FIBONACCI REPRESENTATIONS

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1. INTRODUCTION

We define the Fibonacci numbers as usual by means of

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1} \qquad (n \geq 1).$$

It is known that every positive integer N can be written in the form

$$N = F_{k_1} + F_{k_2} + \cdots + F_{k_r}, \tag{1.1}$$

where

$$k_1 > k_2 > \cdots > k_r \geq 2 \tag{1.2}$$

and r depends on N. We call (1.1) a Fibonacci representation of N. Moreover by the theorem of Zeckendorf, the representation (1.1) is unique provided the k_j satisfy the inequalities

$$k_j - k_{j+1} \geq 2 \quad (j = 1, 2, \cdots, r-1); \qquad k_r \geq 2. \tag{1.3}$$

Such a representation may be called the canonical representation of N.

Now let A_k denote the set of positive integers $\{N\}$ for which $k_r = k$. Then it is clear that the

$$A_k \qquad (k = 2, 3, 4, \cdots)$$

* Supported in part by NSF grant GP-17071.

constitute a partition of the set of positive integers. The chief object of the present paper is to describe the numbers in A_k in terms of the greatest integer function. We shall show that

$$(1.4) \qquad A_{2t} = \{ab^{t-1}a(n) \mid n = 1, 2, 3, \cdots\} \qquad (t = 1, 2, 3, \cdots),$$

$$(1.5) \qquad A_{2t+1} = \{b^t a(n) \mid n = 1, 2, 3, \cdots\} \qquad (t = 1, 2, 3, \cdots),$$

where

$$(1.6) \qquad a(n) = [\alpha n], \qquad b(n) = [\alpha^2 n], \qquad \alpha = (1 + \sqrt{5})/2$$

and $[x]$ denotes the greatest integer $\leq x$. As is customary, powers and juxtaposition of functions should be interpreted as composition.

Moreover, we shall show that

$$A(2t, \overline{2t+2}) = \{ab^{t-1}a^2(n) \mid n = 1, 2, 3, \cdots\}$$
$$A(2t, 2t+2) = \{ab^{t-1}ab(n) \mid n = 1, 2, 3, \cdots\}$$
$$A(2t+1, \overline{2t+3}) = \{b^t a^2(n) \mid n = 1, 2, 3, \cdots\}$$
$$A(2t+1, 2t+3) = \{b^t ab(n) \mid n = 1, 2, 3, \cdots\},$$

where $A(s, s+2)$ denotes the set of positive integers with canonical representation

$$F_{k_1} + \cdots + F_{k_r} + F_{s+2} + F_s ,$$

while $A(s, \overline{s+2})$ denotes the set with canonical representation

$$F_{k_1} + \cdots + F_{k_r} + F_s \qquad (k_r > s+2).$$

Using any Fibonacci representation of N

$$N = F_{k_1} + F_{k_2} + \cdots + F_{k_r} ,$$

we define

$$e(N) = F_{k_1-1} + F_{k_2-1} + \cdots + F_{k_r-1} . \tag{1.7}$$

The fact that $e(N)$ is independent of the Fibonacci representation chosen for N was proved in [2].

The following theorems, which will be used in Section 4, were also established in [2].

Theorem 1. For every N, $e(N+1) \geq e(N)$ with equality if and only if N is in A_2. (See [2], p. 216, Theorem 5 and proof.)

Theorem 2. If N is in A_2 then neither $N-1$ nor $N+1$ is in A_2. (See [2], p. 217, comments following Theorem 5.)

2. THE ARRAY R

As in [3] we form the 3-rowed array R as follows: In the first row we put the positive integers in natural order. We begin the second row with 1. To get an entry of the third row, we add the entries appearing above it in the first and second rows. We get further entries in the second row by choosing the smallest integer which has not appeared so far in the second or third rows.

(2.1)

1	2	3	4	5	6	7	8	9	10	...
1	3	4	6	8	9	11	12	14	16	...
2	5	7	10	13	15	18	20	23	26	...

Note that R is uniquely determined by the following properties:

(2.2) Every positive integer appears exactly once in row 2 or row 3.

(2.3) Each row is a monotone sequence.

(2.4) The sum of the first two rows is the third row.

Now also consider the 3-rowed array R'.

1	2	3	4	...
a(1)	a(2)	a(3)	a(4)	...
b(1)	b(2)	b(3)	b(4)	...

where a(n), b(n) are defined by (1.6). Since $\alpha + 1 = \alpha^2$, properties (2.3) and (2.4) are obviously satisfied by R'. To see that every number appears in R', let $N \geq 2$ be arbitrary. We will show that either $a([N/\alpha])$ or $b([N/\alpha^2])$ is $N - 1$. Suppose not. Then they are both too small; that is,

$$\alpha [N/\alpha] < N - 1 .$$

and

$$\alpha^2 [N/\alpha^2] < N - 1 .$$

Dividing the first inequality by α, the second by α^2, remembering that

$$\frac{1}{\alpha} + \frac{1}{\alpha^2} = 1 ,$$

and adding, we get

$$[N/\alpha] + [N/\alpha^2] < N - 1 .$$

But this is a contradiction since $N/\alpha + N/\alpha^2 = N$.

Now to see that the ranges of a and b are disjoint, suppose for some numbers N, M and P, we had $a(N) = b(M) = P$. Then

$$\alpha N - 1 < P < \alpha N$$

and

$$\alpha^2 M - 1 < P < \alpha^2 M .$$

Again dividing and adding, we get

$$N + M - 1 < P < N + M ,$$

a contradiction. The fact that no number appears twice in the same row follows simply because both α and α^2 are greater than 1. Note that (2.2) was proved using only the fact that α and α^2 are irrational and

$$\frac{1}{\alpha} + \frac{1}{\alpha^2} = 1 .$$

The result is not new, of course.

We have established that $R = R'$.

3. SOME PROPERTIES OF a(n) AND b(n)

In this section we prove several equalities involving the functions a(n) and b(n). In our proof we use only the properties (2.2), (2.3) and (2.4) of R(=R') from Section 2. Of course, the equalities could, with much more effort, be proved from the definitions (1.6).

$$N + a(N) = b(N) \tag{3.1}$$

$$b(N) = a(a(N)) + 1 \tag{3.2}$$

$$a(N) + b(N) = b(a(N)) + 1 \tag{3.3}$$

$$a(b(N)) = b(a(N)) + 1 \tag{3.4}$$

$$a(N) + b(N) = a(b(N)) \tag{3.5}$$

$$b^2(N) = aba(N) + 2 \tag{3.6}$$

$$ab^2(N) = b^2a(N) + 3 \tag{3.7}$$

(3.8) $$b^r(N) = ab^{r-1}a(N) + F_{2r-1} \qquad (r = 1, 2, \cdots)$$

(3.9) $$ab^r(N) = b^r a(N) + F_{2r} \qquad (r = 1, 2, \cdots)$$

(3.10) $$b^r(1) = F_{2r+1} \qquad (r = 1, 2, \cdots) .$$

Proof. Equation (3.1) is (2.4). For (3.2), note that in R, in the third row, to $b(N)$, or the second row to $a(J) = b(N) - 1$, occur all the numbers $1, 2, \cdots, b(N)$. Hence $J + N = b(N)$. Therefore, by (3.1) $J = a(N)$; that is, $a(a(N)) = b(N) - 1$. Equation (3.3) comes from (3.1) and (3.2). To prove (3.4), note that $b(a(N))$ is the $a(N)^{th}$ entry in the third row of R, and $a(b(N))$ is the $b(N)^{th}$ entry in the second row. Then the total number of entries is $a(N) + b(N) = b(a(N)) + 1$. Hence $b(a(N))$ cannot be the largest so $a(b(N))$ must be and every integer $\leq b(a(N)) + 1$ must have appeared. Hence $a(b(N)) = b(a(N)) + 1$. Equation (3.5) is obvious from (3.3) and (3.4). Equation (3.6) is obtained by adding (3.2) and (3.4) and using (3.1) and (3.5). Similarly we get (3.7) by adding (3.4) and (3.6). Equations (3.8) and (3.9) arise by induction. If we set $N = 1$ in (3.8) we get

$$b(b^{r-1}(1)) = a(b^{r-1}(1)) + F_{2r-1} ,$$

so, by (3.1),

$$b^{r-1}(1) = F_{2r-1} .$$

4. THE SETS A_k

We begin with some preliminary theorems.

Theorem 3. If N is in A_2, then $N + 1$ is in A_k with k odd.

Proof. By Theorem 2,

(4.1) $$N + 1 = F_{k_r} + F_{k_{r-1}} + \cdots + F_{k_1} \qquad k_r > 2 .$$

For convenience we let

$$N' = F_{k_{r-1}} + \cdots + F_{k_1} .$$

Then

$$N + 1 = F_{k_r} + N' = F_{k_r-2} + F_{k_r-1} + N'$$
$$= F_{k_r-4} + F_{k_r-3} + F_{k_r-1} + N' .$$

Continuing, we see that $N + 1$ is either

$$F_3 + F_4 + F_6 + \cdots + F_{k_r-1} + N'$$

or

$$F_2 + F_3 + F_5 + \cdots + F_{k_r-1} + N' .$$

If the latter, N would be in A_3. Hence

$$N = F_2 + F_4 + \cdots + F_{k_r-1} + N'$$

and k_r is odd.

Theorem 4. If N and M are in A_2 and $e(e(N)) = e(e(M))$, then $N = M$.

Proof. Suppose $N \neq M$. If $e(N) = e(M)$ then by Theorem 1, N and M are consecutive integers and by Theorem 2 could not both be in A_2. So suppose $e(N) < e(M)$. Then by Theorem 1, $e(N)$ is in A_2 and $e(M) = e(N) + 1$. Hence by Theorem 3, $e(M)$ is in A_{k_r} with k_r odd:

$$e(M) = F_{k_r} + F_{k_{r-1}} + \cdots \qquad (k_r \text{ odd}) .$$

Let

$$P = F_{k_r+1} + F_{k_{r-1}+1} + \cdots .$$

Now $e(P) = e(M)$, but P is in A_{k_r+1} so $P \neq M$. Hence, by Theorem 1 we must have $P = M + 1$. Hence k_r is odd, a contradiction. This proves the theorem.

Theorem 5. Let Q_j be the j^{th} largest number in A_2. Then

$$e(e(Q_j)) = j .$$

Proof. We can easily see by induction that there are exactly F_{n-1} numbers in A_2 whose canonical representations involve only $F_2, F_3, \cdots, F_n$, for let C_n be that set of numbers; i.e., $N \in C_n$ if and only if

$$N = F_2 + \cdots + F_{k_1} \qquad (k_1 \leq n) .$$

We want to show that card $(C_n) = F_{n-1}$ and that if $N \in C_n$, $N < F_{n+1}$. This is easily checked for small n. Suppose it is true up to n. Then

$$C_{n+1} = C_n \cup (C_{n-1} + F_{n+1}) .$$

Since this union is disjoint, by the induction hypothesis, the conclusion follows readily.

The point is that $1 + F_{n+1}$ $(n > 3)$ is the $(1 + F_{n-1})^{th}$ number in A_2. But

$$e(e(1 + F_{n+1})) = 1 + F_{n-1} ,$$

i.e., the value of $e(e(\cdot))$ on the $(1 + F_{n-1})^{th}$ number of A_2 is $1 + F_{n-1}$.

Hence, since $e(e(\cdot))$ is monotone and 1 - 1 on A_2 (Theorems 1 and 4), we see that $e(e(\cdot))$ simply counts the members of A_2; that is,

$$e(e(Q_j)) = j .$$

Now let N_i be defined by the requirements

$$(4.3) \qquad e(N_i) = i, \qquad e(N_i - 1) \neq i .$$

(Set $e(0) = 0$, so that $N_1 = 1$, $N_2 = 3$, etc.)

Theorem 6. For any N, $e(a(N)) = N$ and $e(b(N)) = a(n)$. The numbers $(N_1, N_2, \cdots)$ and $(Q_1 + 1, Q_2 + 1, \cdots)$ are the second and third rows of the array R_1.

Proof. Note that by Theorem 1, $e((Q_i + 1) - 1) = e(Q_i + 1)$ so that the sets $\{N_i\}$ and $\{Q_i + 1\}$ are disjoint. Furthermore, again by Theorem 1, together they exhaust all positive integers. Now to establish the theorem we only have to show property (2.4) of Section 2 and then that $e(Q_j + 1) = N_j$. Suppose for some j that the latter is false. Then, since

$$e(e(Q_j + 1)) = e(e(Q_j)) = j = e(N_j) ,$$

we must have

$$e(Q_j + 1) = N_j + 1$$

(since $e(N_j - 1) \neq j$, by (4.3)). Furthermore N_j must be in A_2. Therefore $e(Q_j + 1) \in A_{k_r}$, k_r odd, so that

$$e(Q_j + 1) = F_{k_r} + \cdots + F_{k_1} \qquad (k_r \text{ odd}) .$$

But then

$$Q_j + 1 = F_{k_r+1} + \cdots + F_{k_1+1} \qquad (k_r + 1 \text{ even}) .$$

Theorem 3 implies that Q_j is not in A_2, a contradiction. Hence $e(Q_j + 1) = N_j$.

Now suppose

$$N_j = F_{k_s} + F_{k_{s-1}} + \cdots + F_{k_1}$$

is the canonical representation of N_j. Then, since $Q_j + 1$ is not in A_2,

$$Q_j + 1 = F_{k_s+1} + F_{k_{s-1}+1} + \cdots + F_{k_1+1} ,$$

so that

(4.4) $$j + N_j = e(N_j) + N_j = Q_j + 1.$$

This proves the theorem.

Theorem 7. We have $A_2 = a^2(\mathbb{N})$ where $\mathbb{N}$ is the set of positive integers. Further,

(4.5) $$A_{2t+1} = b^t a(\mathbb{N}) \qquad (t = 1, 2, 3, \cdots)$$

and

(4.6) $$A_{2t} = ab^{t-1}a(\mathbb{N}) \qquad (t = 1, 2, 3, \cdots) .$$

Proof. We have seen that for any N,

$$e(b(N)) = e(a^2(N)) = a(N) .$$

Hence since $b(N) \neq a^2(N)$ and $Q_N + 1 = b(N)$, we get $Q_N = a^2(N)$. This shows that $A_2 = a^2(\mathbb{N})$. Now suppose N is in A_3. Then $e(N)$ is in A_2 and $e(N) = a^2(M)$ for some M. Hence N is either $ba(M)$ or $a^3(M)$. The latter is impossible since N is in A_3, not A_2. Hence $A_3 = ba(\mathbb{N})$.

Continuing in this way, we complete the proof of the theorem by induction.

5. SOME ADDITIONAL PROPERTIES

Since

(5.1) $$\mathbb{N} = a(\mathbb{N}) \cup b(\mathbb{N})$$

it follows from Theorem 7 that

(5.2) $$a(\mathbb{N}) = \bigcup_{t=1}^{\infty} A_{2t}$$

and

(5.3) $$b(\mathbb{N}) = \bigcup_{t=1}^{\infty} A_{2t+1} .$$

Again, by (5.1)

(5.4) $$a^2(\mathbb{N}) = a^2(\mathbb{N}) \cup a^2b(\mathbb{N}) .$$

By (3.2)

$$a^3(n) = ba(n) - 1 .$$

Since, by (4.5),

(5.5) $$ba(\mathbb{N}) = A_3 ,$$

it follows that

(5.6) $$a^3(\mathbb{N}) = A(2, \overline{4}) ,$$

where the right member denotes the set of positive integers with canonical representation

$$F_{k_1} + \cdots + F_{k_r} + F_2 \qquad (k_r > 4) .$$

Thus by (5.4), we have

(5.7) $$a^2b(\mathbb{N}) = A(2, 4) ,$$

where the right member denotes the set of positive integers with canonical representation

$$F_{k_1} + \cdots + F_{k_r} + F_4 + F_2 \qquad (k_r > 5) .$$

Generally if we let $A(s, s + 2)$ denote the set of positive integers with canonical representation

$$F_{k_1} + \cdots + F_{k_r} + F_{s+2} + F_s \qquad (k_r > s + 3)$$

and $A(s, \overline{s+2})$ the set with canonical representation

$$F_{k_1} + \cdots + F_{k_r} + F_s \qquad (k_r > s + 2)$$

then we may state

Theorem 8. For $t \geq 1$ we have

$$ab^{t-1}a^2(\mathbf{N}) = A(2t, 2t + 2) , \tag{5.8}$$

$$ab^{t-1}ab(\mathbf{N}) = A(2t, 2t + 2) , \tag{5.9}$$

$$b^t a^2(\mathbf{N}) = A(2t + 1, \overline{2t + 3}) , \tag{5.10}$$

$$b^t ab(\mathbf{N}) = A(2t + 1, 2t + 3) . \tag{5.11}$$

The proof is by induction on t. For $t = 1$, Eqs. (5.8) and (5.9) reduce to (5.6) and (5.7), respectively. Next by (5.5)

$$A_3 = ba^2(\mathbf{N}) \cup bab(\mathbf{N}) . \tag{5.12}$$

Let $n \in ba^2(\mathbf{N})$; then

$$e(n) \in a^3(\mathbf{N}) = A(2, \overline{4}) ,$$

that is,

$$e(n) = F_2 + \epsilon F_5 + \cdots ,$$

where $\epsilon = 0$ or 1. This implies either

$$n = F_2 + \epsilon F_6 + \cdots \qquad \text{or} \qquad F_3 + \epsilon F_6 + \cdots$$

The first possibility contradicts (5.3), so that

(5.13) $$ba^2(\mathbb{N}) \subset A(a, \overline{5}) .$$

Now take n bab($\mathbb{N}$), so that

$$e(n) \in a^2b(\mathbb{N}) = A(2, 4) ,$$

$$e(n) = F_2 + F_4 + \epsilon F_6 + \cdots .$$

This implies either

$$n = F_2 + F_5 + \epsilon F_7 + \cdots \quad \text{or} \quad F_3 + F_5 + \epsilon F_7 + \cdots .$$

The first possibility cannot occur, so that

(5.14) $$bab(\mathbb{N}) \subset A(3, 5) .$$

Clearly (5.13) and (5.14) prove (5.10) and (5.11) for $t = 1$.

We now assume that (5.8), $\cdots$, (5.11) hold up to and including the value $t - 1$. Let $n \in ab^{t-1}a^2(\mathbb{N})$, so that

$$e(n) \in b^{t-1}a^2(\mathbb{N}) .$$

By the inductive hypothesis this gives

$$e(n) \in A(2t - 1, \overline{2t + 1}) ,$$

that is,

$$e(n) = F_{2t-1} + \epsilon F_{2t+2} + \cdots .$$

This implies

$$n = F_{2t} + \epsilon F_{2t+3} + \cdots ,$$

so that

(5.15) $$ab^{t-1}a^2(\mathbf{N}) \subset A(2t, \overline{2t + 2}) .$$

Now take $n \in ab^{t-1}ab(N)$, so that

$$e(n) \in b^{t-1}ab(\mathbf{N}) .$$

Hence by the inductive hypothesis

$$e(n) \in A(2t - 1, 2t + 1) ,$$

that is,

$$e(n) = F_{2t-1} + F_{2t+1} + \epsilon F_{2t+3} + \cdots .$$

This implies

$$n = F_{2t} + F_{2t+2} = \epsilon F_{2t+4} + \cdots ,$$

so that

(5.16) $$ab^{t-1}ab(\mathbf{N}) \subset A(2t, 2t + 2) ,$$

In the next place, take $n \in b^t a^2(\mathbf{N})$, so that

$$e(n) \in ab^{t-1}a^2(\mathbf{N}) .$$

By (5.15) this gives

$$e(n) \in A(2t, \overline{2t + 2}) ,$$

that is,

$$e(n) = F_{2t} + \epsilon F_{2t+3} + \cdots .$$

Then either

$$n = F_{2t+1} + \epsilon F_{2t+4} + \cdots$$

or

$$n = F_2 + F_4 + \cdots + F_{2t} + \epsilon F_{2t+4} + \cdots .$$

The second possibility is ruled out, so that

$$ab^{t-1}a^2(\mathbb{N}) \subset A(2t+1, \overline{2t+3}) . \tag{5.17}$$

Finally take $n \in b^t ab(\mathbb{N})$, so that

$$e(n) \in ab^{t-1}ab(\mathbb{N}) .$$

Then by (5.16),

$$e(n) \in A(2t, 2t+2) ,$$

that is,

$$e(n) = F_{2t} + F_{2t+2} + \epsilon F_{2t+4} + \cdots .$$

Then either

$$n = F_{2t+1} + F_{2t+3} + \epsilon F_{2t+5} + \cdots$$

or

$$n = F_2 + F_4 + \cdots + F_{2t} + F_{2t+3} + \epsilon F_{2t+5} + \cdots .$$

Again the second possibility is ruled out, so that

$$b^t ab(\mathbb{N}) \in A(2t+1, 2t+3) . \tag{5.18}$$

Combining (5.15), (5.16), (5.17), (5.18), it is clear that we have completed the induction.

We define a function $\lambda(N)$ by means of $\lambda(1) = 0$ and $\lambda(N) = t$, where $N > 1$ and t is the smallest integer such that

$$e^t(N) = 1. \tag{5.19}$$

<u>Theorem 9.</u> Let

$$N = F_{k_1} + F_{k_2} + \cdots + F_{k_r}, \tag{5.20}$$

where

$$k_j - k_{j+1} \geq 2 \qquad (j = 1, \cdots, r - 1); \qquad k_r \geq 2,$$

be the canonical representation of N. Then

$$\lambda(N) = \begin{cases} k_r - 2 & (r = 1) \\ k_r - 1 & (r \geq 1) \end{cases} . \tag{5.21}$$

<u>Proof.</u>

1. $r = 1$. Clear.
2. $r = 2$, $N = F_{k_1} + F_{k_2}$.

$$e^{k_2-2}(N) = F_{k_1-k_2+2} + F_2$$

$$e^{k_r k_2 - 2} e^{k_2-2}(N) = F_4 + F_2$$

$$e^{k_1-3}(N) = F_3 + F_2 = F_4$$

$$e^{k_1-2}(N) = F_3$$

$$e^{k_1-1}(N) = F_2 = 1 .$$

3. $r > 2$. By induction.

Let Λ_t denote the set of positive integers N such that

$$\lambda(N) = t . \tag{5.22}$$

Theorem 10. Λ_t consists of the integers N such that

$$F_{t+1} < N \le F_{t+2} . \tag{5.23}$$

Thus

$$|\Lambda_t| = F_t . \tag{5.24}$$

Proof. Let N satisfy (4.22) and assume that N has the canonical representation (5.20). By (5.21) the value $N = F_{t+2}$ satisfies (5.22). For all other values of N, it is clear that $r > 1$. Moreover since

$$F_2 + F_4 + \cdots + F_{2s} = F_{2s+1} - 1,$$
$$F_3 + F_4 + \cdots + F_{2s-1} = F_{2s} - 1 ,$$

it is clear that N must satisfy

$$F_{t+1} < N < F_{t+2} . \tag{5.25}$$

Conversely all N that satisfy (5.25) are of the form (5.20) with $r > 1$. This evidently completes the proof.

Finally we state

Theorem 11. Let $\{x\} = x - [x]$ denote the fractional part of the real number x. Then

$$N \in a(\mathbf{N}) \rightleftarrows 0 < \left\{\frac{N}{\alpha^2}\right\} < \frac{1}{\alpha} \tag{5.26}$$

$$N \in b(\mathbf{N}) \rightleftarrows \frac{1}{\alpha} < \left\{\frac{N}{\alpha^2}\right\} < 1 . \tag{5.27}$$

Proof. We recall that

$$a(n) = [\alpha n], \qquad b(n) = [\alpha^2 n] .$$

Thus $N = b(n)$ is equivalent to

$$\alpha^2 n = N + \epsilon \qquad (0 < \epsilon < 1) ,$$

so that

$$\frac{N}{\alpha^2} = n - \frac{\epsilon}{\alpha^2} .$$

Thus

$$1 \geq \left\{\frac{N}{\alpha^2}\right\} = 1 - \frac{\epsilon}{\alpha^2} > 1 - \frac{1}{\alpha^2} = \frac{1}{\alpha} .$$

Conversely if

$$\frac{1}{\alpha} < \left\{\frac{N}{\alpha^2}\right\}$$

then

$$\frac{N}{\alpha^2} = m + \epsilon, \qquad \frac{1}{\alpha} < \epsilon < 1 .$$

Thus

$$N = \alpha^2 m + \alpha^2 \epsilon ,$$

so that

$$\alpha^2(m + 1) = N + \alpha^2(1 - \epsilon) .$$

Since

$$\alpha^2(1 - \epsilon) < \alpha^2\left(1 - \frac{1}{\alpha}\right) = \alpha - 1 < 1 ,$$

it follows that $b(m + 1) = N$.

This proves (5.27). The equivalence (5.26) follows from (5.27) since

$$a(\mathbb{N}) \cup b(\mathbb{N}) = \mathbb{N} .$$

6. WORD FUNCTIONS

By a word function (or briefly a word) is meant any monomial in the a's and b's. It is convenient to include 1 as a word. Clearly if u, v are any words, then $au \neq bv$. Also if $au = av$ or $bu = bv$ then $u = v$. It follows readily that any word is uniquely represented as a product of "primes" a, b.

We define the weight of a word by means of

(6.1) $$p(1) = 0, \qquad p(a) = 1, \qquad p(b) = 2$$

together with

(6.2) $$p(uv) = p(u) + p(v) ,$$

where u, v are arbitrary words. Thus there is exactly one word of weight 1, two of weight 2, and three of weight 3. Let N_p denote the number of words of weight p. If w is any word of weight p, then, for $p > 2$, $w = au$ or bv, where u is of weight $p - 1$, v of weight $p - 2$. Hence

$$N_p = N_{p-1} + N_{p-2} \qquad (p \geq 2) .$$

It follows that

(6.3) $$N_p = F_{p+1} \qquad (p \geq 0) ,$$

the number of words of weight p is equal to the Fibonacci number F_{p+1}.

Consider the equation

$$uv = vu . \tag{6.4}$$

We may assume without loss of generality that $p(u) \geq p(v)$. It then follows from the unique factorization property that $u = vz$, where z is some word. Thus $vzv = v^2z$, so that $zv = vz$. Thus by an easy induction on the total weight of uv we get the following theorem.

Theorem 12. The words u, v satisfy (6.4) if and only if there is a word w such that $u = w^r$, $v = w^s$, where r, s are nonnegative integers.

We show next that any word is "almost" linear. More precisely we prove

Theorem 13. Any word w of weight p is uniquely representable in the form

$$u(n) = F_p a(n) + F_{p-1} n - \lambda_u , \tag{6.5}$$

where λ_u is independent of n.

Proof. We have

$$b(n) = a(n) + n ,$$

$$a^2(n) = a(n) + n - 1 ,$$

$$ab(n) = 2a(n) + n ,$$

$$ba(n) = 2a(n) + n - 1 .$$

We accordingly assume the truth of (6.5) for words u of weight $< p$. There are two cases to consider. (i) if $u = va$, then v is of weight $p - 1$, so that (6.5) gives

$$v(n) = F_{p-1} a(n) + F_{p-2} n - \lambda_v .$$

Hence

$$u(n) = va(n) = F_{p-1}a^2(n) + F_{p-2}a(n) - \lambda_v$$
$$= F_p a(n) + F_{p-1}n - \lambda_v - F_{p-1} ,$$

(ii) if $u = vb$, v is of weight $p - 2$, so that

$$v(n) = F_{p-2}a(n) + F_{p-3}n - \lambda_v .$$

Then

$$\begin{aligned} u(n) = vb(n) &= F_{p-2}ab(n) + F_{p-3}b(n) - \lambda_v \\ &= F_{p-2}(2a(n) + n) + F_{p-3}(a(n) + n) - \lambda_v \\ &= (2F_{p-2} + F_{p-3})a(n) + (F_{p-2} + F_{p-3})n - \lambda_v \\ &= F_p a(n) + F_{p-1}n - \lambda_v . \end{aligned}$$

This completes the induction.

We now show that the representation (6.5) is unique. Otherwise there exist numbers r, s, t such that

$$ra(n) + sn = t .$$

Taking $n = 1, 2, 3$ we get

$$\begin{cases} r + s = t \\ 3r + 2s = t \\ 4r + 3s = t \end{cases}$$

and therefore $r = s = t = 0$.

Incidentally, we have proved that λ_u satisfies

$$(6.6) \qquad \lambda_{va} = \lambda_v + F_p, \qquad \lambda_{vb} = \lambda_v ,$$

where v is of weight p. Note that

$$\lambda_{vab} = \lambda_{va} = \lambda_v + F_p, \qquad \lambda_{vba} = \lambda_{vb} + F_{p+1} = \lambda_v + F_{p+1} .$$

Note also that (6.5) implies

$$\lambda_u = F_{p+1} - u(1) . \tag{6.7}$$

As an immediate corollary of Theorem 13 we have

Theorem 14. For arbitrary words, u, v, we have

$$uv - vu = C , \tag{6.8}$$

where C is independent of n.

It may be of interest to mention a few special cases of (6.5):

$$a^k(n) = F_k a(n) + F_{k-1}n - F_{k+1} + 1, \tag{6.9}$$

$$b^k(n) = F_{2k}a(n) + F_{2k-1}n , \tag{6.10}$$

$$b^k(n) = a^{2k}(n) + F_{2k+1} - 1 , \tag{6.11}$$

$$(ab)^k(n) = F_{3k}a(n) + F_{3k-1}n - \tfrac{1}{2}(F_{3k-1} - 1) , \tag{6.12}$$

$$(ba)^k(n) = F_{3k}a(n) + F_{3k-1}n - F_{3k-1} , \tag{6.13}$$

$$(ab)^k(n) - (ba)^k(n) = \tfrac{1}{2}(F_{3k-1} + 1) , \tag{6.14}$$

$$a^k b^j(n) = F_{2j+k}a(n) + F_{2j+k-1}n - F_{k+1} + 1 , \tag{6.15}$$

$$b^j a^k(n) = F_{2j+k}a(n) + F_{2j+k-1}n - F_{2j+k+1} + F_{2j+1} , \tag{6.16}$$

$$a^k b^j(n) - b^j a^k(n) = F_{2j+k+1} - F_{2j+1} - F_{k+1} + 1 . \tag{6.17}$$

7. GENERATING FUNCTIONS

Put

$$(7.1)\qquad \phi_j(x) = \sum_{n\in A_j} x^n \qquad (j = 2, 3, 4, \cdots) .$$

In view of (4.5) and (4.6), Eq. (5.1) is equivalent to

$$(7.2)\qquad \phi_{2r}(x) = \sum_{n=1}^{\infty} x^{ab^{r-1}a(n)}$$

and

$$(7.3)\qquad \phi_{2r+1}(x) = \sum_{n=1}^{\infty} x^{b^r a(n)} .$$

Also it is clear that

$$(7.4)\qquad \frac{x}{1-x} = \sum_{j=0}^{\infty} \phi_j(x) .$$

It follows from the definition of A_r that

$$(7.5)\qquad \phi_r(x) = x^{F_r}\left\{1 + \sum_{j=r+2}^{\infty} \phi_j(x)\right\} \qquad (r = 2, 3, 4, \cdots) .$$

This evidently implies

$$(7.6)\qquad x^{-F_r}\phi_r(x) - x^{-F_{r+1}}\phi_{r+1}(x) = \phi_{r+2}(x) \qquad (r = 2, 3, 4, \cdots) .$$

In particular, by (7.5),

$$\phi_2(x) = x\left\{1 + \sum_{j=4}^{\infty} \phi_j(x)\right\} .$$

Combining this with (5.4), we get

$$(7.7) \qquad (1 + x)\phi_2(x) + x\phi_3(x) = \frac{x}{1 - x} .$$

It is convenient to define

$$(7.8) \qquad \phi(x) = \sum_{n=1}^{\infty} x^{a(n)} .$$

Since the set $a(\mathbf{N})$ is the union of the sets $a^2(\mathbf{N})$ and $ab(\mathbf{N})$, it follows from (3.4) that

$$(7.9) \qquad \phi(x) = \phi_2(x) + x\phi_3(x) .$$

Therefore by (7.7), we have

$$(7.10) \qquad x\phi_2(x) = \frac{x}{1 - x} - \phi(x)$$

and

$$(7.11) \qquad x^2\phi_3(x) = \frac{x}{1 - x} + (1 + x)\phi(x) .$$

Making use of (7.5), (7.10) and (7.11) we can express all $\phi_j(x)$ in terms of $\phi(x)$. For example, since

$$x^{-1}\phi_2(x) - x^{-2}\phi_3(x) = \phi_4(x) ,$$

we get

$$x^4\phi_4(x) = \frac{x + x^3}{1 - x} - (1 + x + x^2)\phi(x) . \tag{7.12}$$

Generally we have

$$x^{F_{r+1}-1}\phi_r(x) = (-1)^r\left\{\frac{xA_r(x)}{1 - x} - B_r(x)\phi(x)\right\} , \tag{7.13}$$

where $A_r(x)$, $B_r(x)$ are polynomials that satisfy

$$\begin{cases} A_{r+2}(x) = A_{r+1}(x) + x^{F_{r+1}}A_r(x) \\ B_{r+2}(x) = B_{r+1}(x) + x^{F_{r+1}}B_r(x) \end{cases} \tag{7.14}$$

together with the initial conditions

$$\begin{cases} A_2(x) = 1, & A_3(x) = 1, \\ B_2(x) = 1, & B_3(x) = 1 + x . \end{cases}$$

It follows readily that

$$B_r(x) = \frac{1 - x^{F_r}}{1 - x} \tag{7.15}$$

while

$$xA_r(x) = \sum_{j=1}^{F_{r-1}} x^{a(j)} . \tag{7.16}$$

In conclusion we shall show that the function $\phi(x)$ cannot be continued across the unit circle. Indeed by a known theorem [1, p. 315], either $\phi(x)$ is rational or it has the unit circle for a natural boundary. Moreover, it is rational if and only if, for some positive integer m, (7.17) $$(1 - x^m)\phi(x) = P(x) ,$$

where P(x) is a polynomial. Clearly the coefficients of P(x) are rational integers. It follows that

(7.18) $$\lim_{x = 1} (1 - x)\phi(x) = C,$$

where C is rational. On the other hand, if we put

$$\phi(x) = \sum_{k=1}^{\infty} c_k x^k ,$$

so that $c_k = 0$ or 1, it is evident from (7.8) that

$$\sum_{k=1}^{n} c_k \sim \frac{n}{\alpha} .$$

Since this implies

$$\lim_{x = 1} (1 - x)\phi(x) = \frac{1}{\alpha}$$

we have a contradiction with (7.18).

8. APPENDIX

In addition to the canonical representation (1.1) we have another representation described in the following

Theorem 15. Every integer N is uniquely represented in the form

(8.1) $$N = F_{k_1} + \cdots + F_{k_r} + F_{2k+1} \qquad (k \geq 0) ,$$

where

$$k_j - k_{j+1} \geq 2 \quad (j = 1, \cdots, r-1), \qquad k_r - (2k+1) \geq 2 .$$

Proof. By (5.2),

$$a(\mathbb{N}) = \bigcup_{t=1}^{\infty} A_{2t} . \tag{8.2}$$

Hence, by the first proof of Theorem 6,

$$\mathbb{N} = \bigcup_{t=1}^{\infty} A_{2t-1} .$$

This evidently proves the theorem.

We may refer to (8.1) as the second canonical representation of N.

In view of Theorem 15, we let $\overline{A}_{2k+1}$ denote the set of positive integers $\{N\}$ of the form (8.1). Then the sets

$$\overline{A}_{2k+1} \qquad (k = 0, 1, 2, \cdots)$$

constitute a partition of the positive integers. Clearly

$$\overline{A}_{2k+1} = A_{2k+1} \qquad (k = 1, 2, 3, \cdots) , \tag{8.3}$$

while

$$\overline{A}_1 = \bigcup_{t=1}^{\infty} A_{2t} = a(\mathbb{N}). \tag{8.4}$$

For $N \in \overline{A}_1$, if

$$N = F_{k_1} + \cdots + F_{k_r} + F_1 \tag{8.5}$$

$$k_j - k_{j+1} \geq 2 \qquad (j = 1, \cdots, r-1), \qquad k_r > 3 ,$$

then clearly we may replace F_1 by F_2 and (8.5) reduces to the first canonical representation. In this case, then, $N \in A_2$. However, if $k_r = 3$, the situation is less simple. For example

$$8 = F_6 = F_5 + F_3 + F_1 .$$

Generally, since

$$F_1 + F_3 + F_5 + \cdots + F_{2s-1} = F_{2s} ,$$

it follows that if the number N has the second canonical representation

$$N = F_1 + F_3 + \cdots + F_{2s-1} + F_{k_1} + F_{k_2} + \cdots ,$$

where

$$k_{j+1} - k_j \geq 2 \qquad (j \geq 1), \qquad k_1 \geq 2s + 2 ,$$

then $N \in A_{2s}$ and conversely.

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