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ON PSEUDO-FIBONACCI NUMBERS OF THE FORM $2S^2$, WHERE S IS AN INTEGER

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If the pseudo-Fibonacci numbers are defined by

$$(1) \qquad u_1 = 1,\ u_2 = 4,\ u_{n+2} = u_{n+1} + u_n,\ n > 0,$$

then we can show that $u_1 = 1$, $u_2 = 4$, and $u_4 = 9$ are the only square pseudo-Fibonacci numbers.

In this paper we will describe a method to show that none of the pseudo-Fibonacci numbers are of the form $2S^2$, where S is an integer.

Even if we remove the restriction $n > 0$, we do not obtain any number of the form $2S^2$, where S is an integer.

It can be easily shown that the general solution of the difference equation (1) is given by

$$(2) \qquad u_n = \frac{7}{5.2^n}(\alpha^n + \beta^n) - \frac{1}{5.2^{n-1}}(\alpha^{n-1} + \beta^{n-1}),$$

where

$$\alpha = 1 + \sqrt{5},\ \beta = 1 - \sqrt{5},\ \text{and } n \text{ is an integer.}$$

Let

$$\eta_r = \frac{\alpha^r + \beta^r}{2^r};\quad \xi_r = \frac{\alpha^r - \beta^r}{2^r\sqrt{5}}$$

Then we easily obtain the following relations:

(3) $$u_n = \frac{1}{5}(7\eta_n - \eta_{n-1}),$$

(4) $$\eta_r = \eta_{r-1} + \eta_{r-2},\ \eta_1 = 1,\ \eta_2 = 3$$

(5) $$\xi_r = \xi_{r-1} + \xi_{r-2},\ \xi_1 = 1,\ \xi_2 = 1$$

(6) $$\eta_r^2 - 5\xi_r^2 = (-1)^r 4,$$

(7) $$\eta_{2r} = \eta_r^2 + (-1)^{r+1}2,$$

(8) $$2\eta_{m+n} = 5\xi_m\xi_n + \eta_m\eta_n,$$

(9) $$2\xi_{m+n} = \eta_n\xi_m + \eta_n\xi_m,$$

(10) $$\xi_{2r} = \eta_r\xi_r$$

The following congruences hold:

(11) $$u_{n+2r} \equiv (-1)^{r+1}u_n \pmod{\eta_r 2^{-s}},$$

(12) $$u_{n+2r} \equiv (-1)^r u_n \pmod{\xi_r 2^{-s}},$$

where $S = 0$ or 1.

We also have the following table of values:

n	0	1	2	3	4	5	6	7	8	9	12	14	16
u_n	3	1	4	5	9	14	23	37	60	97	411	1076	2817

t	4	5	8	10	t	5
ξ_t	3	5	$3\cdot 7$	$5\cdot 11$	η_t	11

Let

(13) $$2x^2 = u_n, \text{ where } x \text{ is an integer.}$$

The proof is now accomplished in eighteen stages.

(a) (13) is impossible if $n \equiv 0 \pmod{16}$, for, using (12) we find that

$$\begin{aligned} u_n &\equiv u_0 \pmod{\xi_8} \\ &\equiv 3 \pmod 7, \text{ since } 7/\xi_8 \\ &\equiv 10 \pmod 7. \end{aligned}$$

Thus, we find that

$$\frac{u_n}{2} \equiv 5 \pmod 7, \text{ since } (2,7) = 1,$$

and since $\left(\frac{5}{7}\right) = -1$, (13) is impossible.

(b) (13) is impossible if $n \equiv 1 \pmod 8$, for, using (12) in this case

$$\begin{aligned} u_n &\equiv u_1 \pmod{\xi_4} \\ &\equiv 1 \pmod 3 \\ &\equiv 4 \pmod 3. \end{aligned}$$

Thus,

$$\frac{u_n}{2} \equiv 2 \pmod 3, \text{ since } (2,3) = 1,$$

and since $\left(\frac{2}{3}\right) = -1$, (13) is impossible.

(c) (13) is impossible if $n \equiv 2 \pmod 8$, for, using (12) we find that

$$\begin{aligned} u_n &\equiv u_2 \pmod{\xi_4} \\ &\equiv 4 \pmod 3. \end{aligned}$$

Thus, we find that

$$\frac{u_n}{2} \equiv 2 \pmod 3, \text{ since } (2,3) = 1,$$

and since $\left(\frac{2}{3}\right) = -1$, (13) is impossible.

(d) (13) is impossible if $n \equiv 3 \pmod{16}$, for, using (12) in this case

$$\begin{aligned} u_n &\equiv u_3 \pmod{\xi_8} \\ &\equiv 5 \pmod 7, \text{ since } 7/\xi_8 \\ &\equiv 12 \pmod 7. \end{aligned}$$

Thus,

$$\frac{u_n}{2} \equiv 6 \pmod 7, \text{ since } (2,7) = 1,$$

and since $\left(\frac{6}{7}\right) = -1$, (13) is impossible.

(e) (13) is impossible if $n \equiv 4 \pmod{10}$, for, using (12) we find that

$$\begin{aligned} u_n &\equiv \pm u_4 \pmod{\xi_5} \\ &\equiv \pm 9 \pmod 5 \\ &\equiv \pm 4 \pmod 5. \end{aligned}$$

Thus, we find that

$$\frac{u_n}{2} \equiv \pm 2 \pmod 5, \text{ since } (2,5) = 1,$$

and since $\left(\frac{-2}{5}\right) = \left(\frac{2}{5}\right) = -1$, (13) is impossible.

(f) (13) is impossible if $n \equiv 5 \pmod{10}$, for, using (12) in this case

$$\begin{aligned} u_n &\equiv \pm u_5 \pmod{\xi_5} \\ &\equiv \pm 14 \pmod 5. \end{aligned}$$

Thus,

$$\frac{u_n}{2} \equiv \pm 7 \pmod 5, \text{ since } (2,5) = 1,$$

and since $\left(\frac{-7}{5}\right) = \left(\frac{7}{5}\right) = -1$, (13) is impossible.

(g) (13) is impossible if $n \equiv 6 \pmod{20}$, for, using (12) we find that

$$\begin{aligned} u_n &\equiv u_6 \pmod{\xi_{10}} \\ &\equiv 23 \pmod{11}, \text{ since } 11/\xi_{10} \\ &\equiv 12 \pmod{11}. \end{aligned}$$

Thus, we find that

$$\frac{u_n}{2} \equiv 6 \pmod{11}, \text{ since } (2,11) = 1,$$

and since $\left(\frac{6}{11}\right) = -1$, (13) is impossible.

(h) (13) is impossible if $n \equiv 7 \pmod 8$, for, using (12) in this case

$$\begin{aligned} u_n &\equiv u_7 \pmod{\xi_4} \\ &\equiv 37 \pmod 3 \\ &\equiv 34 \pmod 3. \end{aligned}$$

Thus,

$$\frac{u_n}{2} \equiv 17 \pmod 3, \text{ since } (2,3) = 1,$$

and since $\left(\frac{17}{3}\right) = -1$, (13) is impossible.

(i) (13) is impossible if $n \equiv 8 \pmod{10}$, for, using (11) we find that

$$\begin{aligned} u_n &\equiv u_8 \pmod{\eta_5} \\ &\equiv 60 \pmod{11}. \end{aligned}$$

Thus, we find that

$$\frac{u_n}{2} \equiv 30 \pmod{11}, \text{ since } (2,11) = 1,$$

and since $\left(\frac{30}{11}\right) = -1$, (13) is impossible.

(j) (13) is impossible if $n \equiv 1 \pmod{10}$, for, using (12) in this case

$$\begin{aligned} u_n &\equiv \pm u_1 \pmod{\xi_5} \\ &\quad \pm 1 \pmod 5 \\ &\quad \pm 4 \pmod 5. \end{aligned}$$

Thus,

$$\frac{u_n}{2} \equiv \pm 2 \pmod 5, \text{ since } (2,5) = 1,$$

and since $\left(\frac{-2}{5}\right) = \left(\frac{2}{5}\right) = -1$, (13) is impossible.

(k) (13) is impossible if $n \equiv 12 \pmod{16}$, for, using (12) we find that

$$\begin{aligned} u_n &\quad u_{12} \pmod{\xi_8} \\ &\quad 411 \pmod 7, \text{ since } 7/\xi_8 \\ &\quad 404 \pmod 7. \end{aligned}$$

Thus,

$$\frac{u_n}{2} \equiv 202 \pmod 7, \text{ since } (2,7) = 1,$$

and since $\left(\frac{202}{7}\right) = -1$, (13) is impossible.

(l) (13) is impossible if $n \equiv 3 \pmod{10}$, for, using (11) in this case

$$\begin{aligned} u_n &\equiv u_3 \pmod{\eta_5} \\ &\equiv 5 \pmod{11} \\ &\equiv 16 \pmod{11}. \end{aligned}$$

Thus,

$$\frac{u_n}{2} \equiv 8 \pmod{11}, \text{ since } (2,11) = 1,$$

and since $\left(\frac{8}{11}\right) = -1$, (13) is impossible.

(m) (13) is impossible if $n \equiv 14 \pmod{16}$, for, using (12) we find that

$$\begin{aligned} u_n &\equiv u_{14} \pmod{\xi_8} \\ &\equiv 1076 \pmod{7}, \text{ since } 7/\xi_8. \end{aligned}$$

Thus,

$$\frac{u_n}{2} \equiv 538 \pmod{7}, \text{ since } (2,7) = 1,$$

and since $\left(\frac{538}{7}\right) = -1$, (13) is impossible.

(n) (13) is impossible if $n \equiv 0 \pmod{10}$, for, using (11) in this case

$$\begin{aligned} u_n &\equiv u_0 \pmod{\eta_5} \\ &\equiv 3 \pmod{11} \\ &\equiv 14 \pmod{11}. \end{aligned}$$

Thus, we find that

$$\frac{u_n}{2} \equiv 7 \pmod{11}, \text{ since } (2,11) = 1,$$

and since $\left(\frac{7}{11}\right) = -1$, (13) is impossible.

(o) (13) is impossible if $n \equiv 16 \pmod{20}$, for, using (12) we find that

$$\begin{aligned} u_n &\equiv u_{16} \pmod{\xi_{10}} \\ &\equiv 2817 \pmod{11}, \text{ since } 11/\xi_{10} \\ &\equiv 2806 \pmod{11}. \end{aligned}$$

Thus,

$$\frac{u_n}{2} \equiv 1403 \pmod{11}, \text{ since } (2,11) = 1,$$

and since $\left(\frac{1403}{11}\right) = -1$, (13) is impossible.

(p) (13) is impossible if $n \equiv 2 \pmod{10}$, for, using (11) in this case

$$\begin{aligned} u_n &\equiv \pm u_2 \pmod{\xi_5} \\ &\equiv \pm 4 \pmod{5}. \end{aligned}$$

Thus, we find that

$$\frac{u_n}{2} \equiv 2 \pmod{5}, \text{ since } (2,5) = 1,$$

and since $\left(\frac{-2}{5}\right) = \left(\frac{2}{5}\right) = -1$, (13) is impossible.

(q) (13) is impossible if $n \equiv 7 \pmod{10}$, for, using (11) in this case

$$\begin{aligned} u_n &\equiv u_7 \pmod{\eta_5} \\ &\equiv 37 \pmod{11} \\ &\equiv 26 \pmod{11}. \end{aligned}$$

Thus,

$$\frac{u_n}{2} \equiv 13 \pmod{11}, \text{ since } (2,11) = 1,$$

and since $\left(\frac{13}{11}\right) = -1$, (13) is impossible.

(r) (13) is impossible if $n \equiv 9 \pmod{10}$, for, using (11) we find that

$$\begin{aligned} u_n &\equiv u_9 \pmod{\eta_5} \\ &\equiv 97 \pmod{11} \\ &\equiv 86 \pmod{11}. \end{aligned}$$

Thus, we find that

$$\frac{u_n}{2} \equiv 43 \pmod{11}, \text{ since } (2,11) = 1,$$

and since $\left(\frac{43}{11}\right) = -1$, (13) is impossible.

Hence, none of the pseudo-Fibonacci numbers are of the form $2S^2$, where S is an integer.

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INFINITE SERIES WITH FIBONACCI AND LUCAS POLYNOMIALS

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In [7], D. A. Millin poses the problem of showing that

$$\sum_{n=0}^{\infty} F_{2^n}^{-1} = \frac{7 - \sqrt{5}}{2} \tag{1}$$

where F_k is the kth Fibonacci number. A proof of (1) by I. J. Good is given in [5], while in [3], Hoggatt and Bicknell demonstrate ten different methods of finding the same sum. Furthermore, the result of (1) is extended by Hoggatt and Bicknell in [4], where they show that