ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
FLORIAN LUCA

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG X3, WITS 2050, JOHANNESBURG, SOUTH AFRICA or by e-mail at the address florian.luca@wits.ac.za as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-813 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania

If $x_k > 0$ for $k = 1, \ldots, n$ and $m \geq 0$ is an integer, prove that

$$\left(\sum_{k=1}^{n} \frac{1}{x_k}\right) \sum_{\text{cyclic}} \frac{x_1 x_2 x_3}{L_m x_2 x_3 + L_{m+1} x_3 x_1 + L_{m+2} x_1 x_2} \geq \frac{n^2}{2L_{m+2}}$$

and that the same inequality holds with the Lucas numbers replaced by the Fibonacci numbers.

H-814 Proposed by Ray Melham, Sydney, Australia

Define the Tribonacci numbers, for all integers n, by $T_n = T_{n-1} + T_{n-2} + T_{n-3}$, with $T_{-1} = 0$, $T_0 = 0$, and $T_1 = 1$. If k and n are integers, prove that

$$\begin{aligned} -T_{2k}T_{n-2}^2 - T_{2k-2}T_{n-1}^2 - 2T_{2k-1}T_n^2 + 2(T_{2k} + T_{2k+1})T_{n+1}^2 &+ (T_{2k} + 2T_{2k+1})T_{n+2}^2 \\ &+ T_{2k+2}T_{n+3}^2 = 2T_{2n+2k+4}. \end{aligned}$$

H-815 Proposed by Mehtaab Sawhney, Commack, NY

Let p be a prime congruent to 1 modulo 4. Prove that

$$\sum_{n=0}^{p-1} 2^n \binom{3n}{n} \equiv 0 \pmod{p}.$$

H-816 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania

Prove that for a positive integer n

$$\frac{F_1}{(F_1^2+F_2^2)^2}+\frac{F_2}{(F_1^2+F_2^2+F_3^2)^2}+\cdots+\frac{F_n}{(F_1^2+F_2^2+\cdots+F_{n+1}^2)^2}\geq\frac{1}{F_{n+2}}-\frac{1}{F_{n+2}^2}.$$

SOLUTIONS

An identity with Fibonomial coefficients

H-779 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 53, No. 4, November 2015)

Let $\binom{n}{k}_F$ denote the Fibonomial coefficient. For integers $n\geq 1$ and $r\neq 0$ with $n+r\neq 0$, prove that

$$\sum_{k=0}^{n}(-1)^{k(k+1)/2}F_{k+r}\left(\frac{F_r}{F_{n+r}}\right)^k\binom{n}{k}_F=0.$$

Solution by the proposer

It is known that

$$\sum_{k=0}^{n}(-1)^{k(k+1)/2}\binom{n}{k}_F x^k=\prod_{k=0}^{n-1}(1-\alpha^{n-k-1}\beta^k x) \tag{1}$$

(see [1]). Let $c=F_r/F_{n+r}$. We have

$$\begin{aligned}
\sum_{k=0}^{n}(-1)^{k(k+1)/2}F_{k+r}c^k\binom{n}{k}_F &= \sum_{k=0}^{n}(-1)^{k(k+1)/2}\frac{\alpha^r(c\alpha)^k-\beta^r(c\beta)^k}{\sqrt{5}}\binom{n}{k}_F\\
&=\frac{\alpha^r}{\sqrt{5}}\prod_{k=0}^{n-1}(1-c\alpha^{n-k}\beta^k)-\frac{\beta^r}{\sqrt{5}}\prod_{k=0}^{n-1}(1-c\alpha^{n-k-1}\beta^{k+1}) \qquad \text{(by (1))}\\
&=\frac{\alpha^r}{\sqrt{5}}\prod_{k=0}^{n-1}(1-c\alpha^{n-k}\beta^k)-\frac{\beta^r}{\sqrt{5}}\prod_{k=1}^{n}(1-c\alpha^{n-k}\beta^k)\\
&=\frac{1}{\sqrt{5}}(\alpha^r(1-c\alpha^n)-\beta^r(1-c\beta^n))P(n),
\end{aligned}$$

where $P(1)=1$ and $P(n)=\prod_{k=1}^{n-1}(1-c\alpha^{n-k}\beta^k)$ for $n\geq 2$.

Here, we have

$$\begin{aligned}
&\alpha^r(1-c\alpha^n)-\beta^r(1-c\beta^n)=\alpha^r-c\alpha^{r+n}-\beta^r+c\beta^{r+n}\\
&=\sqrt{5}(F_r-cF_{n+r})=\sqrt{5}(F_r-F_r)=0.
\end{aligned}$$

Therefore, we obtain the desired identity.

Note: In the same manner, for integers $n\geq 1$ and r, we have

$$\sum_{k=0}^{n}(-1)^{k(k+1)/2}L_{k+r}\left(\frac{L_r}{L_{n+r}}\right)^k\binom{n}{k}_F=0.$$

[1] L. Carlitz, *The characteristic polynomial of a certain matrix of binomial coefficients*, The Fibonacci Quarterly, **3.2** (1965), 81–89.

A closed form for a certain sum

H-780 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 53, No. 4, November 2015)

Given real numbers r and $t > 0$ and an integer $n \geq 0$, find a closed form expression for the sum:

$$\sum_{k=0}^{n} \frac{1}{f_k(L_{2^k}^r + t)(L_{2^{k+1}}^r + t)\cdots(L_{2^n}^r + t)},$$

where $f_0 = t/(t+1)$ and $f_k = F_{2^{k+1}}^r$ for $k \geq 1$.

Solution by the proposer

We find the identity

$$\sum_{k=0}^{n} \frac{1}{f_k(L_{2^k}^r + t)(L_{2^{k+1}}^r + t)\cdots(L_{2^n}^r + t)} = \frac{1}{tF_{2^{n+1}}^r}. \tag{2}$$

The proof of (2) is by mathematical induction on n. For $n = 0$, both sides are equal to $1/t$. Assume that (2) holds for n. For $n + 1$, we have

$$\begin{aligned}
&\sum_{k=0}^{n+1} \frac{1}{f_k(L_{2^k}^r + t)(L_{2^{k+1}}^r + t)\cdots(L_{2^{n+1}}^r + t)} \\
&= \frac{1}{f_{n+1}(L_{2^{n+1}}^r + t)} + \frac{1}{(L_{2^{n+1}}^r + t)} \sum_{k=0}^{n} \frac{1}{f_k(L_{2^k}^r + t)(L_{2^{k+1}}^r + t)\cdots(L_{2^n}^r + t)} \\
&= \frac{1}{F_{2^{n+2}}^r(L_{2^{n+1}}^r + t)} + \frac{1}{(L_{2^{n+1}}^r + t)} \times \frac{1}{tF_{2^{n+1}}^r} \\
&= \frac{F_{2^{n+2}}^r + tF_{2^{n+1}}^r}{tF_{2^{n+1}}^r F_{2^{n+2}}^r (L_{2^{n+1}}^r + t)} = \frac{F_{2^{n+1}}^r(L_{2^{n+1}}^r + t)}{tF_{2^{n+1}}^r F_{2^{n+2}}^r (L_{2^{n+1}}^r + t)} = \frac{1}{tF_{2^{n+2}}^r}.
\end{aligned}$$

Thus, (2) holds for $n + 1$.

Also solved by Dmitry Fleischman.

More closed form expressions

H-781 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 53, No. 4, November 2015)

Find a closed form expression for the sums:

(i) $\displaystyle\sum_{k=1}^{n}(L_{2^k} \pm \sqrt{5})(L_{2^{k+1}} \pm \sqrt{5})\cdots(L_{2^n} \pm \sqrt{5})$ for $n \geq 1$;

(ii) $\displaystyle\sum_{k=m+1}^{n}(L_{2^k} \pm L_{2^m})(L_{2^{k+1}} \pm L_{2^m})\cdots(L_{2^n} \pm L_{2^m})$ for $n > m \geq 1$.

Solution by the proposer

We use the identity

$$L_m^2 = L_{2m} + 2(-1)^m \qquad \text{(see [1](17}c\text{))}. \tag{3}$$

For $n \geq 1$, we have

$$x^2 + x - 2 + (L_{2^n} - x)(L_{2^n} + x) = L_{2^n}^2 + x - 2 = L_{2^{n+1}} + x \qquad \text{(by (3))}.$$

If $a_n = L_{2^n} - x,\ b_n = L_{2^n} + x$, and $c = x^2 + x - 2$, then we have $b_{n+1} = c + a_n b_n$. Using this identity repeatedly for $n \geq m+2 \geq 2$, we have

$$\begin{aligned} b_{n+1} &= c + a_n b_n = c + a_n(c + a_{n-1}b_{n-1}) = \cdots \\ &= c + a_n(c + a_{n-1}(c + a_{n-2}(c + \cdots a_{m+2}(c + a_{m+1}b_{m+1})\cdots))) \\ &= c + \sum_{k=m+2}^{n} c \prod_{j=k}^{n} a_j + b_{m+1} \prod_{j=m+1}^{n} a_j. \end{aligned}$$

Therefore, we obtain

$$(x^2 + x - 2) \sum_{k=m+2}^{n} \prod_{j=k}^{n} (L_{2^j} - x) + (L_{2^{m+1}} + x) \prod_{j=m+1}^{n} (L_{2^j} - x) = L_{2^{n+1}} - x^2 + 2. \qquad (4)$$

(i) If $m = 0$ and $x = \mp\sqrt{5}$ in (4), for $n \geq 2$, we have

$$(3 \mp \sqrt{5}) \sum_{k=2}^{n} \prod_{j=k}^{n} (L_{2^j} \pm \sqrt{5}) + (3 \mp \sqrt{5}) \prod_{j=1}^{n} (L_{2^j} \pm \sqrt{5}) = L_{2^{n+1}} - 3.$$

Therefore, we obtain

$$\sum_{k=1}^{n} \prod_{j=k}^{n} (L_{2^j} \pm \sqrt{5}) = \frac{L_{2^{n+1}} - 3}{3 \mp \sqrt{5}}.$$

This identity holds also for $n = 1$, since then,

$$RHS = \frac{L_4 - 3}{3 \mp \sqrt{5}} = 3 \pm \sqrt{5} = L_2 \pm \sqrt{5} = LHS.$$

(ii) If $m \geq 1$ and $x = \mp L_{2^m}$ in (4), for $n \geq m+2$, we have

$$\begin{aligned} &(L_{2^m}^2 \mp L_{2^m} - 2) \sum_{k=m+2}^{n} \prod_{j=k}^{n} (L_{2^j} \pm L_{2^m}) + (L_{2^{m+1}} \mp L_{2^m}) \prod_{j=m+1}^{n} (L_{2^j} \pm L_{2^m}) \\ &= L_{2^{n+1}} - L_{2^m}^2 + 2. \end{aligned}$$

Using (3), we have

$$(L_{2^{m+1}} \mp L_{2^m}) \sum_{k=m+1}^{n} \prod_{j=k}^{n} (L_{2^j} \pm L_{2^m}) = L_{2^{n+1}} - L_{2^{m+1}}.$$

Therefore, we obtain

$$\sum_{k=m+1}^{n} \prod_{j=k}^{n} (L_{2^j} \pm L_{2^m}) = \frac{L_{2^{n+1}} - L_{2^{m+1}}}{L_{2^{m+1}} \mp L_{2^m}}.$$

The identity holds for $n = m+1$ as well, since then,

$$RHS = \frac{L_{2^{m+2}} - L_{2^{m+1}}}{L_{2^{m+1}} \mp L_{2^m}} = \frac{L_{2^{m+1}}^2 - L_{2^m}^2}{L_{2^{m+1}} \mp L_{2^{m+1}}} = L_{2^{m+1}} \pm L_{2^m} = RHS,$$

where in the above chain of equalities we used (3).

[1] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section*, Dover, 2008.

And yet more closed form formulas

H-782 **Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 54, No. 1, February 2016)**

Given positive integers r and s, find formulas for the sums

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{srn}}{\alpha^{(s-1)rn} F_{rn} F_{r(n+1)} F_{r(n+2)} \cdots F_{r(n+s)}}$;

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{srn}}{\alpha^{(s-1)rn} L_{rn} L_{r(n+1)} L_{r(n+2)} \cdots L_{r(n+s)}}$.

Solution by the proposer

(i) We have

$$\begin{aligned}
&\frac{\beta^{srn}}{F_{rn}F_{r(n+1)}\cdots F_{r(n+s-1)}} - \frac{\beta^{sr(n+1)}}{F_{r(n+1)}F_{r(n+2)}\cdots F_{r(n+s)}} = \frac{\beta^{srn}(F_{r(n+s)} - \beta^{sr}F_{rn})}{F_{rn}F_{r(n+1)}\cdots F_{r(n+s)}} \\
&= \frac{(-\alpha^{-1})^{srn}(\alpha^{r(n+s)} - \beta^{r(n+s)} - \beta^{sr}(\alpha^{rn} - \beta^{rn}))}{\sqrt{5}F_{rn}F_{r(n+1)}\cdots F_{r(n+s)}} = \frac{(-1)^{srn}\alpha^{rn}(\alpha^{sr} - \beta^{sr})}{\sqrt{5}\alpha^{srn}F_{rn}F_{r(n+1)}\cdots F_{r(n+s)}} \\
&= \frac{(-1)^{srn}F_{sr}}{\alpha^{(s-1)rn}F_{rn}F_{r(n+1)}\cdots F_{r(n+s)}}.
\end{aligned}$$

Using the above identity, we have

$$\begin{aligned}
\sum_{n=1}^{m} \frac{(-1)^{srn}}{\alpha^{(s-1)rn}\prod_{i=n}^{n+s} F_{ri}} &= \frac{1}{F_{sr}} \sum_{n=1}^{m} \left(\frac{\beta^{srn}}{\prod_{i=n}^{n+s-1} F_{ri}} - \frac{\beta^{sr(n+1)}}{\prod_{i=n+1}^{n+s} F_{ri}} \right) \\
&= \frac{1}{F_{sr}} \left(\frac{\beta^{sr}}{\prod_{i=1}^{s} F_{ri}} - \frac{\beta^{sr(m+1)}}{\prod_{i=m+1}^{m+s} F_{ri}} \right).
\end{aligned}$$

Therefore, we obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^{srn}}{\alpha^{(s-1)rn}F_{rn}F_{r(n+1)}F_{r(n+2)}\cdots F_{r(n+s)}} = \frac{\beta^{sr}}{F_{sr}(F_r F_{2r} F_{3r}\cdots F_{sr})}.$$

(ii) We have

$$\begin{aligned}
&\frac{\beta^{srn}}{L_{rn}L_{r(n+1)}\cdots L_{r(n+s-1)}} - \frac{\beta^{sr(n+1)}}{L_{r(n+1)}L_{r(n+2)}\cdots L_{r(n+s)}} = \frac{\beta^{srn}(L_{r(n+s)} - \beta^{sr}L_{rn})}{L_{rn}L_{r(n+1)}\cdots L_{r(n+s)}} \\
&= \frac{(-\alpha^{-1})^{srn}(\alpha^{r(n+s)} + \beta^{r(n+s)} - \beta^{sr}(\alpha^{rn} + \beta^{rn}))}{L_{rn}L_{r(n+1)}\cdots L_{r(n+s)}} = \frac{(-1)^{srn}\alpha^{rn}(\alpha^{sr} - \beta^{sr})}{\alpha^{srn}L_{rn}L_{r(n+1)}\cdots L_{r(n+s)}} \\
&= \frac{(-1)^{srn}\sqrt{5}F_{sr}}{\alpha^{(s-1)rn}L_{rn}L_{r(n+1)}\cdots L_{r(n+s)}}.
\end{aligned}$$

Using the above identity, we have

$$\begin{aligned}
\sum_{n=1}^{m} \frac{(-1)^{srn}}{\alpha^{(s-1)rn}\prod_{i=n}^{n+s} L_{ri}} &= \frac{1}{\sqrt{5}F_{sr}} \sum_{n=1}^{m} \left(\frac{\beta^{srn}}{\prod_{i=n}^{n+s-1} L_{ri}} - \frac{\beta^{sr(n+1)}}{\prod_{i=n+1}^{n+s} L_{ri}} \right) \\
&= \frac{1}{\sqrt{5}F_{sr}} \left(\frac{\beta^{sr}}{\prod_{i=1}^{s} L_{ri}} - \frac{\beta^{sr(m+1)}}{\prod_{i=m+1}^{m+s} L_{ri}} \right).
\end{aligned}$$

Therefore, we obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^{srn}}{\alpha^{(s-1)rn} L_{rn} L_{r(n+1)} L_{r(n+2)} \cdots L_{r(n+s)}} = \frac{\beta^{sr}}{\sqrt{5} F_{sr} (L_r L_{2r} L_{3r} \cdots L_{sr})}.$$

Example. If $s = 4$ and $r = 1$, then we have

$$\sum_{n=1}^{\infty} \frac{1}{\alpha^{3n} F_n F_{n+1} F_{n+2} F_{n+3} F_{n+4}} = \frac{7 - 3\sqrt{5}}{36};$$

$$\sum_{n=1}^{\infty} \frac{1}{\alpha^{3n} L_n L_{n+1} L_{n+2} L_{n+3} L_{n+4}} = \frac{-15 + 7\sqrt{5}}{2520}.$$

Also solved by Dmitry Fleischman.