# ELEMENTARY PROBLEMS AND SOLUTIONS

**Edited by**
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Dept. of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within four months of the publication date.

Contributors (in the United States) who desire acknowledgement of receipt of their contributions are asked to enclose self-addressed stamped postcards.

DEFINITIONS. The Fibonacci numbers $F_n$ and the Lucas numbers $L_n$ satisfy $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$, $L_0 = 2$, $L_1 = 1$.

## PROBLEMS PROPOSED IN THIS ISSUE

*B-250 Proposed by Guy A. R. Guillotte, Montreal, Quebec, Canada.*

$$\begin{array}{r} \text{DO} \\ \text{YOU} \\ \underline{\text{LIKE}} \\ \text{SUSY} \end{array}$$

In this alphametic, each letter stands for a particular but different digit, nine digits being shown here. What do you make of the perfect square sum SUSY?

*B-251 Proposed by Paul S. Bruckman, San Rafael, California*

A and B play a match consisting of a sequence of games in which there are no ties. The odds in favor of A winning any one game is m. The match is won by A if the number of games won by A minus the number won by B equals 2n before it equals -n. Find m in terms of n given that the match is a fair one, i.e., the probability is 1/2 that A will win the match.

*B-252 Proposed by Wray G. Brady, Slippery Rock State College, Slippery Rock, Pennsylvania.*

Prove that

$$\sum_{i+j+k=n} \frac{(-1)^k}{i!j!k!} = \frac{1}{n!} .$$

*B-253 Proposed by Wray G. Brady, Slippery Rock State College, Slippery Rock, Pennsylvania.*

Prove that

$$\sum_{i+j+k=n} \frac{(-1)^k L_{j+2k}}{i!j!k!} = 0 = \sum_{i+j+k=n} \frac{(-1)^k F_{j+2k}}{i!j!k!} .$$

*B-254 Proposed by Clyde A. Bridger, Springfield, Illinois.*

Let $A^n = a^n + b^n + c^n$ and $B^n = d^n + e^n + f^n$ where a, b, and c are the roots of $x^3 - 2x - 1$ and d, e, and f are the roots of $x^3 - 2x^2 + 1$. Find recursion formulas for the $A_n$ and for the $B_n$. Also express $B_n$ in terms of $A_n$.

*B-255 Proposed by L. Carlitz and Richard Scoville, Duke University, Durham, North Carolina.*

Show that

$$\sum_{2k \leq n} k\binom{n-k}{k} = \sum_{k=0}^{n} F_k F_{n-k} = [(n-1)F_{n+1} + (n+1)F_{n-1}]/5.$$

## SOLUTIONS

## FIBONACCI SUM OF FOUR SQUARES

*B-226 Proposed by R. M. Grassl, University of New Mexico, Albuquerque, New Mexico.*

Find the smallest number in the Fibonacci sequence 1, 1, 2, 3, 5, ⋯ that is not the sum of the squares of three integers.

*Solution by Paul S. Bruckman, San Rafael, California.*

It is a well-known result in number theory (see, for example, The Higher Arithmetic, by H. Davenport, p. 127, Harper Torchbooks, 1960) that any number of the form $4^u(8v + 7)$ is not representable as the sum of three squares, whereas all other numbers are representable. The first few numbers in this sequence are as follows:

$$7, 15, 23, 28, 31, 39, 47, 55, \cdots.$$

The smallest number of this set which is also a Fibonacci number is 55, which is therefore the solution to the problem.

*Also solved by Ralph Fecke, J. A. H. Hunter, Peter A. Lindstrom, C. B. A. Peck, Stephen Rayport, and the Proposer.*

## GENERALIZATION OF RECKE'S FORMULA

*B-227 Proposed by H. V. Krishna, Manipal Engineering College, Manipal, India.*

Let $H_0, H_1, H_2, \cdots$ be a generalized Fibonacci sequence satisfying $H_{n+2} = H_{n+1} + H_n$ (and any initial conditions $H_0 = q$ and $H_1 = p$). Prove that

$$F_1H_3 + F_2H_6 + F_3H_9 + \cdots + F_nH_{3n} = F_nF_{n+1}H_{2n+1}\,.$$

*Solution by John W. Milsom, Butler County Community College, Butler, Pennsylvania.*

This is a generalization of Problem B-153 in which it was established that

$$F_1F_3 + F_2F_6 + F_3F_9 + \cdots + F_nF_{3n} = F_nF_{n+1}F_{2n+1}\,.$$

An induction proof follows.

$$\sum_{i=1}^{n} F_iH_{3i} = F_nF_{n+1}H_{2n+1}$$

for $n = 1$. Assume that for some positive integer $k$ that

$$\sum_{i=1}^{k} F_iH_{3i} = F_kF_{k+1}H_{2k+1}\,.$$

The difference between

$$\sum_{i=1}^{k+1} F_i H_{3i}$$

and

$$\sum_{i=1}^{k} F_i H_{3i}$$

is $F_{k+1} H_{3k+3}$. If it can be shown that

$$F_{k+1}F_{k+2}H_{2k+3} - F_kF_{k+1}H_{2k+1} = F_{k+1}H_{3k+3}\,,$$

then it will follow that

$$\sum_{i=1}^{k+1} F_i H_{3i} = F_{k+1} F_{k+2} H_{2k+3} \,.$$

$$\begin{aligned} F_{k+1} F_{k+2} H_{2k+3} - F_k F_{k+1} H_{2k+1} &= F_{k+1}(F_{k+2} H_{2k+3} - F_k H_{2k+1}) \\ &= F_{k+1} \left[ (F_{k+1} + F_k)(H_{2k+1} + H_{2k+2}) - F_k H_{2k+1} \right] \\ &= F_{k+1} (F_{k+1} H_{2k+3} + F_k H_{2k+2}) \\ &= F_{k+1} H_{3k+3} \,. \end{aligned}$$

This last statement follows from the known statement of equality

$$H_{n+r} = F_{r-1} H_n + F_r H_{n+1}$$

with $n = k + 1$ and $r = 2k + 2$. Thus it can be said for all positive integral values of $n$ that

$$F_1 H_3 + F_2 H_6 + F_3 H_9 + \cdots + F_n H_{3n} = F_n F_{n+1} H_{2n+1} \,.$$

*Also solved by Paul S. Bruckman, A. Carroll, Herta T. Freitag, Ralph Garfield, Pierre J. Malraison, Jr., C. B. A. Peck, A. Sivasubramanian, David Zeitlin, and the Proposer.*

## A CYCLICALLY SYMMETRIC FORMULA

*B-228 Proposed by Wray G. Brady, Slippery Rock State College, Slippery Rock, Pennsylvania.*

Extending the definition of the $F_n$ to negative subscripts using $F_{-n} = (-1)^{n-1} F_n$, prove that for all integers $k$, $m$, and $n$

$$(-1)^k F_n F_{m-k} + (-1)^m F_k F_{n-m} + (-1)^n F_m F_{k-n} = 0 \,.$$

*Solution by Paul S. Bruckman, San Rafael, California*

Using the Binet definitions of the Fibonacci and Lucas numbers,

$$F_n = (a^n - b^n)/\sqrt{5}, \qquad L_n = a^n + b^n,$$

where

$$a = \tfrac{1}{2}(1 + \sqrt{5}), \qquad b = \tfrac{1}{2}(1 - \sqrt{5}) \,;$$

$$\begin{aligned} (-1)^k F_n F_{m-k} &= (-1)^k (a^n - b^n)(a^{m-k} - b^{m-k}) \div 5 \\ &= (-1)^k (a^{m+n-k} - b^{n-m+k}(ab)^{m-k} - a^{n-m+k}(ab)^{m-k} + b^{m+n-k})/5 \\ &= \tfrac{1}{5}(-1)^k L_{m+n-k} - \tfrac{1}{5}(-1)^m L_{n-m+k} \,, \end{aligned}$$

since $ab = -1$. Similarly,

$$(-1)^m F_k F_{n-m} = \frac{1}{5}(-1)^m L_{n+k-m} - \frac{1}{5}(-1)^n L_{k-n+m}$$

and

$$(-1)^n F_m F_{k-n} = \frac{1}{5}(-1)^n L_{m+k-n} - \frac{1}{5}(-1)^k L_{m-k+n} \quad .$$

Adding these three expressions, the term on the R. H. S. vanish, yielding the desired result.

*Also solved by Herta T. Freitag, R. Garfield, C. B. A. Peck, David Zeitlin, and the Proposer.*

### AN ANALOGUE OF B-228 GENERALIZED

*B-229 Proposed by Wray G. Brady, Slippery Rock State College, Slippery Rock, Pennsylvania.*

Using the recursion formulas to extend the definition of $F_n$ and $L_n$ to all integers n, prove that for all integers k, m, and n

$$(-1)^k L_n F_{m-k} + (-1)^m L_k F_{n-m} + (-1)^n L_m F_{k-n} = 0 .$$

*Solution by David Zeitlin, Minneapolis, Minnesota.*

To solve B-228 and B-229 simultaneously, we let $\{H_n\}$ satisfy $H_{n+2} = H_{n+1} + H_n$. Then it is well known that

(1) $$(-1)^a H_i F_j = H_{a+i} F_{a+j} - H_{a+i+j} F_a .$$

In (1) we let $(a, i, j) = (k, n, m-k)$, $(m, k, n-m)$, and $(n, m, k-n)$ and add the results to obtain

$$(-1)^k H_n F_{m-k} + (-1)^m H_k F_{n-m} + (-1)^n H_m F_{k-n} = 0 ,$$

which contains B-228 and B-229 as special cases.

*Also solved by Paul S. Bruckman, Herta T. Freitag, R. Garfield, C. B. A. Peck, and the Proposer.*

### A SIMPLE RESULT, GENERALIZED

*B-230 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.*

Let $\{C_n\}$ satisfy

$$C_{n+4} - 2C_{n+3} - C_{n+2} + 2C_{n+1} + C_n = 0$$

and let

$$G_n = C_{n+2} - C_{n+1} - C_n.$$

Prove that $\{G_n\}$ satisfies $G_{n+2} = G_{n+1} + G_n$ .

*Solution by David Zeitlin, Minneapolis, Minnesota.*

Theorem 1. Let A and B be real constants, and let

$$W_{n+4} = AW_{n+3} + BW_{n+2} + (3 - B - 2A)W_{n+1} + (2 - A - B)W_n$$

for $n = 0, 1, \cdots$. Let

$$Q_{n+2} = W_{n+2} + (1 - A)W_{n+1} + (2 - A - B)W_n .$$

Then

$$Q_{n+2} = Q_{n+1} + Q_n, \qquad n = 0, 1, \cdots .$$

Theorem 1 is proved easily and gives the desired result for $A = 2$ and $B = 1$. We also have

Theorem 2. Let A be a real constant and let

$$W_{n+3} = AW_{n+2} + (2 - A)W_{n+1} + (1 - A)W_n$$

for $n = 0, 1, \cdots$. Let

$$Q_n = W_{n+1} + (1 - A)W_n .$$

Then

$$Q_{n+2} = Q_{n+1} + Q_n, \qquad n = 0, 1, \cdots .$$

*Also solved by Paul S. Bruckman, Herta T. Freitag, R. Garfield, Peter A. Lindstrom, John W. Milsom, C. B. A. Peck, Richard W. Sielaff, A. Sivasubramanian, and the Proposer.*

## GENERALIZED FIBONACCI SEQUENCES

*B-231 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.*

A GFS (generalized Fibonacci sequence) $H_0, H_1, H_2, \cdots$ satisfies the same recursion formula $H_{n+2} = H_{n+1} + H_n$ as the Fibonacci sequence but may have any intial values. It is known that

$$H_n H_{n+2} - H_{n+1}^2 = (-1)^n c ,$$

where the constant c is characteristic of the sequence. Let $\{H_n\}$ and $\{K_n\}$ be GFS and let

$$C_n = H_0K_n + H_1K_{n-1} + H_2K_{n-2} + \cdots + H_nK_0 .$$

Show that

$$C_{n+2} = C_{n+1} + C_n + G_n ,$$

where $\{G_n\}$ is a GFS whose characteristic is the product of those of $\{H_n\}$ and $\{K_n\}$.

*Solution by Paul S. Bruckman, San Rafael, California.*

Let $G_n = C_{n+2} - C_{n+1} - C_n$. By the definition of $C_n$, we obtain:

$$\begin{aligned} G_n &= \sum_{i=0}^{n+2} H_iK_{n+2-i} - \sum_{i=0}^{n+1} H_iK_{n+1-i} - \sum_{i=0}^{n} H_iK_{n-i} \\ &= H_{n+2}K_0 + H_{n+1}K_1 - H_{n+1}K_0 + \sum_{i=0}^{n} H_i\,(K_{n+2-i} - K_{n+1-i} - K_{n-i}) \\ &= H_{n+2}K_0 + H_{n+1}K_1 - H_{n+1}K_0 \end{aligned}$$

(since the terms in the summation vanish)

$$= (H_{n+1} + H_n)K_0 + H_{n+1}K_1 - H_{n+1}K_0 = H_{n+1}K_1 + H_nK_0 .$$

Substituting the latter expression for $G_n$ in the following, we obtain:

$$\begin{aligned} G_{n+1}G_{n-1} - G_n^2 &= (H_{n+2}K_1 + H_{n+1}K_0)(H_nK_1 + H_{n-1}K_0) - (H_{n+1}K_1 + H_nK_0)^2 \\ &= H_{n+2}H_nK_1^2 + H_nH_{n+1}K_0K_1 + H_{n+2}H_{n-1}K_0K_1 + H_{n+1}H_{n-1}K_0^2 \\ &\quad - H_{n+1}^2K_1^2 - 2H_nH_{n+1}K_0K_1 - H_n^2K_0^2 \\ &= K_1^2(H_{n+2}H_n - H_{n+1}^2) + K_0K_1(H_nH_{n+1} + H_{n+2}H_{n-1} - 2H_nH_{n+1}) \\ &\quad + K_0^2(H_{n+1}H_{n-1} - H_n^2) . \end{aligned}$$

The coefficient of $K_1^2$ in the above expression, by hypothesis, is equal to $(-1)^nc$. The coefficient of $K_0K_1$ may be expressed as:

$$\begin{aligned} H_{n+2}H_{n-1} - H_nH_{n+1} &= (H_{n+1} + H_n)H_{n-1} - H_n(H_n + H_{n-1}) \\ &= H_{n+1}H_{n-1} - H_n^2 = (-1)^{n-1}c = -(-1)^nc . \end{aligned}$$

The coefficient of $K_0^2$ is also equal to $-(-1)^nc$. Therefore,

$$\begin{aligned} G_{n+1}G_{n-1} - G_n^2 &= (-1)^nc(K_1^2 - K_0K_1 - K_0^2) = (-1)^nc\ K_1^2 - K_0(K_1 + K_0) \\ &= (-1)^nc(K_1^2 - K_0K_2) = (-1)^{n-1}cd , \end{aligned}$$

where d is the characteristic of the sequence $\{K_n\}$. It remains now to prove that $\{G_n\}$ is a GFS. Using the expression $G_n = H_{n+1}K_1 + H_nK_0$, derived above, we see that

$$G_{n+2} - G_{n+1} - G_n = (H_{n+3} - H_{n+2} - H_{n+1})K_1 + (H_{n+2} - H_{n+1} - H_n)K_0 = 0 .$$

*Also solved by R. Garfield, C. B. A. Peck, and the Proposer.*

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[Continued from page 84.]

(IX)
$$\sum_{k=0}^{p} \binom{p}{k} c_1^{r(p-k)} c_2^{rk} f(x + c_1^{m(p-k)} c_2^{mk}) = \sum_{n=0}^{\infty} \frac{V^p_{mn+r}}{n!} D^n f(x) ,$$

(X)
$$\sum_{k=0}^{p} \left[ (-1)^k \binom{p}{k} c_1^{r(p-k)} c_2^{rk} f(x + c_1^{m(p-k)} c_2^{mk}) \right] \Big/ (c_1 - c_2)^p$$

$$= \sum_{n=0}^{\infty} \frac{U^p_{mn+r}}{n!} D^n f(x) .$$

David Zeitlin
Minneapolis, Minnesota

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Dear Editor:

I recently noted problem H-146 in Vol. 6, No. 6 (December 1968), p. 352, by J. A. H. Hunter of Toronto. (I am a slow reader.) I don't know whether you have printed a solution as yet; in any case, the answer is in a paper by Wilhelm Ljunggren, Vid. -Akad. Avhandlinger I, NR. 5 (Oslo 1942).

Indeed, $P_7 = 169$ is the only non-trivial square Pell number.

Ernst M. Cohn
Washington, D.C.

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Renewal notices, normally sent out to subscribers in November or December, are now sent by bulk mail. This means that if your address has changed the notice will not be forwarded to you. If you have a change of address, please notify:

Brother Alfred Brousseau
St. Mary's College
St. Mary's College, Calif.