Hans Kappus, Joseph J. Kostal, Almas Rumov, H.-J. Seiffert, J. Suck, David Zeitlin, and the proposers.

Golden Argument of Tenth Roots of Unity

B-743 ***Proposed by Richard André-Jeannin, Longwy, France (Vol. 31, no. 3, August 1993)***

Find the modulus and the argument of the complex numbers

$$u=\frac{\beta+i\sqrt{\alpha+2}}{2} \quad \text{and} \quad v=\frac{\alpha+i\sqrt{\beta+2}}{2}.$$

Solution by H.-J. Seiffert, Berlin, Germany

From Problem B-674 (proposed by Richard André-Jeannin in *The Fibonacci Quarterly* **29.3** [1991]:280), we know that $\cos(\pi/5)=\alpha/2$ and $\cos(3\pi/5)=\beta/2$. Since $\sin(\pi/5)>0$ and $\sin(3\pi/5)>0$, we find

$$\sin\frac{\pi}{5}=\sqrt{1-\cos^2\frac{\pi}{5}}=\frac{\sqrt{4-\alpha^2}}{2}=\frac{\sqrt{3-\alpha}}{2}=\frac{\sqrt{\beta+2}}{2},$$

where we have used $\alpha^2=\alpha+1$ and $\alpha=1-\beta$. Similarly, we find $\sin(3\pi/5)=\sqrt{\alpha+2}/2$. Therefore, we have

$$u=\cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5} \quad \text{and} \quad v=\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}.$$

This shows that u and v both have modulus 1, and the argument of u and v is $3\pi/5$ and $\pi/5$, respectively.

Flanigan notes that u and v are primitive tenth roots of unity.

Also solved by M. A. Ballieu, Paul S. Bruckman, Leonard A. G. Dresel, Russell Euler, Piero Filipponi, F. J. Flanigan, Pentti Haukkanen, Russell Jay Hendel, Norbert Jensen, Hans Kappus, Carl Libis, Bob Prielipp, J. Suck, and the proposer.

A Sum Divisible

B-744 ***Proposed by Herta T. Freitag, Roanoke, Virginia (Vol. 31, no. 3, August 1993)***

Let n and k be even positive integers. Prove that $L_{2n}+L_{4n}+L_{6n}+\cdots+L_{2kn}$ is divisible by L_n.

Solution 1 by Piero Filipponi, Fond. U. Bordoni, Rome, Italy

Let $S_k(n)=\sum_{j=1}^{k}L_{2nj}$ with even k. We shall prove that $S_k(n)\equiv 0 \pmod{L_n}$ if n is even and $S_k(n)\equiv 0 \pmod{5F_n}$ if n is odd.

Rewrite $S_k(n)$ as

$$S_k(n)=\sum_{j=1}^{k/2}[L_{(4j-1)n-n}+L_{(4j-1)n+n}]$$

and use the identities

$$L_{n+p}+L_{n-p}=\begin{cases}L_nL_p, & p \text{ even,}\\ 5F_nF_p, & p \text{ odd}\end{cases}$$

(formulas 17a and 17b from [2]) to obtain

$$S_k(n)=\begin{cases}L_n\sum_{j=1}^{k/2}L_{(4j-1)n}, & n \text{ even,}\\ 5F_n\sum_{j=1}^{k/2}L_{(4j-1)n}, & n \text{ odd.}\end{cases}$$

Solution 2 by Norbert Jensen, Kiel, Germany

We prove the stronger result that $L_{2n}+L_{4n}+L_{6n}+\cdots+L_{2kn}$ is divisible by L_n^2 when k and n are even.

Pairing the terms up two at a time, we find that in each pair, with j odd,

$$L_{2jn}+L_{2(j+1)n}=[\alpha^{nj}+\beta^{nj}]^2+[\alpha^{n(j+1)}-\beta^{n(j+1)}]^2=L_{nj}^2+5F_{n(j+1)}^2.$$

Since, when j is odd, L_n divides L_{nj} and L_n divides $F_{n(j+1)}$ ([1], Theorems 4 and 5, p. 40), we see that each pair is divisible by L_n^2 and so is the entire sum.

Bruckman notes that $S_k(n)=L_{n(k+1)}F_{nk}/F_n$ for all k and n.

References

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Boston: Houghton Mifflin, 1969; rpt. Santa Clara, Calif: The Fibonacci Association, 1979.
2. S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section.* Chichester: Halsted, 1989.

Also solved by Paul S. Bruckman, Leonard A. G. Dresel, F. J. Flanigan, Russell Jay Hendel, Chris Long, Richard McGuffin, Bob Prielipp, Almas Rumov, H.-J. Seiffert, Lawrence Somer, J. Suck, and the proposer.

Erratum: Paul S. Bruckman was inadvertently omitted as a solver of Problem B-726.

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