NEWTON'S METHOD AND RATIOS OF FIBONACCI NUMBERS

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ABSTRACT

The sequence $\{F_{n+1}/F_n\}$ of ratios of consecutive Fibonacci numbers converges to the golden mean $\varphi = \frac{1}{2}(1+\sqrt{5})$, the positive root of $x^2 - x - 1 = 0$. Newton's method for the equation $x^2 - x - 1 = 0$ with initial approximation 1 produces the subsequence $\{F_{2^n+1}/F_{2^n}\}$ of Fibonacci ratios. The secant method for this equation with initial approximations 1 and 2 produces the subsequence $\{F_{F_n+1}/F_{F_n}\}$. These results generalize to quadratic equations with roots of unequal magnitudes.

It is well known that the ratios of successive Fibonacci numbers converge to the golden mean. We recall that the Fibonacci numbers $\{F_n\}$ are defined by the recurrence $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$ and $F_1 = 1$.* The golden mean, $\varphi = \frac{1}{2}(1+\sqrt{5}) \approx 1.618$, is the positive solution of the equation $x^2 - x - 1 = 0$.

The ratios $\{F_{n+1}/F_n\}$ of consecutive Fibonacci numbers are a sequence of rational numbers converging to φ *linearly*; that is, the number of digits of F_{n+1}/F_n which agree with φ is approximately a linear function of n. In fact, there are constants α, $\beta > 0$ and $\varepsilon < 1$ such that $\alpha\varepsilon^n < |F_{n+1}/F_n - \varphi| < \beta\varepsilon^n$.

We can obtain sequences of rational numbers converging more rapidly to φ by using procedures of numerical analysis for approximating solutions of the equation $x^2 - x - 1 = 0$. Two common methods for solving an equation $f(x) = 0$ numerically are Newton's method and the secant method (*regula falsi*) [1, 3]. Each method generates a sequence $\{x_n\}$ converging to a solution of $f(x) = 0$. For Newton's method,

$$x_n = \mathrm{NEWTON}(x_{n-1}) = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}. \tag{1}$$

The secant method is obtained from Newton's method by replacing $f'(x_{n-1})$ by a difference quotient:

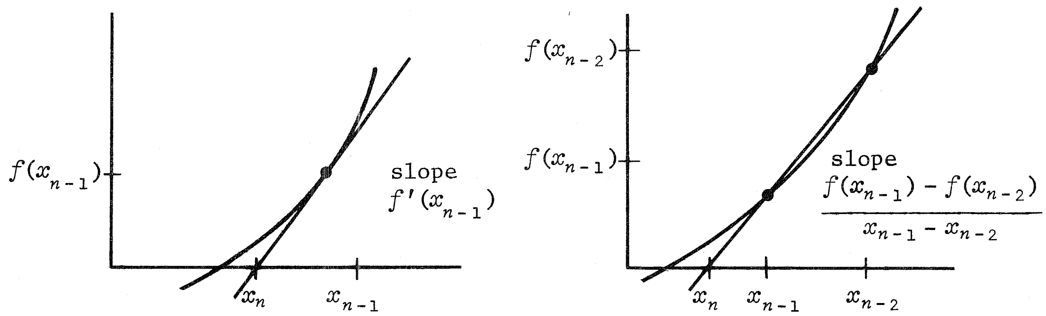
$$\begin{aligned} x_n = \mathrm{SECANT}(x_{n-1}, x_{n-2}) &= x_{n-1} - \frac{f(x_{n-1})(x_{n-1} - x_{n-2})}{f(x_{n-1}) - f(x_{n-2})} \\ &= \frac{x_{n-2}f(x_{n-1}) - x_{n-1}f(x_{n-2})}{f(x_{n-1}) - f(x_{n-2})}. \end{aligned} \tag{2}$$

[The first expression for $\mathrm{SECANT}(x_{n-1}, x_{n-2})$ is more useful for numerical calculations, while the second expression reveals the symmetric roles of x_{n-1} and x_{n-2}.] The familiar geometric interpretations of Newton's method and the secant method are given in Figure 1.

Newton's method requires an initial approximation x_0; the secant method requires two approximations x_0 and x_1. If the initial values are sufficiently close to a solution ξ of $f(x) = 0$, then the sequences $\{x_n\}$ defined by either method converge to ξ. Suppose that $f'(\xi) \neq 0$; that is, ξ is a simple zero of f. Then, the convergence of Newton's method is quadratic [1]: the number of correct digits of x_n is about twice that of x_{n-1}, since $|x_n - \xi| \approx \alpha|x_{n-1} - \xi|^2$

**For future reference, we note the first few Fibonacci numbers: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597.*

for some $\alpha > 0$. Similarly, the order of convergence of the secant method is $\varphi \approx 1.618$. since $|x_n - \xi| \approx \alpha|x_{n-1} - \xi|^{\varphi}$ for some $\alpha > 0$ [3].



(a) Newton's method *(b) Secant method*

Fig. 1. Geometric interpretations of Newton's method and secant method

Both these methods applied to the equation $x^2 - x - 1 = 0$ yield sequences converging to φ more rapidly than $\{F_{n+1}/F_n\}$. For this equation, we calculate easily that

$$\text{(3)} \qquad \text{NEWTON}(x_{n-1}) = \frac{x_{n-1}^2 + 1}{2x_{n-1} - 1} \quad \text{and} \quad \text{SECANT}(x_{n-1}, x_{n-2}) = \frac{x_{n-1}x_{n-2} + 1}{x_{n-1} + x_{n-2} - 1}.$$

For initial approximations to φ, it is natural to choose Fibonacci ratios. For example, with $x_0 = 1$, Newton's method produces the sequence,

$$1,\ 2/1,\ 5/3,\ 34/21,\ 1597/987,\ \ldots,$$

which we recognize (see note on page 1) as a subsequence of Fibonacci ratios. From a few more sample calculations [e.g.,

$$\text{NEWTON}(3/2) = 13/8 \quad \text{or} \quad \text{NEWTON}(8/5) = 89/55]$$

we infer the identity:

$$\text{(4)} \qquad \text{NEWTON}(F_{n+1}/F_n) = F_{2n+1}/F_{2n}.$$

The sequence $\{x_n\}$ generated by Newton's method with $x_0 = 1$ is defined by $x_n = F_{2^n+1}/F_{2^n}$. Now it is obvious that the convergence of $\{x_n\}$ is quadratic, since there are constants α, $\beta > 0$ and $\varepsilon < 1$ such that $\alpha\varepsilon^{2^n} < |x_n - \varphi| < \beta\varepsilon^{2^n}$.

We can similarly apply the secant method with Fibonacci ratios as initial approximations. From examples such as

$$\text{SECANT}(1, 2) = 3/2,\ \text{SECANT}(2, 3/2) = 8/5,\ \text{and}\ \text{SECANT}(3/2, 8/5) = 34/21,$$

we infer the general rule:

$$\text{(5)} \qquad \text{SECANT}(F_{m+1}/F_m, F_{n+1}/F_n) = F_{m+n+1}/F_{m+n}.$$

In particular, if $x_1 = 1$ and $x_2 = 2$, then the sequence $\{x_n\}$ generated by the secant method is given by $x_n = F_{F_n+1}/F_{F_n}$. Since F_n is asymptotic to $\varphi^n/\sqrt{5}$, there are constants α, $\beta > 0$ and $\varepsilon < 1$ such that $\alpha\varepsilon^{\varphi^n} < |x_n - \varphi| < \beta\varepsilon^{\varphi^n}$, which dramatically illustrates that the order of convergence of the secant method is φ.

Equations (4) and (5) are interesting because they imply that the sequences of rational approximations to φ produced by Newton's method and by the secant method are simple subsequences of Fibonacci ratios.

We now verify (4) and (5). In fact, these identities are valid in general for any sequence $\{u_n\}$ defined by a second-order linear difference equation with $u_0 = 0$ and $u_1 = 1$, provided the sequence $\{u_{n+1}/u_n\}$ is convergent.

Lemma: Let $\{u_n\}$ be defined by $au_n + bu_{n-1} + cu_{n-2} = 0$ with $u_0 = 0$ and $u_1 = 1$. Then $au_{m+1}u_{n+1} - cu_m u_n = au_{m+n+1}$ for all m, $n \geq 0$.

Proof: By induction on n. For $n = 0$, the lemma holds for all m since

$$au_{m+1}u_1 - cu_m u_0 = au_{m+1}.$$

Now assume that for $n - 1$ the lemma is true for all m. Then

$$\begin{aligned} au_{m+1}u_{n+1} - cu_m u &= (-bu_n - cu_{n-1})u_{m+1} + (au_{m+2} + bu_{m+1})u_n \\ &= au_{m+2}u_n - cu_{m+1}u_{n-1} \\ &= au_{(m+1)+(n-1)+1} \\ &= au_{m+n+1}. \ \square \end{aligned}$$

The lemma generalizes the Fibonacci identity [2]:

$$F_{m+1}F_{n+1} + F_m F_n = F_{m+n+1}.$$

Suppose that $ax^2 + bx + c$ has distinct zeros λ_1 and λ_2. Any sequence $\{u_n\}$ satisfying the recurrence $au_n + bu_{n-1} + cu_{n-2} = 0$ is of the form

$$u_n = k_1\lambda_1^n + k_2\lambda_2^n,$$

where k_1 and k_2 are constants determined by the initial values u_0 and u_1. If $|\lambda_1| > |\lambda_2|$ and $k_1 \neq 0$, then u_n is asymptotic to $k_1\lambda_1^n$, and so $\{u_{n+1}/u_n\}$ converges linearly to λ_1. We now show that if $u_0 = 0$ and $u_1 = 1$, then Newton's method and the secant method, starting with ratios from $\{u_{n+1}/u_n\}$, generate subsequences of $\{u_{n+1}/u_n\}$.

Theorem: Let $\{u_n\}$ be defined by $au_n + bu_{n-1} + cu_{n-2} = 0$ with $u_0 = 0$ and $u_1 = 1$. If the characteristic polynomial $f(x) = ax^2 + bx + c$ has zeros λ_1 and λ_2 with $|\lambda_1| > |\lambda_2|$, then:

(i) $u_n \neq 0$ for all $n > 0$;

(ii) $\lim\limits_{n\to\infty} u_{n+1}/u_n = \lambda_1$;

(iii) $\mathrm{NEWTON}(u_{n+1}/u_n) = u_{2n+1}/u_{2n}$;

(iv) $\mathrm{SECANT}(u_{m+1}/u_m,\ u_{n+1}/u_n) = u_{m+n+1}/u_{m+n}$.

Proof:

(i) It is easily verified that $u_n = k(\lambda_1^n - \lambda_2^n)$, where $k = \pm a/\sqrt{b^2 - 4ac}$. (The sign of k depends on the signs of a and b.) Since $|\lambda_1| > |\lambda_2|$, if $n > 0$, then $|\lambda_1^n| > |\lambda_2^n|$ and, therefore, $u_n \neq 0$.

(ii) We note, as an aside, that the sequence $\{u_{n+1}/u_n\}$ satisfies the first-order recurrence $x_n = -(bx_{n-1} + c)/ax_{n-1}$. To verify (ii):

$$\frac{u_{n+1}}{u_n} = \frac{k(\lambda_1^{n+1} - \lambda_2^{n+1})}{k(\lambda_1^n - \lambda_2^n)} = \lambda_1\frac{1 - (\lambda_1/\lambda_2)^{n+1}}{1 - (\lambda_1/\lambda_2)^n} \to \lambda_1 \text{ as } n \to \infty, \text{ since } |\lambda_1/\lambda_2| < 1.$$

(iii) For the equation $ax^2 + bx + c = 0$, Newton's method and the secant method are given by

$$\text{(6)} \quad \mathrm{NEWTON}(x_{n-1}) = \frac{ax_{n-1}^2 - c}{2ax_{n-1} + b} \quad \text{and} \quad \mathrm{SECANT}(x_{n-1},\ x_{n-2}) = \frac{ax_{n-1}x_{n-2} - c}{a(x_{n-1} + x_{n-2}) + b}.$$

Therefore, NEWTON(x_{n-1}) = SECANT(x_{n-1}, x_{n-1}), and so (iii) follows from (iv). Note that this identity holds for any polynomial equation $f(x) = 0$.

(iv) By (6),

$$\begin{aligned}\text{SECANT}(u_{m+1}/u_m,\ u_{n+1}/u_n) &= \frac{a(u_{m+1}/u_m)(u_{n+1}/u_n) - c}{a(u_{m+1}/u_m + u_{n+1}/u_n) + b}\\ &= \frac{au_{m+1}u_{n+1} - cu_mu_n}{au_{m+1}u_n + au_mu_{n+1} + bu_mu_n}\\ &= \frac{au_{m+1}u_{n+1} - cu_mu_n}{au_{m+1}u_n - cu_mu_{n-1}}\\ &= au_{m+n+1}/au_{m+n} \quad \text{(by the lemma)}\\ &= u_{m+n+1}/u_{m+n}. \ \square\end{aligned}$$

Remarks:

1. The theorem does not generalize to polynomials of degree higher than 2.

2. Not only do the ratios of the consecutive Fibonacci numbers converge to φ, they are the "best" rational approximation to φ; i.e., if $n > 1$, $0 < F \leq F_n$ and $P/F \neq F_{n+1}/F_n$, then $|F_{n+1}/F_n - \varphi| < |P/F - \varphi|$ by [4]. Since Newton's method and the secant method produce subsequences of Fibonacci ratios, they also produce the best rational approximation to φ.

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A CHARACTERIZATION OF THE FUNDAMENTAL SOLUTIONS TO PELL'S EQUATION $u^2 - Dv^2 = C$

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Due to a confusion originating with Euler, the diophantine equation

$$u^2 - Dv^2 = C, \tag{1}$$

where D is a positive integer that is not a perfect square and C is a nonzero integer, is usually called *Pell's equation*. In a previous article [1, Theorem 2], the following theorem was proved.

Theorem 1: Let $x_1 + y_1\sqrt{D}$ be the fundamental solution to $x^2 - Dy^2 = 1$. If $k =$