STAR OF DAVID THEOREM (I)

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The greatest Common divisor property of the binomial coefficients, namely,

$$\text{✡}\quad GCD\left\{\binom{n-1}{k-1}, \binom{n}{k+1}, \binom{n+1}{k}\right\} = GCD\left\{\binom{n+1}{k+1}, \binom{n}{k-1}, \binom{n-1}{k}\right\}$$

was conjectured and named as the Star of David Property by H. Gould in 1972 [1]. So far, three solutions appeared [2, 3, 4]. All three proofs were based on the exponents of primes in binomial coefficients of ✡.

An integer matrix multiplication of the integer vectors,

$$\begin{bmatrix}\binom{n-1}{k-1}\\ \binom{n}{k+1}\\ \binom{n+1}{k}\end{bmatrix} = \begin{bmatrix} k+1 & k-n-1 & -n-1\\ -k & n-k+1 & n\\ k+1 & k-n & -n\end{bmatrix}\begin{bmatrix}\binom{n+1}{k+1}\\ \binom{n}{k-1}\\ \binom{n-1}{k}\end{bmatrix},$$

which together with its inverse, i.e.,

$$\begin{bmatrix}\binom{n+1}{k+1}\\ \binom{n}{k-1}\\ \binom{n-1}{k}\end{bmatrix} = \begin{bmatrix} -n & -k & n-k+1\\ n & k+1 & k-n\\ -n-1 & -k-1 & n-k+1\end{bmatrix}\begin{bmatrix}\binom{n-1}{k-1}\\ \binom{n}{k+1}\\ \binom{n+1}{k}\end{bmatrix}$$

shows that a common factor of numbers that appear on one side of ✡ also divides each number of the other side. This proves the Star of David property ✡.

REFERENCES

1. H. W. Gould, "A Greatest Common Divisor Property of Binomial Coefficients," *The Fibonacci Quarterly,* Vol. 10, No. 5 (December 1972), pp. 579–584, 628. Announced in Abstract 72T-A248, *Notices AMS* 19 (1972), A-685.
2. A.P. Hillman and V.E. Hoggatt, Jr., "A Proof of Gould's Pascal Hexagon Conjecture," *The Fibonacci Quarterly,* Vol. 10, No. 5 (December 1972), pp. 565–568, 598.
3. E.G. Straus, "On the Greatest Common Divisor of Some Binomial Coefficients," *The Fibonacci Quarterly,* Vol. 11, No. 1 (February 1973), pp. 25–26.
4. David Singmaster, "Notes on Binomial Coefficients IV," *The Fibonacci Quarterly,* Vol. 11, No. 3 (October 1973), pp. 282–284.
5. A.P. Hillman and V.E. Hoggatt, Jr., "Exponents of Primes in Generalized Binomial Coefficients," *Journal für die reine und Angewandte Mathematik,* pp. 262–263 (1973), 375–380.

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