LUCAS REPRESENTATIONS

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1. INTRODUCTION

We define the Fibonacci and Lucas numbers as usual by means of

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1} \qquad (n \geq 1),$$

$$L_0 = 2, \quad L_1 = 1, \quad L_{n+1} = L_n + L_{n-1} \qquad (n \geq 1).$$

We recall that every positive integer N can be written uniquely in the form

$$N = F_{k_1} + F_{k_2} + \cdots + F_{k_r}, \tag{1.1}$$

where

$$k_j - k_{j+1} \geq 2 \quad (j = 1, 2, \cdots, n-1); \qquad k_r \geq 2. \tag{1.2}$$

If A_k denotes the set of positive integers $\{N\}$ for which $k_r = k$, it is clear that the sets

$$\{A_k\} \qquad (k = 2, 3, 4, \cdots) \tag{1.3}$$

constitute a partition of the set of positive integers. We may refer to (1.3) as a Fibonacci partition of the positive integers. It is proved in [2] that the numbers in A_k can be described in terms of the greatest integer function. More precisely, if

$$\alpha = \frac{1}{2}(1 + \sqrt{5})$$

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and we put

$$a(n) = [\alpha n], \qquad b(n) = [\alpha^2 n] , \tag{1.4}$$

then we have

$$A_{2t} = \{ab^{t-1}a(n) \mid n = 1, 2, 3, \cdots\} \qquad (t = 1, 2, 3, \cdots) , \tag{1.5}$$

$$A_{2t+1} = \{b^t a(n) \mid n = 1, 2, 3, \cdots\} \qquad (t = 1, 2, 3, \cdots) . \tag{1.6}$$

As is customary, powers and juxtaposition of functions should be interpreted as composition.

Turning next to representations as sums of Lucas numbers, we show first that every positive integer is uniquely representable either in the form

$$N = L_{k_1} + \cdots + L_{k_r} + L_0 , \tag{1.7}$$

where

$$k_j = k_{j+1} \ge 2 \quad (j = 1, 2, \cdots, r - 1); \qquad k_r \ge 3 \tag{1.8}$$

or in the form

$$N = L_{k_1} + \cdots + L_{k_r} , \tag{1.9}$$

where now

$$k_j - k_{j+1} \ge 2 \quad (j = 1, 2, \cdots, r - 1); \qquad k_r \ge 1 ; \tag{1.10}$$

but not in both (1.7) and (1.9).

Let B_0 denote the set of positive integers representable in the form (1.7) and let B_k denote the set of positive integers representable in the form (1.9) with $k_r = k$. Then as above the sets

$$B_k \qquad (k = 0, 1, 2, \cdots) \tag{1.11}$$

constitute a partition of the positive integers which may be called a Lucas partition. In the next section we shall prove the following.

$$B_0 = \{a^2(n) + n \mid n = 1, 2, 3, \cdots\}, \tag{1.12}$$

$$B_1 = \{a^2(n) + n - 1 \mid n = 1, 2, 3, \cdots\}, \tag{1.13}$$

and

$$B_{2t+1} = \{ab^{t-1}a(n) + ab^t a(n) \mid n = 1, 2, 3, \cdots\} \quad (t = 1, 2, 3, \cdots), \tag{1.14}$$

$$B_{2t} = \{b^{t-1}a(n) + b^t a(n) \mid n = 1, 2, 3, \cdots\} \quad (t = 1, 2, 3, \cdots). \tag{1.15}$$

It is not difficult to show that an integer N is in B_0 if and only if it is not representable in the form

$$N = L_{k_1} + \cdots + L_{k_r}, \tag{1.16}$$

where

$$k_1 > k_2 > \cdots > k_r \geq 1.$$

Let $\nu(n)$ denote the number of integers $\leq n$ that are not representable in the form (1.17). Hoggatt has conjectured that

$$\nu(L_n) = F_{n-1} \tag{1.17}$$

and that, for fixed k,

$$\nu(kL_n) = kF_{n-1}, \tag{1.18}$$

if n is sufficiently large. The conjecture (1.17) was proved by Klarner; we shall prove (1.18) in Section 3 below.

2. SOME PROPERTIES OF THE LUCAS REPRESENTATION

Let P_n be the set of numbers that can be written in the form (1.7) with $k_1 \le n$, and let Q_n be those that can be written in the form (1.9) with $k_1 \le n$. Then we have

$$\begin{aligned} P_3 &= \{2,\ 6\} \\ Q_3 &= \{1,\ 3,\ 4,\ 5\} \\ P_4 &= \{2,\ 6,\ 9\} \\ Q_4 &= \{1,\ 3,\ 4,\ 5,\ 7,\ 8,\ 10\}\ . \end{aligned} \tag{2.1}$$

By induction we obtain the following theorem.

Theorem 1. Every positive integer can be uniquely represented in either the form (1.7) or the form (1.9), but not both. Moreover,

$$P_n \cup Q_n = \{1,\ 2,\ \cdots,\ L_{n+1} - 1\} \tag{2.2}$$

$$\text{card}\,(P_n) = F_n \tag{2.3}$$

$$\text{card}\,(Q_n) = F_{n+2} - 1\ . \tag{2.4}$$

Proof. We will prove (2.2)—(2.4) and also

$$P_n \cap Q_n = \phi \tag{2.5}$$

by induction. Hence let us assume (2.2)–(2.5) up to and including the value n. Now by definition

$$P_{n+1} = P_n \cup (P_{n-1} + L_{n+1})$$
$$Q_{n+1} = Q_n \cup (Q_{n-1} + L_{n+1}) \cup \{L_{n+1}\}$$

and these unions are disjoint; if for instance, $N \in P_{n-1} + L_{n+1}$, then $N > L_{n+1}$ and by (2.2) $N \notin P_n$, etc. Hence

$$\operatorname{card}(P_{n+1}) = \operatorname{card}(P_n) + \operatorname{card}(P_{n-1}) = F_{n+1}$$

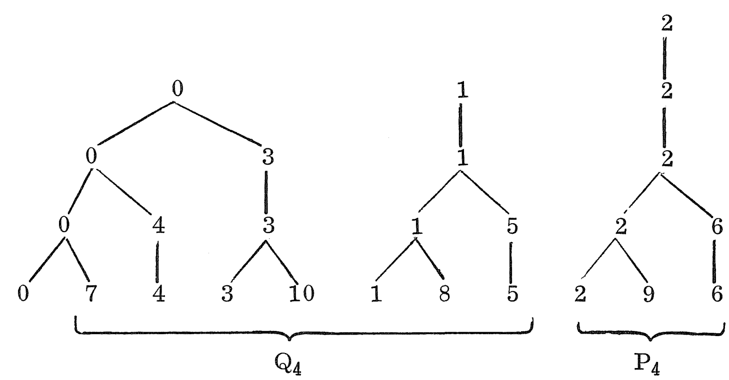
and

$$\operatorname{card}(Q_{n+1}) = F_{n+2} - 1 + F_{n+1} - 1 + 1 = F_{n+3} - 1 .$$

The other properties are easily checked.

The following tree may aid the reader.

n	L_n
0	2
1	1
2	3
3	4
4	7



We turn next to the relations (1.12)–(1.15). We make use of the function e defined in [1]. The properties we need are the following (see [1] and [2]):

(i) If

$$N = F_{k_1} + \cdots + F_{k_r}, \qquad k_r \geq 2, \quad k_1 > k_2 > \cdots > k_r ,$$

then

$$e(n) = F_{k_1-1} + \cdots + F_{k_r-1} .$$

(ii) For every N,

$$e(a(N)) = N$$

and

$$e(b(N)) = a(N) .$$

Theorem 2. The following relations hold.

$$B_0 = \{a^2(n) + n \mid n = 1, 2, 3, \cdots\} \tag{2.6}$$

$$B_1 = \{a^2(n) + n - 1 \mid n = 1, 2, 3, \cdots\} \tag{2.7}$$

$$B_{2t} = \{b^{t-1}a(n) + b^t a(n) \mid n = 1, 2, 3, \cdots\} \quad (t = 1, 2, 3, \cdots) \tag{2.8}$$

$$B_{2t+1} = \{ab^{t-1}a(n) + ab^t a(n) \mid n = 1, 2, 3, \cdots\} \quad (t = 1, 2, 3, \cdots) . \tag{2.9}$$

Proof. Let N be an arbitrary positive integer. By (1.5), we have $a^2(N) \in A_2$. Hence

$$a^2(N) = F_2 + \epsilon_4 F_4 + \cdots , \tag{2.10}$$

where ϵ_i may assume the values 0 or 1. Applying e twice, we get

$$N = F_1 + \epsilon_4 F_2 + \cdots . \tag{2.11}$$

Adding (2.10) and (2.11), we get

$$a^2(N) + N = 2 + \epsilon_4 L_3 + \cdots \in B_0 . \tag{2.12}$$

On the other hand, suppose

$$M = L_0 + \epsilon_3 L_3 + \epsilon_4 L_4 + \cdots \tag{2.13}$$

is in B_0. Let

$$K = F_2 + \epsilon_3 F_4 + \epsilon_4 F_5 + \cdots .$$

Since $K \in A_2$, by (1.5) K must be of the form $a^2(M)$ for some M. Also $M = e^2(a^2(M))$. Hence

$$a^2(M) = F_2 + \epsilon_3 F_4 + \epsilon_4 F_5 + \cdots$$

$$M = F_1 + \epsilon_3 F_2 + \epsilon_4 F_3 + \cdots$$

and

$$N = M + a^2(M) .$$

This proves (2.6). Equation (2.7) is clear from the definition. To prove (2.8), let N be arbitrary. Then

$$b^t a(N) \in A_{2t+1} ,$$

by (1.6), so

$$b^t a(N) = F_{2t+1} + \epsilon_{2t+3} F_{2t+3} + \cdots .$$

Applying e twice and adding we get

$$b^t a(N) + b^{t-1}(N) = L_{2t} + \epsilon_{2t+2} L_{2t+2} + \cdots \in B_{2t} .$$

Conversely, suppose $N \in B_{2t}$, so that

$$N = L_{2t} + \epsilon_{2t+2} L_{2t+2} + \cdots .$$

Put

$$M = F_{2t+1} + \epsilon_{2t+2} F_{2t+3} + \cdots .$$

Then, by (1.6), $M = b^t a(K)$ for some K. Moreover, since

$$e^2(M) = b^{t-1} a(K) ,$$

we have

$$N = b^t a(K) + b^{t-1} a(K) .$$

This proves (2.8), and the proof of (2.9) is similar.

3. PROOF OF HOGGATT'S CONJECTURES

Theorem 3. An integer N is in B_0 if and only if it is not representable in the form

$$N = L_{j_1} + \cdots + L_{j_s} , \tag{3.1}$$

where

$$j_1 > j_2 > \cdots > j_s > 1 . \tag{3.2}$$

Proof. If

$$j_t - j_{t+1} \geq 2 \qquad (t = 1, \cdots, s - 1) , \tag{3.3}$$

then Theorem 3 is an immediate consequence of Theorem 1. Let u be the least positive integer such that

$$j_u - j_{u+1} = 1 .$$

In (3.1), replace

$$L_{j_u} + L_{j_{u+1}} \qquad \text{by} \qquad L_{j_u+1}$$

and then repeat the process. Since

$$L_1 + L_2 + \cdots + L_k = L_{k+2} - 3,$$

we ultimately reach a representation of the form (3.1) that satisfies (3.3). This evidently proves the theorem.

Let $\nu(n)$ denote the number of positive integers $N \leq n$ that are not representable in the form (3.1), so that by the theorem just proved, $\nu(n)$ is also the number of integers $\leq n$ in B_0.

Theorem 4. We have

$$\nu(n) = \left[\frac{n+2}{\alpha^2+1}\right] \qquad (3.4)$$

Proof. By Theorem 2,

$$\begin{aligned} B_0 &= \{aa(k) + k \mid k = 1, 2, 3, \cdots\} \\ &= \{b(k) + k - 1 \mid k = 1, 2, 3, \cdots\}. \end{aligned}$$

Thus $\nu(n)$ is the largest integer k such that

$$b(k) + k \leq n + 1 .$$

Since $b(k) = [\alpha^2 k]$, $\nu(n)$ is the largest k such that

$$[(\alpha^2 + 1)k] \leq n + 1 ,$$

that is, the largest k such that

$$(\alpha^2 + 1)k < n + 2 .$$

Thus (3.4) follows at once.

Theorem 5. We have

$$\nu(L_n) = F_{n-1} \qquad (n \geq 1) . \qquad (3.5)$$

Proof. Since

$$L_n = \alpha^n + \beta^n \qquad (\alpha\beta = -1) ,$$

it follows that

$$\frac{L_n + 2}{\alpha^2 + 1} = \frac{\alpha^n + \beta^n + 2}{\alpha^2 + 1} = \frac{\alpha^{n-1} - 2\beta - \beta^{n+1}}{\alpha - \beta}$$

$$= \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} + \frac{-2\beta + \beta^{n-1} - \beta^{n+1}}{\alpha - \beta}$$

$$= F_{n-1} + \frac{2 + \beta^{n-1}}{\alpha^2 + 1} .$$

It is easily verified that

$$0 < \frac{2 + \beta^{n-1}}{\alpha^2 + 1} < 1 \qquad (n \geq 1) .$$

Theorem 6. Let k be a fixed positive integer. Then

$$\nu(kL_n) = kF_{n-1} \tag{3.6}$$

for n sufficiently large.

Proof. We have

$$\frac{kL_n + 2}{\alpha^2 + 1} = \frac{k(\alpha^n + \beta^n) + 2}{\alpha^2 + 1} = \frac{k(\alpha^{n-1} - \beta^{n+1}) - 2\beta}{\alpha - \beta}$$

$$= k\,\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} + \frac{k(\beta^{n-1} - \beta^{n+1}) - 2\beta}{\alpha - \beta}$$

For n sufficiently large it is clear that

$$0 < \frac{k(\beta^{n-1} - \beta^{n+1}) - 2\beta}{\alpha - \beta} < 1$$

so that

$$\left[\frac{kL_n + 2}{\alpha^2 + 1}\right] = kF_{n-1} .$$

This completes the proof of the theorem.

Theorem 7. We have

$$\nu(5F_n) = L_{n-1} \qquad (n > 1) \tag{3.7}$$

and

$$\nu(5kF_n) = kL_{n-1} \tag{3.8}$$

for sufficiently large n.

Proof. To prove (3.7), note that

$$\frac{5F_n + 2}{\alpha^2 + 1} = \frac{(\alpha - \beta)(\alpha^n - \beta^n) + 2}{\alpha^2 + 1} = \frac{(\alpha - \beta)(\alpha^{n-1} + \beta^{n+1}) - 2\beta}{\alpha - \beta}$$

$$= \alpha^{n-1} + \beta^{n-1} - \beta^{n-1}(1 - \beta^2) - \frac{2\beta}{\alpha - \beta}$$

$$= L_{n-1} + \beta^n - \frac{2\beta}{\alpha - \beta} .$$

Since

$$0 < \beta^n - \frac{2\beta}{\alpha - \beta} < 1 \qquad (n \geq 1) ,$$

(3.7) follows.

Next to prove (3.8) we take

$$\frac{5kF_n + 2}{\alpha^2 + 1} = \frac{k(\alpha - \beta)(\alpha^n - \beta^n) + 2}{\alpha^2 + 1} = \frac{k(\alpha - \beta)(\alpha^{n-1} + \beta^{n+1}) - 2\beta}{\alpha - \beta}$$

$$= k(\alpha^{n-1} + \beta^{n+1}) - \frac{2\beta}{\alpha - \beta} = k(\alpha^{n-1} + \beta^{n-1}) + k\beta^n - \frac{2\beta}{\alpha - \beta} .$$

Since

$$0 < k\beta^n - \frac{2\beta}{\alpha - \beta} < 1$$

for n sufficiently large, Eq. (3.8) follows at once.

The last two theorems were also conjectured by Hoggatt.

4. GENERATING FUNCTIONS

Put

$$\psi_j(0) = \sum_{n \in B_j} x^n \qquad (j = 0, 1, 2, \cdots) . \tag{4.1}$$

In view of Theorem 2, Eq. (4.1) is equivalent to

$$\psi_0(x) = \sum_{n=1}^{\infty} x^{a^3(n)+n} , \tag{4.2}$$

$$\psi_1(x) = \sum_{n=1}^{\infty} x^{a^2(n)+n-1} , \tag{4.3}$$

$$\psi_{2t+1}(x) = \sum_{n=1}^{\infty} x^{ab^{t-1}a(n)+ab^{t}a(n)} \qquad (t \geq 1) , \tag{4.4}$$

$$\psi_{2t}(x) = \sum_{n=1}^{\infty} x^{b^{t-1}a(n)+b^{t}a(n)} \qquad (t \geq 1) . \tag{4.5}$$

Clearly

$$\psi_0(x) = x\psi_1(x) \ . \tag{4.6}$$

Also it is evident that

$$\frac{x}{1-x} = \sum_{j=0}^{\infty} \psi_j(x) \ , \tag{4.7}$$

so that, by (4.6),

$$\frac{x}{1-x} = (1+x)\psi_1(x) + \sum_{j=2}^{\infty} \psi_j(x) \ . \tag{4.8}$$

In the next place it follows from the definition of A_r that

$$\psi_r(x) = x^{L_r}\left\{1 + \sum_{j=r+2}^{\infty} \psi_j(x)\right\} \qquad (r \geq 1) \ . \tag{4.9}$$

This implies

$$x^{-L_r}\psi_r(x) - x^{-L_{r+1}}\psi_{r+1}(x) = \psi_{r+2}(x) \qquad (r \geq 1) \ . \tag{4.10}$$

In particular, by (4.9),

$$\psi_1(x) = x\left\{1 + \sum_{j=3}^{\infty} \psi_j(x)\right\} \ .$$

Combining this with (4.8), we get

$$\frac{x}{1-x} = (1 + x + x^2)\psi_1(x) + x\psi_2(x) \ . \tag{4.11}$$

By means of (4.10) and (4.11) we can express all $\psi_j(x)$, $j > 1$, in terms of $\psi_1(x)$. The first few formulas are

$$x\psi_2(x) = \frac{x}{1 - x} - (1 + x + x^2)\psi_1(x)$$

$$x^4\psi_3(x) = -\frac{x}{1 - x} + (1 + x + x^2 + x^3)\psi_1(x)$$

$$x^8\psi_4(x) = \frac{x + x^5}{1 - x} + \frac{1 - x^7}{1 - x}\psi_1(x) \quad .$$

Generally we have

$$x^{L_{r+1}-3}\psi_r(x) = (-1)^r\left\{\frac{xA_r(x)}{1 - x}\right\} - B_r(x)\psi_1(x) \tag{4.12}$$

where

$$\left\{\begin{aligned} A_{r+2}(x) &= A_{r+1}(x) + x^{L_{r+1}}A_r(x) \\ B_{r+2}(x) &= B_{r+1}(x) + x^{L_{r+1}}B_r(x) \end{aligned}\right. \tag{4.13}$$

together with the initial conditions

$$\left\{\begin{array}{ll} A_2(x) = 1, & A_3(x) = 1 \\ B_2(x) = 1 + x + x^2, & B_3(x) = 1 + x + x^2 + x^3 \end{array}\right.$$

It follows that

$$B_r(x) = \frac{1 - x^{L_r}}{1 - x} \tag{4.14}$$

while

[Continued on page 70.]