A RECIPROCAL SERIES OF FIBONACCI NUMBERS WITH SUBSCRIPTS $2^n k$

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A reciprocal series of Fibonacci numbers with subscripts 2^n was summed by I. J. Good [1] and was proposed as a problem by D. A. Millin [2], and there are many proofs in [4] of

$$\sum_{n=0}^{\infty} 1/F_{2^n} = (7-\sqrt{5})/2 .$$

Here, we derive a closely related sum,

$$\sum_{n=0}^{\infty} 1/F_{2^n k} .$$

To sum $1/F_{2^n k}$ we get a good start with early examples, making use of the identity $F_{2k} = F_k L_k$.

$$\frac{1}{F_k} = \frac{1}{F_k}, \qquad \frac{1}{F_k} + \frac{1}{F_{2k}} = \frac{L_k+1}{F_{2k}} = \frac{F_{2k}/F_k+1}{F_{2k}},$$

$$\frac{1}{F_k} + \frac{1}{F_{2k}} + \frac{1}{F_{4k}} = \frac{L_{2k}(L_k+1)+1}{F_{4k}} = \frac{F_{4k}/F_k + L_{2k}+1}{F_{4k}},$$

$$\frac{1}{F_k} + \frac{1}{F_{2k}} + \frac{1}{F_{4k}} + \frac{1}{F_{8k}} = \frac{F_{8k}/F_k + L_{4k}(L_{2k}+1)+1}{F_{8k}} .$$

From

$$L_{m+p} + L_{m-p} = L_m L_p, \quad p \text{ even},$$

and we can rewrite this as

$$\frac{1}{F_k} + \frac{1}{F_{2k}} + \frac{1}{F_{4k}} + \frac{1}{F_{8k}} = \frac{F_{8k}/F_k + (L_{6k}+L_{4k}+L_{2k}+1)}{F_{8k}} .$$

Now, the hinge is the Lucas identity

$$L_{2^n k}\left(L_{(2^n-2)k} + L_{(2^n-4)k} + \cdots + L_{2k} + 1\right) = L_{(2^{n+1}-2)k} + L_{(2^{n+1}-4)k} + \cdots + L_{2k} .$$

Thus,

$$(1) \qquad \sum_{i=0}^{n} \frac{1}{F_{2^i k}} = \frac{F_{2^n k}/F_k + \left(L_{(2^n-2)k} + L_{(2^n-4)k} + \cdots + L_{2k} + 1\right)}{F_{2^n k}} .$$

But,

$$L_{(2^n-2)k} + L_{(2^n-4)k} + \cdots + L_{2k}$$

can be summed and converted to a form using powers of

$$\alpha = (1+\sqrt{5})/2 \qquad \text{and} \qquad \beta = (1-\sqrt{5})/2,$$

making it possible to find the limit as $n \to \infty$.

Using a result of K. Siler [3],

$$\sum_{k=1}^{n} F_{ak-b} = \frac{(-1)^a F_{an-b} - F_{a(n+1)-b} + (-1)^{a-b} F_b + F_{a-b}}{1 - L_a + (-1)^a}$$

whence, with $a = 2k$, $k = j$ and $b = +1$;

$$\sum_{j=1}^{n} F_{2kj-1} = \frac{F_{2kn-1} - F_{2k(n+1)-1} + (-1)^{2k-1} F_1 + F_{2k-1}}{2 - L_{2k}} .$$

Now let $a = 2k$, $k = j$, and $b = -1$;

$$\sum_{j=1}^{n} F_{2kj+1} = \frac{F_{2kn+1} - F_{2k(n+1)+1} + (-1)^{2k+1} F_{-1} + F_{2k-1}}{2 - L_{2k}} .$$

Summing the preceding two series termwise,

$$\sum_{j=1}^{n} L_{2kj} = \frac{L_{2kn} - L_{2k(n+1)} - L_0 + L_{2k}}{2 - L_{2k}} = \frac{L_{2k(n+1)} - L_{2kn} - L_{2k} + 2}{L_{2k} - 2} .$$

Now, let $n = 2^{N-1} - 1$, $n + 1 = 2^{N-1}$ and return to (1):

$$\sum_{n=0}^{N} 1/F_{2^nk} = \frac{F_{2^Nk}/F_k + \left(\sum_{j=1}^{2^{N-1}-1} L_{2kj} \right) + 1}{F_{2^Nk}}$$

$$= \frac{1}{F_k} + \frac{L_{k(2^N)} - L_{(2^N-2)k}}{F_{2^Nk}(L_{2k} - 2)} = A$$

$$\lim_{N \to \infty} A = \frac{1}{F_k} + \lim_{N \to \infty} \frac{L_{k(2^N)} - L_{k(2^N-2)}}{F_{2^Nk}(L_{2k} - 2)} .$$

Trying this for $k = 1$,

$$\lim_{N \to \infty} A = \frac{1}{F_k} + \lim_{N \to \infty} \left(\frac{L_{2^Nk}}{F_{2^Nk}} - \frac{L_{(2^N-2)k}}{L_{(2^N-1)k}} \cdot \frac{L_{(2^N-1)k}}{L_{2^Nk}} \cdot \frac{L_{2^Nk}}{F_{2^Nk}} \right) \left(\frac{1}{L_{2k} - 2} \right) \Bigg|_{k=1}$$

$$= 1 + \sqrt{5} - \sqrt{5}\beta^2 = 1 + \sqrt{5}(1 - \beta^2) = 1 + \sqrt{5}(-\beta)$$

$$= 1 + \sqrt{5}(\sqrt{5} - 1)/2 = (7 - \sqrt{5})/2 ,$$

which is the result of Millin and of Good.

Generally, we get

$$\lim_{N \to \infty} A = \frac{1}{F_k} + \frac{1}{L_{2k} - 2}(\sqrt{5} - \sqrt{5}\beta^{2k}) = \frac{1}{F_k} + \sqrt{5} \left(\frac{1 - (L_{2k} - \sqrt{5}F_{2k})/2}{L_{2k} - 2} \right) = \frac{1}{F_k} - \frac{\sqrt{5}}{2} + \frac{5F_{2k}}{2(L_{2k} - 2)} .$$

We need the identity

$$L_k^2 = L_{2k} + 2(-1)^k \tag{2}$$

which for odd k gives us

$$L_k^2 = L_{2k} - 2.$$

For odd k, then, we can continue

$$\lim_{N\to\infty} A = \frac{1}{F_k} - \frac{\sqrt{5}}{2} + \frac{5F_{2k}}{2L_k^2} = \frac{1}{F_k} - \frac{\sqrt{5}}{2} + \frac{5F_k}{2L_k}, \quad k \text{ odd.}$$

However, if we let k be even, then (2) gives us

$$L_k^2 = L_{2k} + 2, \qquad L_k^2 - 4 = L_{2k} - 2 = 5F_k^2,$$

so that our limit becomes

$$\lim_{N\to\infty} A = \frac{1}{F_k} - \frac{\sqrt{5}}{2} + \frac{5F_{2k}}{2(5F_k^2)} = \frac{1}{F_k} - \frac{\sqrt{5}}{2} + \frac{L_k}{2F_k}, \quad k \text{ even}$$

Finally,

$$\sum_{n=0}^{\infty} 1/F_{2^n k} = \begin{cases} \dfrac{2L_k - F_{2k}\sqrt{5} + 5F_k^2}{2F_{2k}}, & k \text{ odd;} \\[2ex] \dfrac{2 - F_k\sqrt{5} + L_k}{2F_k}, & k \text{ even.} \end{cases}$$

It would seem that the odd and even cases are closely related. First, let k be odd, or, $k = 2s + 1$. Then

$$\sum_{n=0}^{\infty} 1/F_{(2s+1)2^n} = \frac{1}{F_{2s+1}} + \frac{5F_{2(2s+1)}}{2L_{2s+1}^2} - \frac{\sqrt{5}}{2} = B.$$

Now, let k be even. Let $k = 2(2s + 1)$, making

$$\sum_{n=0}^{\infty} 1/F_{2(2s+1)2^n} = \frac{1}{F_{2(2s+1)}} + \frac{L_{2(2s+1)}}{2F_{2(2s+1)}} - \frac{\sqrt{5}}{2} = C.$$

Then, notice that $B = C + 1/F_{2s+1}$.

REFERENCES

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2. D. A. Millin, Problem H-237, *The Fibonacci Quarterly,* Vol. 12, No. 3 (Oct., 1974), p. 309.
3. Ken Siler, "Fibonacci Summations," *The Fibonacci Quarterly,* Vol. 1, No. 3 (Oct., 1963), pp. 67–69.
4. V. E. Hoggatt, Jr., and Marjorie Bicknell, "A Primer for the Fibonacci Numbers, Part XV: Variations on Summing a Series of Reciprocals of Fibonacci Numbers with Subscripts $F_{2^n k}$," *The Fibonacci Quarterly,* Vol. 14, No. 3 (Oct. 1976), pp. 272–276.

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[Cont. from p. 452.]

RESPONSE

We push Pascal to the left, up tight,
To see what else can be brought to light.
In flowers and trees the world around,
The Fibonacci numbers do abound.
Look up to the right while taking sums.
What you find there will strike you dumb.

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