METRIC THEORY OF PIERCE EXPANSIONS

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1. INTRODUCTION

It is well known that every real number admits an essentially unique expansion as a *continued fraction* in the form

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

where the a_i are positive integers (except for a_0, which may be negative or 0).

Many mathematicians have been interested in the length of such expressions; in particular, if $x = p/q$ is rational, the expansion terminates with a_n as the last *partial quotient*, and it is not difficult to show that

$$n = O(\log q).$$

See, for example, [14]. This type of result is of particular interest because continued fractions are closely linked to Euclid's algorithm to compute the greatest common divisor.

Another question that has received attention is how the a_i are related to x, in particular, by equating probabilities with Lebesgue measure, we can consider the $a_i = a_i(x)$ to be random variables, and ask:

1. How are the $a_i(x)$ distributed? What are the means and variances of these distributions?

2. Are the $a_i(x)$ independent, or "almost" independent? What does the distribution of $a_i(x)$ look like as $i \to \infty$?

We could also restate these questions in terms of iteration of an appropriate function. For example, if

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} \quad \text{and} \quad g(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor,$$

then it is easy to see that

$$g(x) = \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}$$

so that $g(x)$ may be viewed as a "shift" operator. Here $\lfloor x \rfloor$ is the greatest integer function.

This so-called "metric theory" of continued fractions has been studied extensively by Kuzmin [16], Lévy [17], Khintchine [12], and others.

We can ask similar questions of other algorithms for expressing real numbers. Engel's series

$$x = \frac{1}{a_1} + \frac{1}{a_1a_2} + \frac{1}{a_1a_2a_3} + \cdots$$

was investigated thoroughly by Erdös, Rényi, and Szüsz [7], and later by Rényi [21] and Deheuvels [5].

Cantor's product

$$x = \left(1 + \frac{1}{a_1}\right)\left(1 + \frac{1}{a_2}\right)\left(1 + \frac{1}{a_3}\right) \cdots$$

was investigated by Rényi [22].

There are also results for Sylvester's series [7] and other expansions of Cantor. For a summary of some of these results, see [9].

The subject of this paper is an expansion that has not received much attention; it is of the form

$$x = \frac{1}{a_1} - \frac{1}{a_1a_2} + \frac{1}{a_1a_2a_3} - \cdots \tag{1}$$

and is due to Pierce [19], who briefly examined its properties. Remez [20] attributes the expansion to M. V. Ostrogradskij and proves some elementary results. There are some metric theory results in [24], but they do not overlap with our results. We call an expansion of the form (1) a *Pierce expansion*, and in this paper we will demonstrate a connection between these expansions and Stirling numbers of the first kind. We obtain some new identities for Stirling numbers, and give a new derivation of a series for $\zeta(3)$. We discuss the distribution of the $a_i = a_i(x)$, and the behavior of the related function

$$f(x) = 1 \bmod x = 1 - x\lfloor 1/x \rfloor,$$

where by $a \bmod b$ we mean $a - b\lfloor a/b \rfloor$.

We also obtain some results on the lengths of finite Pierce expansions.

2. ELEMENTARY CONSIDERATIONS

In this section, we sketch some of the simple properties of Pierce expansions. The proofs are easy and all details are not given.

Any real number $x \in (0, 1]$ can be written uniquely in the form

$$x = \frac{1}{a_1} - \frac{1}{a_1a_2} + \frac{1}{a_1a_2a_3} - \cdots \tag{2}$$

where the a_i form a strictly increasing sequence of positive integers, and the expansion may or may not terminate. If the expansion does terminate with

$$\frac{(-1)^{n+1}}{a_1a_2 \cdots a_n}$$

as the last term, we impose the additional restriction

$$a_{n-1} < a_n - 1.$$

This is to ensure uniqueness, since we could write $1/k$ as

$$\frac{1}{k-1} - \frac{1}{(k-1)k}.$$

We will sometimes abbreviate the expansion (2) as

$$x = \{a_1, a_2, a_3, \ldots\}$$

where appropriate.

Given a real number x, we can obtain the terms of the Pierce expansion using the following algorithm:

[Pierce expansion algorithm]: Given a real number $x \in (0, 1]$, this algorithm produces the sequence of a_i such that $x = \{a_1, a_2, \ldots\}$.

P1. [Initialize]. Set $x_0 \leftarrow x$, set $i \leftarrow 1$.

P2. [Iterate]. Set $a_i \leftarrow \lfloor 1/x_{i-1} \rfloor$; set $x_i \leftarrow 1 - a_i x_{i-1}$.

P3. [All done?]. If $x_i = 0$, stop. Otherwise set $i \leftarrow i + 1$ and return to P2.

If we run this algorithm on the rational number $x = p/q$, it is easy to see that in step P2 we sill replace p by q mod p; this is less than p, and so eventually $x_i = 0$ and the algorithm terminates. On the other hand, if the algorithm terminates, we have

$$x = \{a_1, a_2, \ldots, a_n\}$$

and so x must be rational.

(This argument provides simple irrationality proofs for some numbers of interest. For example, using the Taylor series for e^x, $\sin x$, and $\cos x$, we find:

$$1 - e^{-1/a} = \{a, 2a, 3a, 4a, \ldots\},$$
$$\sin(1/a) = \{a, 6a^2, 20a^2, 42a^2, \ldots\},$$
$$\cos(1/a) = \{1, 2a^2, 12a^2, 30a^2, \ldots\}.$$

Since the expansions do not terminate, these functions take irrational values for any positive integer a.)

Now choose x uniformly from $(0, 1]$, and let $\Pr[X = c]$ be the probability that the random variable X equals c (thinking of probability as Lebesgue measure). Let

$$x = \{a_1, a_2, \ldots\}$$

be the Pierce expansion of x. Then

Theorem 1:

$$\Pr[a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n] = \frac{1}{b_1 b_2 \cdots b_n(b_n + 1)}$$

Proof: Let $b_1, \ldots, b_n$ be chosen. Now it is easy to see that the numbers whose expansions begin $\{b_1, b_2, \ldots, b_n\}$ form a half-open interval whose endpoints are the two numbers

$$x_1 = \{b_1, b_2, \ldots, b_{n-1}, b_n\}$$

and

$$x_2 = \{b_1, b_2, \ldots, b_{n-1}, b_n + 1\}.$$

(The first point is included, but the second is not.) The measure of this interval is just

$$|x_1 - x_2| = \frac{1}{b_1 b_2 \cdots b_n(b_n + 1)}$$

and the result follows.

Theorem 2:

$$\Pr[a_{n+1} = k | a_n = j] = \frac{j + 1}{k(k + 1)}$$

(Compare this with the result in [7] for Engel's series.)

Proof: To prove this, we show it is true for all x that have Pierce expansions that begin $\{b_1, b_2, \ldots, b_{n-1}, j\}$ where the b_i are specified constants. Then

$$\Pr[a_{n+1} = k | a_1 = b_1, \ldots, a_{n-1} = b_{n-1}, a_n = j]$$

$$= \frac{\Pr[a_1 = b_1, \ldots, a_{n-1} = b_{n-1}, a_n = j, a_{n+1} = k]}{\Pr[a_1 = b_1, \ldots, a_{n-1} = b_{n-1}, a_n = j]}$$

$$= \frac{b_1 b_2 \cdots b_{n-1} j(j + 1)}{b_1 b_2 \cdots b_{n-1} jk(k + 1)} = \frac{j + 1}{k(k + 1)}.$$

Now this conditional probability is the SAME for any specified prefix $b_1, \ldots, b_{n-1}$; hence, it is equal to

$$\frac{j + 1}{k(k + 1)}$$

if the b_i are left unspecified. In particular, the conditional probability in this theorem shows that the $a_i = a_i(x)$, considered as a sequence of random variables, form a homogeneous Markov chain.

Theorem 3:

$$\Pr[a_n = k] = \frac{\left[{k \atop n}\right]}{(k + 1)!}$$

where $\left[{k \atop n}\right]$ is a Stirling number of the first kind. See, e.g., [14] or [11].

Proof: By Theorem 1, we can compute the measure of the set of x whose Pierce expansions begin with a specified prefix. Let us fix $a_n = k$, and sum over all possible prefixes, i.e., all strictly increasing sequences of positive integers of length n whose largest element is k.

$$\Pr[a_n = k] = \sum_{1 < a_1 < \cdots < a_{n-1} < k} \frac{1}{a_1 a_2 \cdots a_{n-1} k(k + 1)}$$

(continued)

$$= \sum_{\substack{A \subset \{1, 2, \ldots, k-1\} \\ |A| = n-1}} \frac{1}{\Pi A} \cdot \frac{1}{k(k+1)} = \sum_{\substack{B \subset \{1, 2, \ldots, k-1\} \\ |B| = k-n}} \frac{\Pi B}{(k-1)!} \cdot \frac{1}{k(k+1)}$$

$$= \frac{1}{(k+1)!} \sum_{\substack{B \subset \{1, 2, \ldots, k-1\} \\ |B| = k-n}} \Pi B$$

and the proof is now complete if we observe that the sum over the product of elements of B is in fact the coefficient of x^n in the polynomial

$$x(x+1)(x+2) \cdots (x+k-1)$$

which is just $\begin{bmatrix} k \\ n \end{bmatrix}$, a Stirling number of the first kind.

(Some brief comments about the notation: in the proof above, A and B are sets. $|A|$ is the cardinality of A. The sum is over all subsets with specified cardinality, and ΠA means the product of all elements in A.)

We get two interesting corollaries: using a theorem of Jordan [11] we can estimate the distribution of the a_n. We have

$$\begin{bmatrix} k \\ n \end{bmatrix} \sim \frac{(k-1)!}{(n-1)!}(\log k + \gamma)^{n-1}$$

and so we get

$$\mathbf{Pr}[a_n = k] \sim \frac{(\log k + \gamma)^{n-1}}{k(k+1)(n-1)!}$$

where n is fixed and $k \to \infty$ and γ is Euler's constant. Compare this with the similar result of Békéssy [2] for Engel's series. More detailed asymptotic results can be obtained by using the results of Moser and Wyman [18].

Also, we observe that the events $a_n = 1$, $a_n = 2$, ... are all disjoint and exhaust the space of events. Therefore,

$$\sum_{k=0}^{\infty} \frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k+1)!} = 1, \tag{3}$$

which is another derivation of the formula due to Jordan [11, p. 165].

In the next section, we derive some results on series involving Stirling numbers.

3. IDENTITIES ON STIRLING NUMBERS

Theorem 4:

$$\sum_{j=1}^{\infty} \frac{\begin{bmatrix} j \\ n \end{bmatrix}}{j \cdot j!} = \zeta(n+1)$$

where $\zeta(k)$ is Riemann's zeta function.

Proof: This is a result due to Jordan [11, pp. 164, 194, 339].

Theorem 5:

$$\sum_{k=j}^{\infty} \frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k+1)!} = 1 - \sum_{k=0}^{j-1} \frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k+1)!} = \frac{1}{j!} \sum_{i=1}^{n} \begin{bmatrix} j \\ i \end{bmatrix} \quad \text{for } n \geqslant 1,\ j \geqslant 1.$$

Proof: The proof of the first equality is just formula (3) above. To verify the second, we use induction on j, holding n fixed. It is easy to verify the case $j = 1$. Now assume true for j; we show the identity holds for $j + 1$. We have

$$1 - \sum_{k=0}^{j-1} \frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k+1)!} = \frac{1}{j!} \sum_{i=1}^{n} \begin{bmatrix} j \\ i \end{bmatrix}.$$

Now subtract $\dfrac{\begin{bmatrix} j \\ n \end{bmatrix}}{(j+1)!}$ from both sides to get:

$$\begin{aligned} 1 - \sum_{k=0}^{j} \frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k+1)!} &= \left(\frac{1}{j!} \sum_{i=1}^{n} \begin{bmatrix} j \\ i \end{bmatrix}\right) - \frac{\begin{bmatrix} j \\ n \end{bmatrix}}{(j+1)!} \\ &= \left(\frac{1}{(j+1)!} \sum_{i=1}^{n} (j+1)\begin{bmatrix} j \\ i \end{bmatrix}\right) - \frac{\begin{bmatrix} j \\ n \end{bmatrix}}{(j+1)!} \\ &= \left(\frac{1}{(j+1)!} \sum_{i=1}^{n} j\begin{bmatrix} j \\ i \end{bmatrix}\right) + \left(\frac{1}{(j+1)!} \sum_{i=1}^{n} \begin{bmatrix} j \\ i \end{bmatrix}\right) - \frac{\begin{bmatrix} j \\ n \end{bmatrix}}{(j+1)!} \\ &= \frac{1}{(j+1)!} \sum_{i=1}^{n} \left(\begin{bmatrix} j+1 \\ i \end{bmatrix} - \begin{bmatrix} j \\ i-1 \end{bmatrix} + \begin{bmatrix} j \\ i \end{bmatrix}\right) - \frac{\begin{bmatrix} j \\ n \end{bmatrix}}{(j+1)!} \\ &= \frac{1}{(j+1)!} \sum_{i=1}^{n} \begin{bmatrix} j+1 \\ i \end{bmatrix}, \end{aligned}$$

where we have used telescoping cancellation and the well-known identity on Stirling numbers

$$j\begin{bmatrix} j \\ i \end{bmatrix} = \begin{bmatrix} j+1 \\ i \end{bmatrix} - \begin{bmatrix} j \\ i-1 \end{bmatrix}.$$

This completes the proof of Theorem 5. This is apparently a new identity on Stirling numbers.

Michael Luby made the following clever observation (personal communication): It is possible to prove Theorem 5 without the use of induction, by interpreting the left and right sides combinatorially, in terms of the a_n. The left side, in fact, is just

$$\Pr[a_n \geqslant j]$$

while the right side can be shown to be

$$\Pr[(a_1 \geqslant j) \text{ or } (a_1 < j \text{ and } a_2 \geqslant j) \text{ or } (a_1,\ a_2 < j \text{ and } a_3 \geqslant j) \cdots].$$

Theorem 6:

$$\sum_{k=1}^{\infty} H(k)\frac{\begin{bmatrix}k\\n\end{bmatrix}}{(k+1)!} = \zeta(2) + \zeta(3) + \cdots + \zeta(n+1)$$

where $H(k)$ is the k^{th} harmonic number,

$$H(k) = 1 + \frac{1}{2} + \cdots + \frac{1}{k}.$$

Proof:

$$\sum_{k=1}^{\infty} H(k)\frac{\begin{bmatrix}k\\n\end{bmatrix}}{(k+1)!} = \sum_{k=1}^{\infty} \frac{\begin{bmatrix}k\\n\end{bmatrix}}{(k+1)!}\sum_{j=1}^{k}\frac{1}{j} = \sum_{j=1}^{\infty}\frac{1}{j}\sum_{k=j}^{\infty}\frac{\begin{bmatrix}k\\n\end{bmatrix}}{(k+1)!}$$

$$= \sum_{j=1}^{\infty}\frac{1}{j\cdot j!}\sum_{i=1}^{n}\begin{bmatrix}j\\i\end{bmatrix} = \sum_{i=1}^{n}\sum_{j=1}^{\infty}\frac{\begin{bmatrix}j\\i\end{bmatrix}}{j\cdot j!} = \sum_{i=1}^{n}\zeta(i+1),$$

where we have used Theorems 4 and 5.

The author would like to express his thanks to Richard Fateman and the Vaxima version of the MACSYMA computer algebra system—an early version of Theorem 6 was suggested by experimentation with Vaxima!

Theorem 7:

$$\sum_{k=1}^{\infty}\frac{1}{k}\cdot\frac{\begin{bmatrix}k\\n\end{bmatrix}}{(k+1)!} = \zeta(n+1) - 1$$

Proof: See [11, p. 339].

We can now give a new derivation of a formula for $\zeta(3)$ due to Briggs et al. [3]. Noting that

$$\begin{bmatrix}k\\2\end{bmatrix} = H(k-1)(k-1)!$$

we get

$$\zeta(3) = \sum_{k=1}^{\infty}\left(1+\frac{1}{k}\right)\frac{\begin{bmatrix}k\\2\end{bmatrix}}{(k+1)!} = \sum_{k=1}^{\infty}\frac{H(k-1)}{k^2}$$

or, adding $\sum_{k=1}^{\infty}\frac{1}{k^3}$ to both sides, we get

$$2\zeta(3) = \sum_{k=1}^{\infty}\frac{H(k)}{k^2}.$$

Many similar formulas can be given; for example, by appealing to Theorem 6, we can obtain

$$\sum_{k=1}^{\infty}\frac{H(k)(H(k-1)-1)}{k(k+1)} = \zeta(3).$$

See also [4], [10], [13], and [23].

Theorem 8:

$$\sum_{k=1}^{\infty} H(k+1)\frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k+1)!} = n + 1$$

Proof:

$$\sum_{k=1}^{\infty} H(k+1)\frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k+1)!} = \sum_{k=1}^{\infty} \sum_{j=1}^{k+1} \frac{1}{j} \cdot \frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k+1)!}$$

$$= \sum_{k=1}^{\infty} \frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k+1)!} + \sum_{j=2}^{\infty} \sum_{k=j-1}^{\infty} \frac{1}{j} \cdot \frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k+1)!}$$

$$= 1 + \sum_{j=2}^{\infty} \frac{1}{j} \sum_{i=1}^{n} \frac{\begin{bmatrix} j-1 \\ i \end{bmatrix}}{(j-1)!}$$

$$= 1 + \sum_{i=1}^{n} \sum_{j=1}^{\infty} \frac{\begin{bmatrix} j \\ i \end{bmatrix}}{(j+1)!} = n + 1$$

In the next section, we use these identities on Stirling numbers to derive estimates for the expected value and variance of quantities connected with a_n.

4. EXPECTED VALUES AND VARIANCES

We will use $\mathbf{E}[X]$ and $\mathbf{Var}[X]$ for the expected value and variance of the random variable X.

We are interested in how the a_n are distributed. However, the a_n are distributed such that $\mathbf{E}[a_n] = \infty$ for every n. It is reasonable to expect that the quantity $\log a_n$ rather than a_n gives more information.

Theorem 9:

(a) $\mathbf{E}[H(a_n)] = \zeta(2) + \zeta(3) + \cdots + \zeta(n+1)$

(b) $\mathbf{E}[\log a_n] = n + 1 - \gamma + O(2^{-n})$

Proof:

(a) $\mathbf{E}[H(a_n)] = \sum_{k=1}^{\infty} H(k)\frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k+1)!} = \zeta(2) + \zeta(3) + \cdots + \zeta(n+1)$

using Theorem 6.

(b) To prove part (b) we use the famous estimate

$$H(k) = \log k + \gamma + O\left(\frac{1}{k}\right),$$

and therefore, using Theorems 6 and 7,

$$\mathbf{E}[\log a_n] = \mathbf{E}[H(a_n)] - \gamma + O\left(\mathbf{E}\left[\frac{1}{a_n}\right]\right)$$

$$= \zeta(2) + \zeta(3) + \cdots + \zeta(n+1) - \gamma + O(\zeta(n+1) - 1).$$

Now it is easily shown that

$$\zeta(2) + \zeta(3) + \cdots + \zeta(k) = k + \theta(2^{-k}) \tag{4}$$

and

$$\zeta(k) = 1 + \theta(2^{-k});$$

so, by substitution, we obtain the desired result.

Similar techniques allow us to calculate the variance.

Theorem 10:

(a) $\mathrm{Var}[H(a_n)] = n + O(1)$

(b) $\mathrm{Var}[\log a_n] \doteq n + O(1)$

Proof: We find first that

$$\text{(a)}\quad \mathrm{E}[H(a_n)^2] = \sum_{k=1}^{\infty} H(k)^2 \frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k+1)!} = \sum_{k=1}^{\infty} \left(\sum_{j=1}^{k} \frac{2H(j-1)}{j} + \frac{1}{j^2} \right) \frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k+1)!}$$

$$= \sum_{j=1}^{\infty} \left(\frac{2H(j-1)}{j} + \frac{1}{j^2} \right) \sum_{k=j}^{\infty} \frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k+1)!}$$

$$= \sum_{j=1}^{\infty} \left(\frac{2H(j-1)}{j} + \frac{1}{j^2} \right) \frac{1}{j!} \sum_{i=1}^{n} \begin{bmatrix} j \\ i \end{bmatrix}, \tag{5}$$

where we have used the fact that

$$H(j)^2 = H(j-1)^2 + \frac{2H(j-1)}{j} + \frac{1}{j^2}$$

and Theorem 5. Note that $H(0) = 0$ by definition.

On the other hand, we have already seen that

$$\sum_{k=1}^{\infty} \frac{H(k)}{k+1} \frac{\begin{bmatrix} k \\ n \end{bmatrix}}{k!} = \sum_{k=1}^{\infty} H(k) \frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k+1)!} = n + 1 + O(2^{-n+1}),$$

and therefore,

$$\sum_{k=1}^{\infty} \frac{H(k)}{k+1} \frac{1}{k!} \sum_{i=1}^{n} \begin{bmatrix} k \\ i \end{bmatrix} = \sum_{i=1}^{n} (i + 1 + O(2^{-i+1})) = \frac{n^2 + 3n}{2} + O(1).$$

Hence, we find

$$\sum_{j=1}^{\infty} \frac{2H(j)}{j+1} \frac{1}{j!} \sum_{i=1}^{n} \begin{bmatrix} j \\ i \end{bmatrix} = n^2 + 3n + O(1).$$

The left side of this equation looks very much like the right side of equation (5). In fact, it is easy to show that their difference is bounded by a constant that is independent of n. We have

$$\sum_{j=1}^{\infty} \left(\frac{2H(j-1)}{j} + \frac{1}{j^2} - \frac{2H(j)}{j+1} \right) \frac{1}{j!} \sum_{i=1}^{n} \begin{bmatrix} j \\ i \end{bmatrix} \leqslant \sum_{j=1}^{\infty} \left(\frac{2H(j-1)}{j} + \frac{1}{j^2} - \frac{2H(j)}{j+1} \right) \tag{6}$$

since

$$\sum_{i=1}^{n} \begin{bmatrix} j \\ i \end{bmatrix} \leqslant j!.$$

Now the sum on the right side of (6) can be computed exactly:

$$\begin{aligned}\sum_{j=1}^{\infty}\left(\frac{2H(j-1)}{j} + \frac{1}{j^2} - \frac{2H(j)}{j+1}\right) &= \sum_{j=1}^{\infty}\left(\frac{2H(j-1)}{j} + \frac{1}{j^2} - \frac{2H(j-1)}{j+1} - \frac{2}{j(j+1)}\right) \\ &= \sum_{j=1}^{\infty}\left(\frac{2H(j-1)}{j(j+1)} + \frac{1}{j^2} - \frac{2}{j(j+1)}\right) \\ &= \zeta(2) = O(1).\end{aligned}$$

Thus, we conclude that

$$\mathsf{E}[H(a_n)^2] = n^2 + 3n + O(1).$$

On the other hand, from Theorem 9, we see that

$$\mathsf{E}^2[H(a_n)] = n^2 + 2n + 1 + O(n2^{-n})$$

and therefore,

$$\mathrm{Var}[H(a_n)] = n + O(1)$$

which is the desired result.

(b) To prove part (b), we use the fact that

$$H(k) = \log k + O(1)$$

to get

$$\mathrm{Var}[\log a_n] = \mathrm{Var}[H(a_n)] + O(\mathrm{Var}[1]) = n + O(1).$$

This completes the proof.

In a similar fashion, we can obtain theorems about the expected values of various functions of the a_n. We give some unusual examples.

Let $f(x) = 1 \bmod x = 1 - x\lfloor 1/x \rfloor$. Then it is easy to see that if

$$x = \{a_1, a_2, \ldots\}$$

then

$$f(x) = \{a_2, a_3, \ldots\}.$$

Let us write $f^{(2)}(x) = f(f(x))$, etc. Then we have

Theorem 11:

$$\mathsf{E}[f^{(n)}(x)] = \frac{1}{2}(n + 1 - \zeta(2) - \zeta(3) - \cdots - \zeta(n+1)) = \theta(2^{-n+2})$$

Proof: Suppose $a_n = k$. What is the expected value of $f^{(n)}(x)$? If we restrict our attention to the half-open interval that contains all numbers whose Pierce expansions begin

$$\{a_1,\ a_2,\ \ldots,\ a_{n-1},\ k\},$$

then it is easily seen that $f^{(n)}(x)$ is linear on this interval. The minimum and maximum values that $f^{(n)}(x)$ attains are 0 and $1/(k+1)$ respectively; hence the expected value of $f^{(n)}(x)$ on this specified interval is $1/[2(k+1)]$. But this is independent of the choice of $a_1,\ a_2,\ \ldots,\ a_{n-1}$; hence the expected value of $f^{(n)}(x)$ given that $a_n = k$ is $1/[2(k+1)]$. Therefore,

$$\begin{aligned}\mathrm{E}[f^{(n)}(x)] &= \sum_{k=1}^{\infty}\frac{1}{2(k+1)}\,\frac{\begin{bmatrix}k\\n\end{bmatrix}}{(k+1)!} = \frac{1}{2}\left(\sum_{k=1}^{\infty}(H(k+1)-H(k))\frac{\begin{bmatrix}k\\n\end{bmatrix}}{(k+1)!}\right)\\ &= \frac{1}{2}\left(\sum_{k=1}^{\infty}H(k+1)\frac{\begin{bmatrix}k\\n\end{bmatrix}}{(k+1)!} - \sum_{k=1}^{\infty}H(k)\frac{\begin{bmatrix}k\\n\end{bmatrix}}{(k+1)!}\right)\\ &= \frac{1}{2}(n+1-\zeta(2)-\zeta(3)-\cdots-\zeta(n+1)),\end{aligned}$$

where we have used Theorems 7 and 8.

From equation (4), this quantity is $\theta(2^{-n+2})$, and the proof is complete.

It is of some interest to note that Theorem 11 is a generalization of a result of Dirichlet [6]. He stated that

$$\sum_{k=1}^{n} k\left\lfloor\frac{n}{k}\right\rfloor \sim \frac{\pi^2 n^2}{12}.$$

We can derive this easily. From Theorem 11, we have

$$\begin{aligned}\frac{n}{2}(2-\zeta(2)) &= n\int_0^1 1 \bmod x\, dx = \int_0^1 n \bmod nx\, dx = \frac{1}{n}\int_0^n n \bmod x\, dx\\ &= \frac{1}{n}\int_0^n n - x\lfloor n/x\rfloor dx = n - \frac{1}{n}\int_0^n x\lfloor n/x\rfloor dx,\end{aligned}$$

and we get the desired result by approximating the integral with a sum.

Theorem 12:

$$\sum_{k=1}^{\infty}\frac{1}{a_k}$$

converges for almost all x (i.e., for all but a set of measure 0). The expected value of the sum is 1. The set of exceptions

$$\left\{x\ \middle|\ \sum_{k=1}^{\infty}\frac{1}{a_k}\text{ diverges}\right\}$$

is uncountable and dense.

Proof: From Theorem 7, we have

$$\mathrm{E}\left[\frac{1}{a_n}\right] = \zeta(n+1) - 1 < 2^{1-n},$$

and it is easily seen that the variance $\mathrm{Var}\left[\frac{1}{a_n}\right]$ is also $< 2^{1-n}$.

Then, by Chebyshev's inequality,

$$\Pr\left[\frac{1}{a_n} - 2^{1-n} \geqslant 2^{-n/4}\right] \leqslant \frac{2^{1-n}}{2^{-n/2}}$$

Now, by the Borel-Cantelli lemma, with probability 1 only finitely many of the events

$$\frac{1}{a_n} - 2^{1-n} \geqslant 2^{-n/4}$$

occur, and so the series converges almost everywhere.

We also have

$$E\left[\sum_{k=1}^{\infty} \frac{1}{a_n}\right] = \sum_{k=1}^{\infty} (\zeta(k+1) - 1) = 1.$$

(See, e.g., [11, p. 340].) This proves the result on the expected value.

Now we show that the set of exceptions is uncountable. Let the real number x in the interval $(0, 1)$ be written in base two notation,

$$x = e_1 e_2 e_3 \ldots,$$

where each $e_i = 1$ or 0. Then associate with each such x the real number whose Pierce expansion is given by

$$h(x) = \{1 + e_1,\ 3 + e_2,\ 5 + e_3,\ \ldots\}.$$

Then each of these numbers $h(x)$ is distinct by the uniqueness of Pierce expansions, and for each $h(x)$ we have

$$\sum_{k=1}^{n} \frac{1}{a_k} \geqslant \sum_{k=1}^{n} \frac{1}{2k}$$

and so the series diverges.

The proof that the set of exceptions is dense is left to the reader.

Theorem 13:

$$\sum_{k=1}^{\infty} f^{(k)}(x)$$

converges for almost all x. The expected value of the sum is $\frac{\pi^2}{12} - \frac{1}{2}$. The set of exceptions

$$\left\{x \,\middle|\, \sum_{k=1}^{\infty} f^{(k)}(x) \text{ diverges}\right\}$$

is uncountable and dense.

Proof: We prove only the result on the expected value, leaving the rest to the reader.

$$E\left[\sum_{k=1}^{\infty} f^{(k)}(x)\right] = \sum_{k=1}^{\infty} \frac{1}{2}\left(k + 1 - \sum_{j=2}^{k+1} \zeta(j)\right) = \frac{1}{2} \sum_{k=1}^{\infty} \left(1 - \sum_{j=2}^{k+1} (\zeta(j) - 1)\right)$$

(continued)

$$= \frac{1}{2}\sum_{k=1}^{\infty}\left(1 - \sum_{j=2}^{k+1}\sum_{i=2}^{\infty}\frac{1}{i^j}\right) = \frac{1}{2}\sum_{k=1}^{\infty}\left(1 - \sum_{i=2}^{\infty}\frac{1/i - 1/i^{k+1}}{i - 1}\right)$$

$$= \frac{1}{2}\sum_{k=1}^{\infty}\sum_{i=2}^{\infty}\frac{1}{(i - 1)i^{k+1}} = \frac{1}{2}\sum_{i=2}^{\infty}\frac{1}{i - 1}\sum_{k=2}^{\infty}\frac{1}{i^{k+1}} = \frac{1}{2}\sum_{i=2}^{\infty}\frac{1}{i(i - 1)^2}$$

$$= \frac{1}{2}(\zeta(2) - 1),$$

which is the desired result.

5. DISTRIBUTION OF THE a_n: METHOD OF RÉNYI

So far we have shown that log a_n has an expected value that tends to $n + 1 - \gamma$ as n approaches ∞. We have also seen that the variance is small. In fact, it is possible to prove much stronger results; for example, that

$$\lim_{n\to\infty} a^{1/n} = e$$

for almost all x. We will use a method employed by Rényi in his analysis of Engel's series [21]. We start by identifying some new random variables and we show they are independent.

Define

$$\varepsilon_k(x) = \begin{cases} 1 & \text{if } k \text{ appears in the Pierce expansion of } x, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

Theorem 14: $\mathrm{E}[\varepsilon_k(x)] = \dfrac{1}{k + 1}$

Proof:

$$\mathrm{E}[\varepsilon_k(x)] = \sum_{n=1}^{\infty}\frac{\begin{bmatrix}k\\n\end{bmatrix}}{(k + 1)!} = \frac{1}{(k + 1)!}\sum_{n=1}^{\infty}\begin{bmatrix}k\\n\end{bmatrix} = \frac{k!}{(k + 1)!} = \frac{1}{k + 1},$$

since the events $a_i = k$ and $a_j = k$ are disjoint if $i \neq j$.

Theorem 15: The random variables $\varepsilon_k(x)$ are independent.

Proof: Let

$$\varepsilon_1 = \delta_1, \ \varepsilon_2 = \delta_2, \ \ldots, \ \varepsilon_n = \delta_n$$

represent an assignment of 0's and 1's for the values of ε_i. Let b_i $(1 \leqslant i \leqslant k)$ be such that $\delta_{b_i} = 1$ and all other values of δ_j are 0. Without loss of generality, assume that $\delta_n = 1$. Then the probability that the events

$$\varepsilon_1 = \delta_1, \ \varepsilon_1 = \delta_2, \ \ldots, \ \varepsilon_n = \delta_n$$

simultaneously occur is just the probability that the Pierce expansion for x begins $b_1, b_2, \ldots, b_k$, which we have seen is equal to

$$\frac{1}{b_1 b_2 \cdots b_k(b_k + 1)}.$$

On the other hand, we have

$$\mathrm{Pr}[\varepsilon_i = \delta_i] = \begin{cases} \dfrac{i}{i+1} & \text{if } \delta_i = 0, \\[2ex] \dfrac{1}{i+1} & \text{if } \delta_i = 1. \end{cases}$$

Let us compute

$$\prod_{i=1}^{n} \mathrm{Pr}[\varepsilon_i = \delta_i]. \tag{7}$$

In the numerator of (7) we have those i corresponding to the δ_i that equal 0; in the denominator we have $(n + 1)!$. By canceling in the numerator and denominator, we see that the value of the product (7) is just

$$\frac{1}{b_1 b_2 \cdots b_k (b_k + 1)},$$

which shows the independence of the ε_i.

It is also easy to see that

$$\mathrm{Var}[\varepsilon_k] = \frac{1}{k+1} - \frac{1}{(k+1)^2}$$

Now, let $\mu_N = \mu_N(x)$ denote the number of terms of the sequence $a_n = a_n(x)$ that are $\leqslant N$. In other words, put

$$\mu_N = \sum_{k=1}^{N} \varepsilon_k.$$

Then we see immediately that

$$\mathrm{E}[\mu_N] = \sum_{k=1}^{N} \frac{1}{k+1} = \log N + \gamma - 1 + O\left(\frac{1}{N}\right)$$

and

$$\mathrm{Var}[\mu_N] = \sum_{k=1}^{N} \left(\frac{1}{k+1} - \frac{1}{(k+1)^2}\right) = \log N + \gamma - \frac{\pi^2}{6} + O\left(\frac{1}{N}\right).$$

We can prove the strong law of large numbers for the random variables ε_k. We need the following general form of this law [21]:

If $\xi_1, \xi_2, \ldots$ are independent nonnegative random variables with finite expectation $E_k = \mathrm{E}[\xi_k]$ and variance $V_k = \mathrm{Var}[\xi_k]$ and if putting

$$A_N = \sum_{k=1}^{N} E_k$$

one has

$$\lim_{N \to \infty} A_N = \infty$$

and also

$$\sum_{N=1}^{\infty} \frac{V_N}{A_N^2} < \infty,$$

then with probability 1 we have

$$\lim_{N \to \infty} \frac{\sum_{k=1}^{N} \xi_k}{A_N} = 1.$$

The conditions of this theorem are fulfilled for $\xi_k = \varepsilon_k$, since

$$\sum_{N=1}^{\infty} \frac{\frac{1}{N+1} - \frac{1}{N+1^2}}{(\log N + \gamma - 1)^2}$$

converges by comparison with the integral

$$\int \frac{1}{x(\log x)^2}\, dx = \frac{-1}{\log x}.$$

Thus we obtain

Theorem 16: For almost all x we have

$$\lim_{N \to \infty} \frac{\mu_N}{\log N + \gamma - 1} = 1$$

Using $\mu_{a_n} = n$, we obtain

$$\lim_{n \to \infty} a^{1/n} = e$$

for almost all x.

[We can easily get a similar result for iterates of $f(x) = 1 \bmod x$. Since

$$\frac{1}{1 + a_{n+1}} < f^{(n)}(x) \leqslant \frac{1}{a_{n+1}},$$

we find

$$\lim_{n \to \infty} (f^{(n)}(x))^{1/n} = \frac{1}{e}$$

for almost all x.]

We can use Ljapunov's condition [8] to obtain a central limit theorem for the a_n. We have

$$\mathbf{E}[\varepsilon_k^3] = \frac{1}{k+1}$$

and

$$\frac{\mathbf{E}[\varepsilon_k^3]}{\mathbf{Var}[\mu_k]}$$

is bounded. Also $\sqrt{\mathbf{Var}[\mu_k]} \to \infty$. Hence, we find

Theorem 17:

$$\lim_{N \to \infty} \mathbf{Pr}\left[\frac{\mu_N - \log N}{\sqrt{\log N}} < y\right] = \Phi(y),$$

where $\Phi(y)$ is the normal distribution given by $\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-u^2/2}\, du.$

Now, noting that

$$\Pr[\mu_N < n] = \Pr[a_n > N],$$

we see that an equivalent statement of Theorem 17 is

$$\lim_{n\to\infty} \Pr\left[\frac{\log a_n - n}{\sqrt{n}} < y\right] = \Phi(y).$$

As a corollary, we get

$$\lim_{n\to\infty} \sum_{k=e^{n+\alpha\sqrt{n}}}^{k=e^{n+\beta\sqrt{n}}} \frac{\begin{bmatrix} k \\ n \end{bmatrix}}{(k+1)!} = \Phi(\beta) - \Phi(\alpha).$$

This is similar to the result

$$\lim_{n\to\infty} \sum_{k=\log n+\alpha\sqrt{\log n}}^{k=\log n+\beta\sqrt{\log n}} \frac{\begin{bmatrix} n \\ k \end{bmatrix}}{n!} = \Phi(\beta) - \Phi(\alpha)$$

given in [8].

Similarly, as the conditions given by Kolmogoroff [15] for the law of the iterated logarithm are fulfilled for the variables ε_k, we get

Theorem 18: For almost all x,

$$\limsup_{N\to\infty} \frac{\mu_N - \log N}{\sqrt{2 \log N \cdot \log\log\log N}} = 1$$

and

$$\liminf_{N\to\infty} \frac{\mu_N - \log N}{\sqrt{2 \log N \cdot \log\log\log N}} = -1$$

or, stated equivalently,

$$\limsup_{n\to\infty} \frac{\log a_n - n}{\sqrt{2n \cdot \log\log n}} = 1$$

and

$$\liminf_{n\to\infty} \frac{\log a_n - n}{\sqrt{2n \cdot \log\log n}} = -1.$$

6. SOME RESULTS ON FINITE PIERCE EXPANSIONS

In [7], Erdös et al. put $E_1(a, b) = n$, where

$$\frac{a}{b} = \frac{1}{q_1} + \frac{1}{q_1 q_2} + \cdots + \frac{1}{q_1 q_2 \cdots q_n}$$

(an expansion into Engel's series) and ask for a nontrivial estimation of $E_1(a, b)$.

We prove two results on the length of finite Pierce expansions. Unfortunately, it does not seem possible to use our techniques for Engel's series.

Let us put $L(p, q) = n$, where

$$\frac{p}{q} = \frac{1}{a_1} - \frac{1}{a_1 a_2} + \cdots + \frac{(-1)^{n+1}}{a_1 a_2 \cdots a_n}.$$

Then we have

Theorem 19: $L(p, q) < 2\sqrt{q}$.

Proof: Let us write

$$\frac{p}{q} = \{a_1, a_2, \ldots, a_n\}$$

and, as in the Pierce Expansion Algorithm, put $p_1 = p$ and

$$a_i = \lfloor q/p_i \rfloor,$$

$$p_{i+1} = q - a_i p_i.$$

Without loss of generality, we may assume that $a_1 = 1$. For otherwise we have

$$\frac{q - p}{q} = \{1, a_1, a_2, \ldots, a_n\},$$

which is a longer Pierce expansion.

Then suppose $p_n \geqslant a_n$. Since $a_n p_n = q$, we have $a_n \leqslant \sqrt{q}$. But the a_i are strictly increasing, so $n \leqslant \sqrt{q}$.

Now suppose $p_n < a_n$. Since the p_i are strictly decreasing, and the a_i are strictly increasing, we see that $p_i - a_i$ is a strictly decreasing sequence. But $p_1 - a_1 \geqslant 0$ since $a_1 = 1$, and $p_n - a_n < 0$ by hypothesis. Hence, there must be a unique subscript k such that

$$p_k - a_k \geqslant 0$$

but

$$p_{k+1} - a_{k+1} < 0.$$

Then, since $p_i a_i \leqslant q$ for all i, we see that

$$a_k \leqslant \sqrt{q} \quad \text{and} \quad p_{k+1} < \sqrt{q}.$$

By the monotonicity of these sequences, we see that $k \leqslant \sqrt{q}$ and $n - k < \sqrt{q}$. We add these inequalities to get $n < 2\sqrt{q}$, which is the desired result.

Unfortunately, this bound is not very tight. For example,

$$\frac{470}{743} = \{1, 2, 3, 4, 5, 10, 11, 14, 17, 61, 67, 123, 148, 247, 371, 743\}.$$

This is the longest Pierce expansion with $q < 1000$. We see that $n = 16$, but our estimate guarantees just $n < 54$.

It seems likely that $L(p, q) = O(\log q)$; we cannot expect a much better lower bound. For example, we have the following theorem.

Theorem 20: There exist infinitely many q with $L(p, q) > \dfrac{\log q}{\log \log q}$.

Proof: The proof is constructive. Let $q = n!$, and set

$$p = n!\left(1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + \frac{(-1)^{n+1}}{n!}\right).$$

Then we have

$$\frac{p}{q} = \{1, 2, 3, \ldots, n - 3, n - 2, n\},$$

and therefore, $L(p, q) = n - 1$.

However, it is easily shown that, for n sufficiently large,

$$n - 1 > \frac{\log n!}{\log \log n!}$$

and the desired result easily follows.

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