A SIMPLE METHOD WHICH GENERATES INFINITELY MANY CONGRUENCE IDENTITIES

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1. Introduction

Let $\phi(m)$ be an integer-valued function defined on the set of all positive integers. If $m = p_1^{k_1}p_2^{k_2} \ldots p_r^{k_r}$, where the p_i's are distinct prime numbers, r and the k_i's are positive integers, we define $\Phi_1(1, \phi) = \phi(1)$ and

$$\Phi_1(m, \phi) = \phi(m) - \sum_{i=1}^{r} \phi(m/p_i) + \sum_{i_1 < i_2} \phi(m/(p_{i_1} p_{i_2}))$$
$$- \sum_{i_1 < i_2 < i_3} \phi(m/(p_{i_1} p_{i_2} p_{i_3})) + \cdots + (-1)^r \phi(m/(p_1 p_2 \ldots p_r)),$$

where the summation $\sum_{i_1 < i_2 < \cdots < i_j}$ is taken over all integers $i_1, i_2, \ldots, i_j$ with $1 \le i_1 < i_2 < \cdots < i_j \le r$.

If $m = 2^{k_0}p_1^{k_1}p_2^{k_2} \ldots p_r^{k_r}$, where the p_i's are distinct odd prime numbers, and $k_0 \ge 0$, r, and the k_i's ≥ 1 are integers, we define, similarly,

$$\Phi_2(m, \phi) = \phi(m) - \sum_{i=1}^{r} \phi(m/p_i) + \sum_{i_1 < i_2} \phi(m/(p_{i_1} p_{i_2}))$$
$$- \sum_{i_1 < i_2 < i_3} \phi(m/(p_{i_1} p_{i_2} p_{i_3})) + \cdots + (-1)^r \phi(m/(p_1 p_2 \ldots p_r)).$$

If $m = 2^k$, where $k \ge 0$ is an integer, we define

$$\Phi_2(m, \phi) = \phi(m) - 1.$$

If, for some integer $n \ge 2$, we have $\phi(m) = n^m$ for all positive integers m, then we denote $\Phi_i(m, \phi)$ by $\Phi_i(m, n)$, $i = 1, 2$, to emphasize the role of this integer n.

On the other hand, let S be a subset of the real numbers and let f be a function from S into itself. For every positive integer n, we let f^n denote the n^{th} iterate of f: $f^1 = f$ and $f^n = f \circ f^{n-1}$ for $n \ge 2$. For every $x_0 \in S$, we call the set $\{f^k(x_0) \mid k \ge 0\}$ the orbit of x_0 under f. If x_0 satisfies $f^m(x_0) = x_0$ for some positive integer m, then we call x_0 a periodic point of f and call the smallest such positive integer m the minimal period of x_0 and of the orbit of x_0 (under f). Note that, if x_0 is a periodic point of f with minimal period m, then, for every integer $1 \le k \le m$, $f^k(x_0)$ is also a periodic point of f with minimal period m and they are all distinct, so every periodic orbit of f with minimal period m consists of exactly m distinct points. Since it is obvious that distinct periodic orbits of f are pairwise disjoint, the number (if finite) of distinct periodic points of f with minimal period m is divisible by m and the quotient equals the number of distinct periodic orbits of f with minimal period m. This observation, together with a standard inclusion-exclusion argument, gives the following well-known result.

Theorem 1: Let S be a subset of the real numbers and let $f: S \to S$ be a mapping with the property that, for every positive integer m, the equation $f^m(x) = x$ (or $-x$, respectively) has only finitely many distinct solutions. Let $\phi(m)$ (or $\psi(m)$, respectively) denote the number of these solutions. Then, for every positive integer m, the following hold.

(i) The number of periodic points of f with minimal period m is $\Phi_1(m, \phi)$. So $\Phi_1(m, \phi) \equiv 0 \pmod{m}$.

(ii) If $0 \in S$ and f is odd, then the number of symmetric periodic points (i.e., periodic points whose orbits are symmetric with respect to the origin) of f with minimal period $2m$ is $\Phi_2(m, \psi)$. Thus, $\Phi_2(m, \psi) \equiv 0 \pmod{2m}$.

Successful applications of the above theorem depend of course on a knowledge of the function ϕ or ψ. For example, if we let S denote the set of all real numbers and, for every integer $n \geq 2$ and every odd integer $t = 2k + 1 > 1$, let

$$f_n(x) = a_n \cdot \prod_{j=1}^{n} (x - j)$$

and let

$$g_t(x) = b_t \cdot x \prod_{j=1}^{k} (x^2 - j^2),$$

where a_n and b_t are fixed sufficiently large positive numbers depending only on n and t, respectively. Then it is easy to see that, for every positive integer m, the equation $f_n^m(x) = x$ [$g_t^m(x) = -x$, resp.] has exactly n^m (t^m, resp.) distinct solutions in S. Therefore, if $\phi(m, n) = n^m$ and $\psi(m, t) = t^m$, then we have as a consequence of Theorem 1 the following well-known congruence identities which include Fermat's Little Theorem as a special case.

Corollary 2: (i) Let $m \geq 1$ and $n \geq 2$ be integers. Then $\Phi_1(m, n) \equiv 0 \pmod{m}$.

(ii) Let $m \geq 1$ be an integer and let $n > 1$ be an odd integer. Then $\phi_2(m, n) \equiv 0 \pmod{2m}$.

In this note, we indicate that the method introduced in [1] can also be used to recursively define infinitely many ϕ and ψ and thus produce infinitely many families of congruence identities related to Theorem 1. In Section 2, we will review this method, and to illustrate it we will prove the following result in Section 3.

Theorem 3: For every positive integer $n \geq 3$, let ϕ_n be the integer-valued function on the set of all positive integers defined recursively by letting $\phi_n(m) = 2^m - 1$ for all $1 \leq m \leq n - 1$ and

$$\phi_n(n + k) = \sum_{j=1}^{n-1} \phi_n(n + k - j), \text{ for all } k \geq 0.$$

Then, for every positive integer m, $\Phi_1(m, \phi_n) \equiv 0 \pmod{m}$. Furthermore,

$$\lim_{m \to \infty} [\log \Phi_1(m, \phi_n)]/m = \lim_{m \to \infty} [\log \phi_n(m)]/m = \log \alpha_n,$$

where α_n is the (unique) positive (and the largest in absolute value) zero of the polynomial

$$x^{n-1} - \sum_{k=0}^{n-2} x^k.$$

Note that in the above theorem these numbers $\phi_n(m)$, $m \geq 1$, are generalized Fibonacci numbers [3, 4] and when $n = 3$, these numbers $\phi_3(m)$, $m \geq 1$, are the well-known Lucas numbers: 1, 3, 4, 7, 11, 18, 29, ...

Just for comparison, we also include the following two results which can be verified numerically. The rigorous proofs of these two results which are similar to that of Theorem 3 below can be found in [1, Theorem 2] and [2, Theorem 3], respectively.

Theorem 4: For every positive integer $n \geq 2$, let sequences

$$\langle b_{k,1,j,n} \rangle, \ \langle b_{k,2,j,n} \rangle, \ 1 \leq j \leq n,$$

be defined recursively as follows:

$$b_{1,1,j,n} = 0, \quad 1 \leq j \leq n,$$
$$b_{2,1,j,n} = 1, \quad 1 \leq j \leq n,$$
$$b_{1,2,j,n} = b_{2,2,j,n} = 0, \quad 1 \leq j \leq n-1,$$
$$b_{1,2,n,n} = b_{2,2,n,n} = 1.$$

For $i = 1$ or 2, and $k \geq 1$,

$$b_{k+2,i,j,n} = b_{k,i,1,n} + b_{k,i,j+1,n}, \quad 1 \leq j \leq n-1,$$
$$b_{k+2,i,n,n} = b_{k,i,1,n} + b_{k+1,i,n,n}.$$

Let $b_{k,1,j,n} = 0$ for all $-2n + 3 \leq k \leq 0$ and $1 \leq j \leq n$, and for all positive $n$integers m, let

$$\phi_n(m) = b_{m,2,n,n} + 2 \cdot \sum_{j=1}^{n} b_{m+2-2j,1,j,n}.$$

Then, for every positive integer m, $\Phi_1(m, \phi_n) \equiv 0 \pmod{m}$. Furthermore,

$$\lim_{m \to \infty} [\log \Phi_1(m, \phi_n)]/m = \lim_{m \to \infty} [\log \phi_n(m)]/m = \log \beta_n,$$

where β_n is the (unique) positive (and the largest in absolute value) zero of the polynomial $x^{2n+1} - 2x^{2n-1} - 1$.

Remark 1: For all positive integers m and n, let

$$A_{m,n} = \Phi_1(2m-1, \phi_n)/(2m-1),$$

where ϕ_n is defined as in Theorem 3 for $n = 1$ and as in Theorem 4 for $2 \leq n$. Table 1 lists the first 31 values of $A_{m,n}$ for $1 \leq n \leq 6$. It seems that $A_{m,n} = 2^{m-n-1}$ for $n + 1 \leq m \leq 3n + 2$ and $A_{m,n} > 2^{m-n-1}$ for $m > 3n + 2$. If, for all positive integers m and n, we define sequences $\langle B_{m,n,k} \rangle$ by letting

$$B_{m,n,1} = A_{m+3n+2,n} - 2A_{m+3n+1,n}$$

and

$$B_{m,n,k} = B_{m+2n+1,n,k-1} - B_{m+2n+1,n+1,k-1}$$

for $k > 1$, then more extensive numerical computations seem to show that, for all positive integers k, we have

(i) $B_{1,n,k} = 2$ for all $n \geq 1$,

(ii) $B_{2,n,k} = 4k$ for all $n \geq 1$,

(iii) $B_{3,n,k}$ is a constant depending only on k, and

(iv) for all $1 \leq m \leq 2n + 1$, $B_{m,n,k} = B_{m,j,k}$ for all $j \geq n \geq 1$.

Theorem 5: Fix any integer $n \geq 2$. For all integers i, j, and k with $i = 1, 2$, $1 \leq |j| \leq n$, and $k \geq 1$, we define $c_{k,i,j,n}$ recursively as follows:

$$c_{1,1,n,n} = 1 \text{ and } c_{1,1,j,n} = 0 \quad \text{for } j \neq n,$$
$$c_{1,2,1,n} = 1 \text{ and } c_{1,2,j,n} = 0 \quad \text{for } j \neq 1.$$

For $i = 1, 2$, and $k \geq 1$,

$$c_{k+1,i,1,n} = c_{k,i,1,n} + c_{k,i,-n,n} + c_{k,i,n,n},$$
$$c_{k+1,i,j,n} = c_{k,i,j-1,n} + c_{k,i,n,n} \quad \text{for all } 2 \leq j \leq n,$$
$$c_{k+1,i,-1,n} = c_{k,i,-1,n} + c_{k,i,-n,n} + c_{k,i,n,n},$$
$$c_{k+1,i,-j,n} = c_{k,i,-j+1,n} + c_{k,i,-n,n} \quad \text{for all } 2 \leq j \leq n.$$

Let $c_{k,1,j,n} = 0$ for all integers k, j with $4 - n \leq k \leq 0$ and $1 \leq |j| \leq n$, and, for all positive integers m, let

$$\phi_n(m) = 2\sum_{k=1}^{n-1} c_{m+2-k,1,n+1-k,n} + 2c_{m+1,2,1,n} - 1$$

and

$$\psi_n(m) = 2\sum_{k=1}^{n-1} c_{m+2-k,1,k-n-1,n} + 2c_{m+1,2,-1,n} + 1.$$

Then, for every positive integer m,

$$\Phi_1(m, \phi_n) \equiv 0 \pmod{m} \quad \text{and} \quad \Phi_2(m, \psi_n) \equiv 0 \pmod{2m}.$$

Furthermore,

$$\lim_{m\to\infty}[\log \Phi_1(m, \phi_n)]/m = \lim_{m\to\infty}[\log \phi_n(m)]/m = \lim_{m\to\infty}[\log \psi_n(m)]/m$$
$$= \lim_{m\to\infty}[\log \Phi_2(m, \psi_n)]/m = \log \gamma_n,$$

where γ_n is the (unique) positive (and the largest in absolute value) zero of the polynomial $x^n - 2x^{n-1} - 1$.

Remark 2: For all integers $m \geq 1$ and $n \geq 2$, let

$$D_{m,n} = \Phi_2(m, \psi_n)/(2m),$$

where the ψ_n's are defined as in the above theorem. Table 2 lists the first 25 values of $D_{m,n}$ for $2 \leq n \leq 6$. It seems that $D_{m,n} = 2^{m-n}$ for $n \leq m \leq 3n$, and $D_{m,n} > 2^{m-n}$ for $m > 3n$. If, for all integers $m \geq 1$ and $n \geq 2$, we define the sequences $\langle E_{m,n,k} \rangle$ by letting

$$E_{m,n,1} = D_{m+3n,n} - 2D_{m+3n-1,n}$$

and

$$E_{m,n,k} = E_{m+2n,n,k-1} - E_{m+2n,n+1,k-1}$$

for $k > 1$, then more extensive computations seem to show that, for all positive integers k, we have

(i) $E_{1,n,k} = 2$ for all $n \geq 2$,

(ii) $E_{2,n,k} = 4k$ for all $n \geq 2$,

(iii) $E_{3,n,k}$ and $E_{4,n,k}$ are constants depending only on k, and

(iv) for all $1 \leq m \leq 2n$, $E_{m,n,k} = E_{m,j,k}$ for all $j \geq n \geq 2$.

See Tables 1 and 2 below.

TABLE 1

m	$A_{m,1}$	$A_{m,2}$	$A_{m,3}$	$A_{m,4}$	$A_{m,5}$	$A_{m,6}$
1	1	1	1	1	1	1
2	1	1	1	1	1	1
3	1	0	0	0	0	0
4	1	1	1	1	1	1
5	2	1	0	0	0	0
6	2	2	2	2	2	2
7	4	2	1	0	0	0
8	5	3	3	3	3	3
9	8	4	2	1	0	0
10	11	6	6	6	6	6
11	18	8	4	2	1	0
12	25	11	9	9	9	9
13	40	16	8	4	2	1
14	58	23	18	18	18	18
15	90	32	16	8	4	2
16	135	46	32	30	30	30
17	210	66	32	16	8	4
18	316	94	61	56	56	56
19	492	136	64	32	16	8
20	750	195	115	101	99	99
21	1164	282	128	64	32	16
22	1791	408	224	191	186	186
23	2786	592	258	128	64	32
24	4305	856	431	351	337	335
25	6710	1248	520	256	128	64
26	10420	1814	850	668	635	630
27	16264	2646	1050	512	256	128
28	25350	3858	1673	1257	1177	1163
29	39650	5644	2128	1026	512	256
30	61967	8246	3328	2402	2220	2187
31	97108	12088	4320	2056	1024	512

TABLE 2

m	$D_{m,2}$	$D_{m,3}$	$D_{m,4}$	$D_{m,5}$	$D_{m,6}$
1	0	0	0	0	0
2	1	0	0	0	0
3	2	1	0	0	0
4	4	2	1	0	0
5	8	4	2	1	0
6	16	8	4	2	1
7	34	16	8	4	2
8	72	32	16	8	4
9	154	64	32	16	8
10	336	130	64	32	16
11	738	264	128	64	32
12	1632	538	256	128	64
13	3640	1104	514	256	128
14	8160	2272	1032	512	256
15	18384	4692	2074	1024	512
16	41616	9730	4176	2050	1024
17	94560	20236	8416	4104	2048
18	215600	42208	16980	8218	4096
19	493122	88288	34304	16464	8194
20	1130976	185126	69376	32992	16392
21	2600388	389072	140458	66132	32794
22	5992560	819458	284684	132608	65616
23	13838306	1729296	577592	265984	131296
24	32016576	3655936	1173040	533672	262740
25	74203112	7742124	2384678	1071104	525824

2. Symbolic Representation for Continuous Piecewise Linear Functions

In this section, we review the method introduced in [1]. Throughout this section, let g be a continuous piecewise linear function from the interval $[c, d]$ into itself. We call the set $\{(x_i, y_i) | i = 1, 2, \ldots, k\}$ a set of nodes for (the graph of) $y = g(x)$ if the following three conditions hold:

(1) $k \geq 2$,
(2) $x_1 = c$, $x_k = d$, $x_1 < x_2 < \cdots < x_k$, and
(3) g is linear on $[x_i, x_{i+1}]$ for all $1 \leq i \leq k - 1$ and $y_i = g(x_i)$ for all $1 \leq i \leq k$.

For any such set, we will use its y-coordinates $y_1, y_2, \ldots, y_k$ to represent its graph and call $y_1 y_2 \ldots y_k$ (in that order) a (symbolic) representation for (the graph of) $y = g(x)$. For $1 \leq i < j \leq k$, we call $y_i y_{i+1} \ldots y_j$ the representation for $y = g(x)$ on $[x_i, x_j]$ obtained by restricting $y_1 y_2 \ldots y_k$ to $[x_i, x_j]$. For convenience, we will also call every y_i in $y_1 y_2 \ldots y_k$ a node. If $y_i = y_{i+1}$ for some i (i.e., g is constant on $[x_i, x_{i+1}]$), we will simply write

$$y_1 \cdots y_i y_{i+2} \cdots y_k$$

instead of

$$y_1 \cdots y_i y_{i+1} y_{i+2} \cdots y_k.$$

That is, we will delete y_{i+1} from the (symbolic) representation $y_1 y_2 \ldots y_k$. Therefore, every two consecutive nodes in a (symbolic) representation are distinct. Note that a continuous piecewise linear function obviously has more than one (symbolic) representation. However, as we will soon see that there is no need to worry about that.

Now assume that $\{(x_i, y_i) | i = 1, 2, \ldots, k\}$ is a set of nodes for $y = g(x)$ and $a_1 a_2 \ldots a_r$ is a representation for $y = g(x)$ with

$$\{a_1, a_2, \ldots, a_r\} \subset \{y_1, y_2, \ldots, y_k\}$$

and $a_i \neq a_{i+1}$ for all $1 \leq i \leq r - 1$. If

$$\{y_1, y_2, \ldots, y_k\} \subset \{x_1, x_2, \ldots, x_k\},$$

then there is an easy way to obtain a representation for $y = g^2(x)$ from the one $a_1 a_2 \ldots a_r$ for $y = g(x)$. The procedure is as follows: First, for any two distinct real numbers u and v, let $[u : v]$ denote the closed interval with endpoints u and v. Then let $b_{i,1} b_{i,2} \ldots b_{i,t_i}$ be the representation for $y = g(x)$ on $[a_i : a_{i+1}]$ which is obtained by restricting $a_1 a_2 \ldots a_r$ to $[a_i : a_{i+1}]$. We use the following notation to indicate this fact:

$$a_i a_{i+1} \to b_{i,1} b_{i,2} \cdots b_{i,t_i} \text{ (under } g\text{) if } a_i < a_{i+1},$$

or

$$a_i a_{i+1} \to b_{i,t_i} \cdots b_{i,2} b_{i,1} \text{ (under } g\text{) if } a_i > a_{i+1}.$$

The above representation on $[a_i : a_{i+1}]$ exists since

$$\{a_1, a_2, \ldots, a_r\} \subset \{x_1, x_2, \ldots, x_k\}.$$

Finally, if $a_i < a_{i+1}$, let $z_{i,j} = b_{i,j}$ for all $1 \leq j \leq t_i$. If $a_i > a_{i+1}$, let

$$z_{i,j} = b_{i,t_i+1-j} \text{ for all } 1 \leq j \leq t_i.$$

Let

$$Z = z_{1,1} \cdots z_{1,t_1} z_{2,2} \cdots z_{2,t_2} \cdots z_{r,2} \cdots z_{r,t_r}.$$

(Note that $z_{i,t_i} = z_{i+1,1}$ for all $1 \leq i \leq r - 1$.) Then it is easy to see that Z is a representation for $y = g^2(x)$. It is also obvious that the above procedure can be applied to the representation Z for $y = g^2(x)$ to obtain one for $y = g^3(x)$, and so on.

3. Proof of Theorem 3

In this section we fix an integer $n \geq 3$ and let $f_n(x)$ be the continuous function from the interval $[1, n]$ onto itself defined by

$$f_n(x) = x + 1 \quad \text{for } 1 \leq x \leq n - 1$$

and

$$f_n(x) = -(n - 1)x + n^2 - n + 1 \quad \text{for } n - 1 \leq x \leq n.$$

Using the notations introduced in Section 2, we have the following result.

Lemma 6: Under f_n, we have

$$k(k + 1) \to (k + 1)(k + 2),\ 2 \leq k \leq n - 2, \text{ if } n > 3,$$

$$(k + 1)k \to (k + 2)(k + 1),\ 2 \leq k \leq n - 2, \text{ if } n > 3,$$

$$(n - 1)n \to n(1),\ n(n - 1) \to (1)n,$$

$$n(1) \to (1)n(n - 1) \ldots 432,\ (1)n \to 234 \ldots (n - 1)n(1).$$

In the following, when we say the representation for $y = f_n^k(x)$, we mean the representation obtained, following the procedure as described in Section 2, by applying Lemma 6 to the representation $234 \ldots (n - 1)n(1)$ for $y = f_n(x)$ successively until we get to the one for $y = f_n^k(x)$.

For every positive integer k and all integers i, j with $1 \leq i$, $j \leq n - 1$, let $a_{k,i,j,n}$ denote the number of uv's and vu's in the representation for $y = f_n^k(x)$ whose corresponding x-coordinates are in the interval $[i, i + 1]$, where $uv = 1n$ if $j = 1$, and $uv = j(j + 1)$ if $2 \leq j \leq n - 1$. It is obvious that

$$a_{1,i,i+1,n} = 1 \text{ for all } 1 \leq i \leq n - 2,$$

$$a_{1,n-1,1,n} = 1, \text{ and } a_{1,i,j,n} = 0 \text{ elsewhere.}$$

From the above lemma, we find that these sequences $\langle a_{k,i,j,n} \rangle$ can be computed recursively.

Lemma 7: For every positive integer k and all integers i with $1 \leq i \leq n - 1$, we have

$$a_{k+1,i,1,n} = a_{k,i,1,n} + a_{k,i,n-1,n},$$

$$a_{k+1,i,2,n} = a_{k,i,1,n},$$

$$a_{k+1,i,j,n} = a_{k,i,1,n} + a_{k,i,j-1,n},\quad 3 \leq j \leq n - 1 \text{ if } n > 3.$$

It then follows from the above lemma that the sequences $\langle a_{k,i,j,n} \rangle$ can all be computed from the sequences $\langle a_{k,n-1,j,n} \rangle$.

Lemma 8: For every positive integer k and all integers j with $1 \leq j \leq n - 1$, we have

$$a_{k,n-1,j,n} = a_{k+i,n-1-i,j,n},\quad 1 \leq i \leq n - 2.$$

For every positive integer k, let

$$c_{k,n} = \sum_{i=1}^{n-1} a_{k,i,1,n} + \sum_{i=2}^{n-1} a_{k,i,i,n}.$$

Then it is easy to see that $c_{k,n}$ is exactly the number of distinct solutions of the equation $f_n^k(x) = x$ in the interval $[1, n]$. From the above lemma, we also have, for all $k \geq 1$, the identities:

$$c_{k,n} = \sum_{i=0}^{n-2} a_{k-i,\, n-1,\, 1,\, n} + \sum_{i=0}^{n-3} a_{k-i,\, n-1,\, n-1-i,\, n}$$

provided that $a_{m,n-1,\, j,\, n} = 0$ for all $m \le 0$ and $j > 0$. Since, for every positive integer k,

$$\begin{aligned}
a_{k,\, n-1,\, 1,\, n} &= a_{k-1,\, n-1,\, 1,\, n} + a_{k-1,\, n-1,\, n-1,\, n} \\
&= a_{k-1,\, n-1,\, 1,\, n} + a_{k-2,\, n-1,\, 1,\, n} + a_{k-2,\, n-1,\, n-2,\, n} \\
&= a_{k-1,\, n-1,\, 1,\, n} + a_{k-2,\, n-1,\, 1,\, n} + a_{k-3,\, n-1,\, 1,\, n} \\
&\qquad + a_{k-3,\, n-1,\, n-3,\, n} \\
&= \ldots \\
&= \sum_{i=1}^{n-1} a_{k-i,\, n-1,\, 1,\, n}
\end{aligned}$$

and

$$\begin{aligned}
c_{k,n} &= \sum_{i=0}^{n-2} a_{k-i,\, n-1,\, 1,\, n} + \sum_{i=0}^{n-3} a_{k-i,\, n-1,\, n-1-i,\, n} \\
&= a_{k,\, n-1,\, 1,\, n} + a_{k-1,\, n-1,\, 1,\, n} + \sum_{i=2}^{n-2} a_{k-i,\, n-1,\, 1,\, n} \\
&\qquad + a_{k-1,\, n-1,\, 1,\, n} + a_{k-1,\, n-1,\, n-2,\, n} \\
&\qquad + \sum_{i=1}^{n-3} a_{k-i,\, n-1,\, n-1-i,\, n} \\
&= a_{k,\, n-1,\, 1,\, n} + 2a_{k-1,\, n-1,\, 1,\, n} + \sum_{i=2}^{n-2} a_{k-i,\, n-1,\, 1,\, n} \\
&\qquad + 2a_{k-1,\, n-1,\, n-2,\, n} + \sum_{i=2}^{n-3} a_{k-i,\, n-1,\, n-1-i,\, n} \\
&= \ldots \\
&= \sum_{i=0}^{n-2} (i+1) a_{k-i,\, n-1,\, 1,\, n}
\end{aligned}$$

provided that $a_{m,n-1,\, 1,\, n} = 0$ if $m \le 0$, we obtain that $c_{k,n} = 2^k - 1$ for all $1 \le k \le n-1$ and

$$c_{k,n} = \sum_{i=1}^{n-1} c_{k-i,\, n} \text{ for all integers } k \ge n.$$

If, for every positive integer m, we let $\phi_n(m) = c_{m,n}$, then, by Theorem 1, we have $\Phi_1(m, \phi_n) \equiv 0 \pmod{m}$. The proof of the other statement of Theorem 3 is easy and omitted (see [3] and [4]). This completes the proof of Theorem 3.

References

1. Bau-Sen Du. "The Minimal Number of Periodic Orbits of Periods Guaranteed in Sharkovskii's Theorem." *Bull. Austral. Math. Soc.* *31* (1985):89-103.
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3. Hyman Gabai. "Generalized Fibonacci k-Sequences." *Fibonacci Quarterly 8.1* (1970):31-38.
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Announcement

FOURTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

Monday through Friday, July 30-August 3, 1990

Department of Mathematics and Computer Science

Wake Forest University

Winston-Salem, North Carolina 27109

International Committee

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CALL FOR PAPERS

The FOURTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS will take place at Wake Forest University, Winston-Salem, N.C., from July 30 to August 3, 1990. This conference is sponsored jointly by the Fibonacci Association and Wake Forest University.

Papers on all branches of mathematics and science related to the Fibonacci numbers as well as recurrences and their generalizations are welcome. Abstracts are to be submitted by March 15, 1990. Manuscripts are requested by May 1, 1990. Abstracts and manuscripts should be sent to G.E. Bergum (address below). Invited and contributed papers will appear in the Conference Proceedings, which are expected to be published.

The program for the Conference will be mailed to all participants, and to those individuals who have indicated an interest in attending the conference, by June 15, 1990. All talks should be limited to one hour or less.

For further information concerning the conference, please contact Gerald Bergum, *The Fibonacci Quarterly*, Department of Computer Science, South Dakota State University, P.O. Box 2201, Brookings, South Dakota 57007-0194.