$$\frac{L_m - L_n\sqrt{5}}{2} = [b_0 - a_r - 1, 1, 1, \dot{a}_2, \ldots, a_r, \dot{a}_1].$$

(*d*) If $3 \nmid m$, then

$$\frac{F_m - F_1\sqrt{5}}{2} = \left[\frac{F_m - 3}{2}, 2, \dot{1}\right]$$

and

$$\frac{L_m - F_1\sqrt{5}}{2} = \left[\frac{L_m - 3}{2}, 2, \dot{1}\right].$$

If $3 | m$, then

$$\frac{F_m - F_1\sqrt{5}}{2} = \frac{F_m - F_2\sqrt{5}}{2} = \left[\frac{F_m - 4}{2}, 1, 7, \dot{2}, \dot{8}\right]$$

and

$$\frac{L_m - F_1\sqrt{5}}{2} = \frac{F_m - F_2\sqrt{5}}{2} = \left[\frac{L_m - 4}{2}, 1, 7, \dot{2}, \dot{8}\right].$$

REFERENCES

1. G. H. Hardy & E. M. Wright. *An Introduction to the Theory of Numbers*. London: Oxford University Press, 1954.
2. C. T. Long & J. H. Jordan. "A Limited Arithmetic on Simple Continued Fractions." *The Fibonacci Quarterly* 5 (1967):113-128.
3. C. T. Long & J. H. Jordan. "A Limited Arithmetic on Simple Continued Fractions—II." *The Fibonacci Quarterly* 8 (1970):135-157.
4. C. D. Olds. *Continued Fractions*. New York: Random House, 1963.

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BENFORD'S LAW FOR FIBONACCI AND LUCAS NUMBERS

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Benford's law states that the probability that a random decimal begins (on the left) with the digit p is $\log_{10}(p + 1)/p$. Recent computations by J. Wlodarski [3] and W. G. Brady [1] show that the Fibonacci and Lucas numbers tend to obey both this law and its natural extension: the probability that a random decimal in base b begins with p is $\log_b(p + 1)/p$. By using the fact that the terms of the Fibonacci and Lucas sequences have exponential growth, we prove the following result.

Theorem: The Fibonacci and Lucas numbers obey the extended Benford's law. More precisely, let $b \geq 2$ and let p satisfy $1 \leq p \leq b - 1$. Let $A_p(N)$ be the number of Fibonacci (or Lucas) numbers F_n (or L_n) with $n \leq N$ and whose first digit in base b is p. Then

$$\lim_{N \to \infty} \frac{1}{N} A_p(N) = \log_b\left(\frac{p + 1}{p}\right).$$

Proof: We give the proof for the Fibonacci sequence. The proof for the Lucas sequence is similar.

Throughout the proof, log will mean $\log_b$. Also, $\langle x \rangle = x - [x]$ will denote the fractional part of x.

Let $\alpha = \frac{1}{2}(1 + \sqrt{5})$, so $F_n = (\alpha^n - (-\alpha)^{-n})/\sqrt{5}$. We first need the following:

Lemma: The sequence $\{\langle n \log \alpha \rangle\}_{n=1}^{\infty}$ is uniformly distributed mod 1.

Recall that a sequence $\{a_n\}$ of real numbers satisfying $0 \le a_n < 1$ is uniformly distributed mod 1 if for every pair of numbers c, d with $0 \le c < d \le 1$ we have

$$\lim_{N \to \infty} \frac{1}{N}(\text{number of } a_n \text{ with } n \le N \text{ and } c \le a_n < d) = d - c.$$

In other words, the fraction of a_n in the interval $[c, d)$ is $d - c$.

Since no power of α is an integer it is easy to see that $\log \alpha$ is irrational (otherwise $b^{m/n} = \alpha$, so $\alpha^n = b^m$, which is integral). In fact, $\log \alpha$ is transcendental, but we shall not need this rather deep fact.

A famous theorem of Weyl [2] states the following: If β is irrational, then the sequence $\{\langle n\beta \rangle\}$ is uniformly distributed mod 1. Letting $\beta = \log \alpha$, we obtain the lemma.

We now continue with the proof of the theorem. Let $\varepsilon > 0$ be small. Let p satisfy $1 \le p \le b - 1$. With the above notation, let

$$c = \log \sqrt{5} + \log p + \varepsilon,\ d = \log \sqrt{5} + \log(p + 1) - \varepsilon.$$

Then $[c, d)$ is an interval of length $\log\left(\frac{p+1}{p}\right) - 2\varepsilon$. Therefore, the fraction of n such that $\langle n \log \alpha \rangle$ lies in $[c, d)$ is $\log\left(\frac{p+1}{p}\right) - 2\varepsilon$. For uniformity of exposition, we have used the convention that all intervals are considered mod 1, so an interval such as $[0.7, 1.2)$ is to be considered as the union of the two intervals $[0.7, 1)$ and $[0, 0.2)$. This technicality occurs only when d is greater than 1 and we leave it to the reader to check that our argument may be extended to cover this case. In particular, the interval $[c, d)$ may be broken into two parts and each part may be treated separately.

Let $m = [n \log \alpha]$ = integer part of $n \log \alpha$. If

$$\log \sqrt{5} + \log p + \varepsilon \le \langle n \log \alpha \rangle = n \log \alpha - m,$$

then

$$pb^m b^\varepsilon \le \alpha^n/\sqrt{5}.$$

If n, hence m, is sufficiently large, then

$$pb^m \le pb^m p^\varepsilon - 1 \le \frac{\alpha^n - (-\alpha)^{-n}}{\sqrt{5}} = F_n,$$

since $p^\varepsilon > 1$ and $|\alpha^{-n}| < 1$. Similarly, if $\langle n \log \alpha \rangle < d$ and n is large, then

$$F_n < (p + 1)b^m.$$

But these last two inequalities simply state that F_n in base b begins with the digit p. Therefore, we have shown that if n is large and $\langle n \log \alpha \rangle$ lies in $[c, d)$ then F_n begins with p. Therefore, the fraction of n such that F_n begins with p is at least

$$\log\left(\frac{p+1}{p}\right) - 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we find that this fraction is at least $\log(p + 1)/p$. However, this is true for each p, and

$$\log\left(\frac{2}{1}\right) + \log\left(\frac{3}{2}\right) + \cdots + \log\left(\frac{b}{b-1}\right) = 1.$$

Therefore, the fraction with first digit p can be no larger than $\log(p + 1)/p$; otherwise, these fractions would have sum greater than 1. Thus, the answer is exactly $\log(p + 1)/p$ and the proof of the theorem is complete.

Finally, we will mention one technicality that we have ignored in the above proof. Since we do not know a priori that $\lim \frac{1}{N}A_p(N)$ exists, it is slightly inaccurate to discuss the fraction of F with first digit p. However, what we proved was that $\liminf \frac{1}{N}A_p(N) \geq \log\left(\frac{p+1}{p}\right)$. By the remark at the end of the proof, it is then easy to see that it is impossible to have $\limsup \frac{1}{N}A_p(N)$ greater than $\log\left(\frac{p+1}{p}\right)$ for any p. Therefore, lim sup = lim inf and the limit exists.

REFERENCES

1. W. G. Brady. "More on Benford's Law." *The Fibonacci Quarterly* 16 (1978): 51-52.
2. G. H. Hardy & E. M. Wright. *An Introduction to the Theory of Numbers*. 4th ed. Oxford: Oxford University Press, 1960, p. 390.
3. J. Wlodarski. "Fibonacci and Lucas Numbers Tend To Obey Benford's Law." *The Fibonacci Quarterly* 9 (1971):87-88.

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A SIMPLE DERIVATION OF A FORMULA FOR $\sum_{k=1}^{n} k^r$

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The formula for

$$\sum_{k=1}^{n} k^r \quad (r \text{ and } n \text{ being positive integers})$$

is known (see Barnard & Child [1] and Jordan [2]). However, most undergraduate texts in algebra and calculus give these formulas only for $r = 1, 2,$ and 3. Perhaps the reason is that the known formula for general integral r is a bit involved and requires some background in the theory of polynomials and Bernoulli numbers. In this note we give a very simple derivation of this formula and no background beyond the knowledge of binomial theorem (integral power) and some elementary facts from calculus are needed. Consequently, the author hopes that the general formula can be exposed to undergraduates at some proper level.

Let

$$S_r(n) = \sum_{k=1}^{n} k^r,$$

where $r = 0, 1, \ldots, n = 1, 2, \ldots,$ and note that $S_0(n) = n$. In order to find a formula for $S_r(n)$, we use the following identity: For any integer k we have

$$\begin{aligned}\int_{k-1}^{k} x^r dx &= \frac{1}{r+1}(k^{r+1} - (k-1)^{r+1}) \\ &= \frac{1}{r+1}\sum_{j=0}^{r}\binom{r+1}{j}(-1)^{r+2-j}k^j \\ &= \sum_{j=0}^{r} a_j(r)k^j,\end{aligned}$$

where $a_j(r) = (-1)^{r+2-j}\binom{r+1}{j}\Big/(r+1)$. Hence,