ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG X3, WITS 2050, JOHANNESBURG, SOUTH AFRICA or by e-mail at the address florian.luca@wits.ac.za as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-801 **Proposed by Refik Keskin, Sakarya University, Turkey and Florian Luca, Wits, Johannesburg, South Africa.**

Let $P \geq 3$ be an integer and $(V_n)_{n\geq 0}$ be the sequence given by $V_0 = 2$, $V_1 = P$ and $V_{n+2} = PV_{n+1} - V_n$ for $n \geq 0$. Assume that $3 \nmid n$. Show that:

(i) $P + 1 \mid V_n + 1$;

(ii) If $V_n + 1 = (P + 1)F(P)$, then $F(-1) = n$ if $n \equiv 1 \pmod 3$ and $F(-1) = -n$ if $n \equiv 2 \pmod 3$.

H-802 **Proposed by Hideyuki Ohtsuka, Saitama, Japan.**

Let a, b, c, d be positive integers such that $a \geq b$, $c \geq d$ and b and d have the same parity. Then for all integers $n \geq 1$, prove that

$$\left(\sum_{k=1}^{n} F_{F_k+a}F_{L_k+c}\right)\left(\sum_{k=1}^{n} F_{F_k+b}F_{L_k+d}\right) \geq \left(\sum_{k=1}^{n} F_{F_k+a}F_{L_k+d}\right)\left(\sum_{k=1}^{n} F_{F_k+b}F_{L_k+c}\right).$$

H-803 **Proposed by Ángel Plaza, Gran Canaria, Spain.**

Assume that the consecutive numbers in the Lucas sequence are coordinates of the vertices of a polygon labeled counterclockwise in the Cartesian system:

$$A_1(L_1, L_2);\ A_2(L_3, L_4),\ A_3(L_5, L_6); \ldots ; A_n(L_{2n-1}, L_{2n}).$$

What is the area of such a polygon?

H-804 **Proposed by Hideyuki Ohtsuka, Saitama, Japan.**

Prove that

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\alpha^{n(n-3)} L_2 L_4 L_6 \cdots L_{2n}} = 1;$

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\alpha^{2F_{n-1}} L_{2F_1} L_{2F_2} L_{2F_3} \cdots L_{2F_n}} = \frac{1}{\alpha^2};$

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\alpha^{2L_{n-1}} L_{2L_1} L_{2L_2} L_{2L_3} \cdots L_{2L_n}} = \frac{1}{\alpha^6}.$

SOLUTIONS

Nested Radicals and Fibonacci Numbers

H-767 **Proposed by H. Ohtsuka, Saitama, Japan.**
(Vol. 53, No. 1, February 2015)

Prove that

$$\lim_{n\to\infty} \sqrt{F_2^2 + \sqrt{F_4^2 + \sqrt{F_8^2 + \sqrt{\cdots + \sqrt{F_{2^n}^2}}}}} = 3.$$

Solution by the proposer.

We use Catalan's identity

$$F_m^2 - F_{m+r}F_{m-r} = (-1)^{m-r} F_r^2.$$

Letting $m = 2^n + 2$ and $r = 2^n$ in the above identity we have

$$F_{2^n+2}^2 - F_{2^{n+1}+2} = F_{2^n}^2.$$

That is

$$F_{2^n+2} = \sqrt{F_{2^n}^2 + F_{2^{n+1}+2}}.$$

Using this identity repeatedly we get

$$3 = F_{2^1+2} = \sqrt{F_{2^1}^2 + F_{2^2+2}} = \sqrt{F_{2^1}^2 + \sqrt{F_{2^2}^2 + F_{2^3+2}}}$$

$$= \cdots = \sqrt{F_2^2 + \sqrt{F_4^2 + \sqrt{\cdots \sqrt{F_{2^{n-1}}^2 + \sqrt{F_{2^n}^2 + F_{2^{n+1}+2}}}}}}.$$

So, by the above argument, we have for all n that

$$\sqrt{F_2^2 + \sqrt{F_4^2 + \sqrt{F_8^2 + \sqrt{\cdots + \sqrt{F_{2^n}}^2}}}} < 3.$$

To see that the limit is in fact 3, let $\varepsilon \in (0,3)$ and let $s = 1 - \varepsilon/3$. We then have

$$3 - \varepsilon = 3s = s\sqrt{F_2^2 + \sqrt{F_4^2 + \sqrt{\cdots \sqrt{F_{2^{n-1}}^2 + \sqrt{F_{2^n}^2 + F_{2^{n+1}+2}}}}}}$$

$$= \sqrt{s^2F_2^2 + \sqrt{s^4F_4^2 + \sqrt{\cdots \sqrt{s^{2^{n-1}}F_{2^{n-1}}^2 + \sqrt{s^{2^n}(F_{2^n}^2 + F_{2^{n+1}+2})}}}}}$$

$$< \sqrt{F_2^2 + \sqrt{F_4^2 + \sqrt{\cdots \sqrt{F_{2^{n-1}}^2 + \sqrt{s^{2^n}(F_{2^n}^2 + F_{2^{n+1}+2})}}}}},$$

where for the last inequality we used the fact that $s \in (0,1)$. We have

$$\frac{F_{2^{n+1}+2}}{F_{2^n}^2} = \sqrt{5} \times \frac{\alpha^{2^{n+1}+2} - \beta^{2^{n+1}+2}}{\alpha^{2^{n+1}} + \beta^{2^{n+1}} - 2} = \sqrt{5} \times \frac{\alpha^2 - \beta^{2^{n+2}+2}}{1 + \beta^{2^{n+2}} - 2\beta^{2^{n+1}}} \to \sqrt{5}\alpha^2$$

as $n \to \infty$. Therefore,

$$\frac{F_{2^n}^2}{F_{2^n}^2 + F_{2^{n+1}+2}} \to \frac{1}{1 + \sqrt{5}\alpha^2} \qquad \text{as} \quad n \to \infty.$$

Since $s \in (0,1)$, there is N such that for $n > N$ we have

$$s^{2^n} < \frac{F_{2^n}^2}{F_{2^n}^2 + F_{2^{n+1}+2}}.$$

Thus, for $n > N$, we have

$$\sqrt{F_2^2 + \sqrt{F_4^2 + \sqrt{\cdots + \sqrt{F_{2^n}^2}}}} > \sqrt{F_2^2 + \sqrt{F_4^2 + \sqrt{\cdots + \sqrt{s^{2^n}(F_{2^{n+1}+2} + F_{2^n}^2)}}}}$$
$$> 3 - \varepsilon.$$

Since ε was arbitrary, the conclusion follows.

Editor's comment: The proposer also conjectured that for $c > 0$ we have

$$\lim_{n\to\infty} \sqrt{cF_2^2 + \sqrt{cF_4^2 + \sqrt{cF_8^2 + \sqrt{\cdots + \sqrt{cF_{2^n}^2}}}}} = \frac{3 + \sqrt{4c+5}}{2}.$$

Also solved by Dmitry Fleischman.

<u>**Summation Formulas for Reciprocals of Fibonomials with Fibonacci Coefficients**</u>

<u>**H-768**</u> **Proposed by H. Ohtsuka, Saitama, Japan.**
(Vol. 53, No. 1, February 2015)

Let $\binom{n}{k}_F$ denote the Fibonomial coefficient. For $n \geq 1$, prove that

(i) $\displaystyle\sum_{k=0}^{n} F_{2(n-k)} \binom{2n}{k}_F^{-1} = \frac{F_{2n+1}(F_{2n+2}+1)}{F_{2n+3}} - \frac{F_{n+1}F_{n+3}}{F_{2n+3}} \binom{2n}{n}_F^{-1};$

(ii) $\displaystyle\sum_{k=0}^{n} F_{2(n-k)} \binom{2n}{k}_F^{-2} = \frac{F_{2n+1}^2}{F_{2n+2}} - \frac{F_{n+1}}{L_{n+1}} \binom{2n}{n}_F^{-2}.$

Solution by the proposer.

We use the following identity: For $a+b=c+d$, we have

$$F_aF_b - F_cF_d = (-1)^r(F_{a-r}F_{b-r} - F_{c-r}F_{d-r}) \tag{1}$$

(see [1]). Let s be an even integer.

(i) For $s > n \geq 0$, we show that

$$\sum_{k=0}^{n} \frac{F_{s-2k}}{\binom{s}{k}_F} = \frac{F_{s+1}(F_{s+2}+1)}{F_{s+3}} - \frac{F_{n+1}(F_{s-n+1}+F_{n+2})}{F_{s+3}\binom{s}{n}_F}. \tag{2}$$

The proof is by mathematical induction on n.

- For $n=0$ we have

$$LHS - RHS = F_s - \frac{F_{s+1}F_{s+2}-1}{F_{s+3}} = \frac{F_sF_{s+3} - F_{s+1}F_{s+2}+1}{F_{s+3}} = \frac{(-1)^s(F_0F_3 - F_1F_2)+1}{F_{s+3}}$$

(by (1)), and this last expression is 0.

- We assume that (2) holds for n. For $n+1$, we have

$$\begin{aligned}
&\sum_{k=0}^{n+1} \frac{F_{s-2k}}{\binom{s}{k}_F} - \frac{F_{s+1}(F_{s+2}+1)}{F_{s+3}} \\
&= \frac{F_{s-2(n+1)}}{\binom{s}{n+1}_F} + \sum_{k=0}^{n} \frac{F_{s-2k}}{\binom{s}{k}_F} - \frac{F_{s+1}(F_{s+2}+1)}{F_{s+3}} \\
&= \frac{F_{s-2n-2}}{\binom{s}{n+1}_F} - \frac{F_{n+1}(F_{s-n+1}+F_{n+2})}{F_{s+3}\binom{s}{n}_F} \\
&= \frac{F_{s-2n-2}}{\binom{s}{n+1}_F} - \frac{F_{n+1}(F_{s-n+1}+F_{n+2})}{F_{s+3}\times\frac{F_{n+1}}{F_{s-n}}\binom{s}{n+1}_F} \\
&= \frac{F_{s-2n-2}F_{s+3} - F_{s-n}F_{s-n+1} - F_{s-n}F_{n+2}}{F_{s+3}\binom{s}{n+1}_F} \\
&= \frac{(-1)^{s-2n-2}(F_0F_{2n+5} - F_{n+2}F_{n+3}) - F_{s-n}F_{n+2}}{F_{s+3}\binom{s}{n+1}_f} \qquad \text{(by (1))} \\
&= \frac{-F_{n+2}(F_{s-n}+F_{n+3})}{F_{s+3}\binom{n}{s+1}_F}.
\end{aligned}$$

Thus, (2) holds for $n+1$. Therefore, (2) is proved. Letting $s=2n$ in (2) for $n \geq 1$, we have

$$\sum_{k=0}^{n} \frac{F_{2(n-k)}}{\binom{2n}{k}_F} = \frac{F_{2n+1}(F_{2n+2}+1)}{F_{2n+3}} - \frac{F_{n+1}(F_{n+1}+F_{n+2})}{F_{2n+3}\binom{2n}{n}_F} = \frac{F_{2n+1}(F_{2n+2}+1)}{F_{2n+3}} - \frac{F_{n+1}F_{n+3}}{F_{2n+3}\binom{2n}{n}_F}.$$

(ii) For $s > n \geq 0$, we show that

$$\sum_{k=0}^{n} \frac{F_{s-2k}}{\binom{s}{k}_F^2} = \frac{F_{s+1}^2}{F_{s+2}} - \frac{F_{n+1}^2}{F_{s+2}\binom{s}{n}_F^2}. \tag{3}$$

The proof is by mathematical induction on n.

• For $n = 0$, we have

$$LHS - RHS = F_s - \frac{F_{s+1}^2 - 1}{F_{s+2}} = \frac{F_s F_{s+2} - F_{s+1}^2 + 1}{F_{s+2}} = \frac{(-1)^s(F_0F_2 - F_1^2) + 1}{F_{s+2}}$$

(by (1)), and this last expression is 0.

• We assume that (3) holds for n. For $n+1$, we have

$$\begin{aligned}
&\sum_{k=0}^{n+1} \frac{F_{s-2k}}{\binom{s}{k}_F^2} - \frac{F_{s+1}^2}{F_{s+2}} \\
&= \frac{F_{s-2(n+1)}}{\binom{s}{n+1}_F^2} + \sum_{k=0}^{n} \frac{F_{s-2k}}{\binom{s}{k}_F^2} - \frac{F_{s+1}^2}{F_{s+2}} \\
&= \frac{F_{s-2n-2}}{\binom{s}{n+1}_F^2} - \frac{F_{n+1}^2}{F_{s+2}\binom{s}{n}_F^2} \\
&= \frac{F_{s-2n-2}}{\binom{s}{n+1}_F^2} - \frac{F_{n+1}^2}{F_{s+2} \times \frac{F_{n+1}^2}{F_{s-n}^2}\binom{s}{n+1}_F^2} \\
&= \frac{F_{s-2n-2}F_{s+2} - F_{s-n}^2}{F_{s+2}\binom{s}{n+1}_F^2} \\
&= \frac{(-1)^{s-2n-2}(F_0F_{2n+4} - F_{n+2}^2)}{F_{s+2}\binom{s}{n+1}_F^2} \qquad \text{(by} \quad \text{(1))} \\
&= \frac{-F_{n+2}^2}{F_{s+2}\binom{s}{n+1}_F^2}.
\end{aligned}$$

Thus, (3) holds for $n+1$. Therefore, (3) is proved. Letting $s = 2n$ in (3) for $n \geq 1$, we have

$$\sum_{k=0}^{n} \frac{F_{2(n-k)}}{\binom{2n}{k}_F^2} = \frac{F_{2n+1}^2}{F_{2n+2}} - \frac{F_{n+1}^2}{F_{2n+2}\binom{2n}{n}_F^2} = \frac{F_{2n+1}^2}{F_{2n+2}} - \frac{F_{n+1}}{L_{n+1}\binom{2n}{n}_F^2},$$

which is (ii).

References

[1] R. C. Johnson, *Fibonacci numbers and matrices*, http://www.dur.ac.uk/bob.johnson/fibonacci/.

A Cyclic Sum Inequality

H-769 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.
(Vol. 53, No. 2, May 2015)

Prove that the inequality

$$\frac{F_1^6}{(F_1^4+F_1^2F_2^2+F_2^4)(\sqrt{2}F_1+F_2)}+\frac{F_2^6}{(F_2^4+F_2^2F_3^2+F_3^4)(\sqrt{2}F_2+F_3)}+\cdots$$
$$+\frac{F_{n-1}^6}{(F_{n-1}^4+F_{n-1}^2F_n^2+F_n^4)(\sqrt{2}F_{n-1}+F_n)}+\frac{F_n^6}{(F_n^4+F_n^2F_1^2+F_1^4)(\sqrt{2}F_n+F_1)}$$
$$\geq \frac{\sqrt{2}-1}{3}(F_{n+2}-1)$$

holds for all positive integers n.

Solution by Ángel Plaza.

It is based on the fact that if $x,\ y$ are positive then $\dfrac{x^6}{(x^4+x^2y^2+y^4)(\sqrt{2}x+y)} \geq \dfrac{(\sqrt{2}x-y)}{3}$. The above inequality is equivalent to $3x^6-(x^4+x^2y^2+y^4)(2x^2-y^2)>0$ which holds because the left-hand side is $(x-y)^2(x+y)^2\left(x^2+y^2\right)\geq 0$.

Corollary. *If $x_k>0$ $(k=1,2,\ldots,n)$, then*

$$\sum_{cyclic}\frac{x_k^6}{(x_k^4+x_k^2x_{k+1}^2+x_{k+1}^4)(\sqrt{2}x_k+x_{k+1})}\geq \frac{\sqrt{2}-1}{3}\sum_{k=1}^n x_k.$$

Proof. We have

$$\sum_{cyclic}\frac{x_k^6}{(x_k^4+x_k^2x_{k+1}^2+x_{k+1}^4)(\sqrt{2}x_k+x_{k+1})}\geq \frac{1}{3}\sum_{cyclic}\left(\sqrt{2}x_k-x_{k+1}\right)=\frac{\sqrt{2}-1}{3}\sum_{k=1}^n x_k.$$

□

We now apply the previous result to the sequence $\{x_k\}=\{F_k\}$ and use that $\sum_{k=1}^n F_k = F_{n+2}-1$ to conclude. See [1] for similar inequalities.

References

[1] M. Bencze and D. M. Bătineţu-Giurgiu, *A cathegory of inequalities*, Octogon Mathematical Magazine, **17.1** (2009), 149–163.

Also solved by Kenneth B. Davenport, Dmitry Fleischman, Zbigniew Jakubczyk, Hideyuki Ohtsuka, Nicuşor Zlota, and the proposers.

A Sum of Products of Shifted Lucas Numbers

H-770 Proposed by H. Ohtsuka, Saitama, Japan.
(Vol. 53, No. 2, May 2015)

For an integer $n\geq 0$, find a closed form expression for the sum

$$S(n):=\sum_{k=0}^n\frac{1}{(L_{2^{k+1}}+1)(L_{2^k}+c)(L_{2^{k+1}}+c)\cdots(L_{2^n}+c)},$$

where $c\neq -L_{2^k}$ for $0\leq k\leq n$.

Solution by the proposer.

We find an identity

$$S(n) = \frac{1}{(c+1)(L_{2^{n+1}}+1)}. \tag{1}$$

The proof of (1) is by mathematical induction on n.

- For $n=0$, we have $LHS = RHS = \dfrac{1}{4(c+1)}$.
- We assume that (1) holds for n. For $n+1$, we have

$$\begin{aligned}
S(n+1) &= \frac{1}{(L_{2^{n+2}}+1)(L_{2^{n+1}}+c)} + \frac{S(n)}{L_{2^{n+1}}+c} \\
&= \frac{1}{(L_{2^{n+2}}+1)(L_{2^{n+1}}+c)} + \frac{1}{(L_{2^{n+1}}+c)} \times \frac{1}{(c+1)(L_{2^{n+1}}+1)} \\
&= \frac{(c+1)(L_{2^{n+1}}+1) + L_{2^{n+2}}+1}{(c+1)(L_{2^{n+1}}+1)(L_{2^{n+1}}+c)}.
\end{aligned}$$

Here, the numerator of RHS is

$$\begin{aligned}
&(c+1)(L_{2^{n+1}}+1) + (L_{2^{n+1}}^2 - 1) \qquad (\text{by} \quad L_{2m} = L_m^2 - 2(-1)^m) \\
&= (c+1)(L_{2^{n+1}}+1) + (L_{2^{n+1}}+1)(L_{2^{n+1}}-1) \\
&= (L_{2^{n+1}}+1)(L_{2^{n+1}}+c).
\end{aligned}$$

Therefore,

$$S(n+1) = \frac{1}{(c+1)(L_{2^{n+2}}+1)}.$$

Thus, identity (1) holds.

Editor's Notes: (i) Inequality (iv) from Problem **H-763** (Vol. 52, No. 4, November 2014) should read

$$\sum_{k=1}^{n} \frac{F_k^8}{k^3} \ge \frac{8F_n^4F_{n+1}^4}{n^3(n+1)^3}.$$

(ii) Related to **H-688** (Vol. 47, No. 2, May 2009/2010), Apoloniusz Tiszka points out his paper:

A. Tyszka, *A hypothetical way to compute an upper bound on the heights of solutions to a Diophantine equation with a finite number of solutions*, Proceedings of the Federated Conference on Computer Science and Information Systems, **5** (2015), 709–716.