ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.

Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by February 15, 2025. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1349 (Corrected) Proposed by Toyesh Prakash Sharma (undergraduate), Agra College, Agra, India.

For any integer $n \geq 2$, show that

$$F_n^{L_n} F_{n+1}^{2F_{n+1}} L_n^{F_n} \leq F_{2n}^{2F_{n+1}}.$$

B-1351 Proposed by Hideyuki Ohtskua, Saitama, Japan.

The sequences $\{U_n\}$ and $\{V_n\}$ are defined by

$$U_0 = 0, \quad U_1 = 1, \qquad \text{and} \qquad U_{n+2} = 4U_{n+1} - U_n,$$

and

$$V_0 = 2, \quad V_1 = 4, \qquad \text{and} \qquad V_{n+2} = 4V_{n+1} - V_n,$$

for any integer n. Prove that

$$\sum_{n=1}^{\infty} \tan^{-1} \frac{1}{4U_n^2} = \frac{\pi}{12}, \qquad \text{and} \qquad \sum_{n=-\infty}^{\infty} \tan^{-1} \frac{3}{V_n^2} = \frac{\pi}{3}.$$

B-1352 **Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.**

Find all integers $n \geq 0$ for which $n^2 - F_{2n}$ is a perfect square. Also prove that no integer $n > 0$ exists such that $n^2 + F_{2n}$ is a perfect square.

B-1353 **Proposed by Kenny B. Davenport, Dallas, PA.**

Prove that

(i) $4\sum_{k=1}^{n} F_{6k} + 3\sum_{k=1}^{n} F_{2k} = 5\big(F_{2n+1}^3 - 1\big).$

(ii) $4\sum_{k=1}^{n} L_{6k} - 3\sum_{k=1}^{n} L_{2k} = L_{2n+1}^3 - 1.$

B-1354 **Proposed by Davide Rotondo, Brescia, Italy.**

Define a trapezoid of numbers $a(n,k)$, where $n \geq 0$, and $0 \leq k \leq n+1$, as follows. Define $a(0,0) = 0$, and $a(0,1) = 1$. For $n \geq 1$, define $a(n,0) = a(n,n+1) = 1$, and

$$a(n,k) = a(n-1,k-1) + a(n-1,k) \qquad \text{for } 1 \leq k \leq n.$$

The first eight rows of the trapezoid are shown below.

$n=0$	0	1							
$n=1$	1	1	1						
$n=2$	1	2	2	1					
$n=3$	1	3	4	3	1				
$n=4$	1	4	7	7	4	1			
$n=5$	1	5	11	14	11	5	1		
$n=6$	1	6	16	25	25	16	6	1	
$n=7$	1	7	22	41	50	41	22	7	1

Prove that, if n is prime, then for $2 \leq k \leq n-1$, the integer n divides $a(n,k) - 1$ when k is even, and n divides $a(n,k) + 1$ when k is odd.

B-1355 **Proposed by Albert Stadler, Herrliberg, Switzerland.**

Let $p_n(x)$ be the polynomial of degree n defined by

$$p_n(x) = e^{-x^2} \frac{d^n}{dx^n}\big(e^{x^2}\big).$$

Prove that for all integers $m, n \geq 1$,

$$F_{2m+1}^{n/2}\, p_n(F_{m+2}) = \sum_{k=0}^{n} \binom{n}{k} F_m^k F_{m+1}^{n-k}\, p_k\big(\sqrt{F_{2m+1}}\big)\, p_{n-k}\big(\sqrt{F_{2m+1}}\big).$$

SOLUTIONS

We Only Need the First Term!

B-1331 Proposed by Neculai Stanciu, "George Emil Palade" School, Buzău, Romania. (Vol. 61.3, August 2023)

Prove that $\frac{F_{n+2}^2}{F_{n-1}^2} + \frac{F_{n+2}}{F_n} + \frac{F_{n+2}}{F_{n+1}} \geq 9$ for all integers $n \geq 2$.

Solution by Wei-Kai Lai, University of South Carolina Salkehatchie, Walterboro, SC.

Since $F_t > 0$ and $F_{t+1} \geq F_t$ for all integers $t \geq 1$, we have that

$$\frac{F_{n+2}^2}{F_{n-1}^2} = \left(\frac{F_{n-1} + 2F_n}{F_{n-1}}\right)^2 = \left(1 + 2 \cdot \frac{F_n}{F_{n-1}}\right)^2 \geq 3^2 = 9.$$

Therefore, only the first term is needed for the solution.

Indeed, we can improve the inequality by applying the AM-GM inequality to the two terms on the left side of the inequality. We find that

$$\frac{F_{n+2}}{F_n} + \frac{F_{n+2}}{F_{n+1}} = \left(1 + \frac{F_{n+1}}{F_n}\right)\left(1 + \frac{F_n}{F_{n+1}}\right) = 2 + \frac{F_{n+1}}{F_n} + \frac{F_n}{F_{n+1}} \geq 4.$$

For a fixed n, the equality for $\frac{F_{n+2}^2}{F_{n-1}^2} \geq 9$ requires that $F_n = F_{n-1}$, or $n = 2$. However, the equality for $\frac{F_{n+2}}{F_n} + \frac{F_{n+2}}{F_{n+1}} \geq 4$ requires that $F_n = F_{n+1}$, or $n = 1$. Therefore, the equality of these two inequalities can never happen at the same time. We conclude that

$$\frac{F_{n+2}^2}{F_{n-1}^2} + \frac{F_{n+2}}{F_n} + \frac{F_{n+2}}{F_{n+1}} > 13.$$

Editor's Note: Grimaldi used the addition formula $F_{m+p} = F_{m+1}F_p + F_mF_{p-1}$ to derive the inequality $\frac{F_{n+k}}{F_{n-1}} \geq F_{k+2}$. Following the same method Lai used above, we see that, for $k \geq 2$,

$$\frac{F_{n+k}^2}{F_{n-1}^2} + \frac{F_{n+k}}{F_n} + \frac{F_{n+k}}{F_{n+1}} > F_{k+2}^2 + 4.$$

Also solved by Thomas Achammer, Michel Bataille, Brian D. Beasley, Charles K. Cook and Michael R. Bacon (jointly), Kenny B. Davenport, the Eagle Problem Solvers (Georgia Southern University), I. V. Fedak, Dmitry Fleischman, G. C. Greubel, Ralph P. Grimaldi, Won Kyun Jeong, Hari Kishan, Carl Libis, Hideyuki Ohtsuka, Ángel Plaza, Patrick Rappa (undergradudate), Henry Ricardo, Alice Souza, Gabriela Destazio, Iuri Corrêa, and Laura Silva (undergraduates) (jointly), Albert Stadler, David Terr, Caitlyn Tyson (undergraduate), Daniel Văcaru, Andrés Ventas, and the proposer.

Two Tribonacci Series

B-1332 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 61.3, August 2023)

Let T_n be the nth Tribonacci number, defined by $T_0 = 0$, $T_1 = T_2 = 1$, and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 3$. Prove that

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{T_n T_{n+1} T_{n+2} T_{n+4}} = -\frac{1}{16}$,

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{T_n T_{n+1} T_{n+4}} = \frac{3}{16}$.

Solution by Albert Stadler, Herrliberg, Switzerland.

(i) We have

$$\frac{1}{2T_n T_{n+1} T_{n+2} T_{n+3}} + \frac{1}{2T_{n+1} T_{n+2} T_{n+3} T_{n+4}} = \frac{T_{n+4} + T_n}{2T_n T_{n+1} T_{n+2} T_{n+3} T_{n+4}}$$

$$= \frac{T_{n+3} + T_{n+2} + T_{n+1} + T_n}{2T_n T_{n+1} T_{n+2} T_{n+3} T_{n+4}} = \frac{2T_{n+3}}{2T_n T_{n+1} T_{n+2} T_{n+3} T_{n+4}} = \frac{1}{T_n T_{n+1} T_{n+2} T_{n+4}}.$$

Thus,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{T_n T_{n+1} T_{n+2} T_{n+4}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2T_n T_{n+1} T_{n+2} T_{n+3}} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2T_{n+1} T_{n+2} T_{n+3} T_{n+4}}$$

$$= -\frac{1}{2T_1 T_2 T_3 T_4} = -\frac{1}{16}.$$

(ii) We have

$$\frac{T_n + T_{n+2}}{2T_n T_{n+1} T_{n+2} T_{n+3}} - \frac{T_{n+1} + T_{n+3}}{2T_{n+1} T_{n+2} T_{n+3} T_{n+4}} = \frac{T_{n+4}(T_n + T_{n+2}) - T_n(T_{n+1} + T_{n+3})}{2T_n T_{n+1} T_{n+2} T_{n+3} T_{n+4}}$$

$$= \frac{T_{n+4}(T_n + T_{n+2}) - T_n(T_{n+4} - T_{n+2})}{2T_n T_{n+1} T_{n+2} T_{n+3} T_{n+4}} = \frac{T_{n+2}(T_{n+4} + T_n)}{2T_n T_{n+1} T_{n+2} T_{n+3} T_{n+4}}$$

$$= \frac{2T_{n+2} T_{n+3}}{2T_n T_{n+1} T_{n+2} T_{n+3} T_{n+4}} = \frac{1}{T_n T_{n+1} T_{n+4}}.$$

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{T_n T_{n+1} T_{n+4}} = \sum_{n=1}^{\infty} \frac{T_n + T_{n+2}}{2T_n T_{n+1} T_{n+2} T_{n+3}} - \sum_{n=1}^{\infty} \frac{T_{n+1} + T_{n+3}}{2T_{n+1} T_{n+2} T_{n+3} T_{n+4}}$$

$$= \frac{T_1 + T_3}{2T_1 T_2 T_3 T_4} = \frac{3}{16}.$$

Also solved by Thomas Achammer, Michel Bataille, Dmitry Fleischman, Robert Frontczak, Yunyong Zhang, and the proposer.

A Fibonacci Equation

B-1333 Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany. (Vol. 61.3, August 2023)

Find all solutions of the equation

$$(F_a+1)(F_b+1)(F_c+1)=3F_aF_bF_c.$$

Solution by Michel Bataille, Rouen, France.

If (a,b,c) is a solution, then any permutation of this triple is obviously also a solution. This is understood below. We shall prove that the only solutions (up to permutations) are the triples $(0,-2,c)$, $(1,4,6)$, $(-1,4,6)$, $(2,4,6)$, $(3,3,4)$, $(3,-3,4)$, and $(-3,-3,4)$, for any integer c.

Let (a,b,c) be a solution. If $|F_a|,|F_b|,|F_c|\geq 3$, then $F_a,F_b,F_c\neq 0$, and

$$\left(1+\frac{1}{F_a}\right)\left(1+\frac{1}{F_b}\right)\left(1+\frac{1}{F_c}\right)=3, \tag{1}$$

with $\frac{1}{F_a},\frac{1}{F_b},\frac{1}{F_c}\in\left[-\frac{1}{3},\frac{1}{3}\right]$. We deduce that $1+\frac{1}{F_a},1+\frac{1}{F_b},1+\frac{1}{F_c}\in\left[\frac{2}{3},\frac{4}{3}\right]$, and therefore

$$\left(1+\frac{1}{F_a}\right)\left(1+\frac{1}{F_b}\right)\left(1+\frac{1}{F_c}\right)\leq\frac{64}{27},$$

in contradiction with (1). Thus, $m:=\min(|F_a|,|F_b|,|F_c|)\in\{0,1,2\}$.

- If $m=0$, one of F_a,F_b,F_c equals 0, say $F_a=0$. Then $a=0$, and $(F_b+1)(F_c+1)=0$. Hence, $b=-2$ or $c=-2$, say $b=-2$, so that $(a,b,c)=(0,-2,c)$.
- If $m=1$, we may assume, without loss of generality, that $|F_a|=1$ and $|F_b|,|F_c|\geq 1$. Since $F_aF_bF_c\neq 0$, we have $F_a=1$. Then, $2(F_b+1)(F_c+1)=3F_bF_c$, which can be rewritten as $(F_b-2)(F_c-2)=6$. It is readily seen that $\{F_b-2,F_c-2\}=\{-2,-3\},\{2,3\},\{-1,-6\}$ cannot occur. Hence $\{F_b-2,F_c-2\}=\{1,6\}$, so that $\{F_b,F_c\}=\{3,8\}$, and $\{b,c\}=\{4,6\}$. We thus obtain the triples $(1,4,6)$, $(-1,4,6)$, and $(2,4,6)$.
- If $m=2$, say $|F_a|=2$ and $|F_b|,|F_c|\geq 2$. Then, $F_a=2$ (no Fibonacci number equals -2) so that $3(F_b+1)(F_c+1)=6F_bF_c$, which writes as $(F_b-1)(F_c-1)=2$. Since $\{F_b-1,F_c-1\}=\{-1,-2\}$ cannot occur, we must have $\{F_b,F_c\}=\{2,3\}$. This leads to the triples $(3,3,4)$, $(3,-3,4)$, and $(-3,-3,4)$.

Conversely, all the triples found above are solutions. This completes the proof.

Also solved by Thomas Achammer, the Eagle Problem Solvers (Georgia Southern University), I. V. Fedak, Dmitry Fleischman, Raphael Schumacher (graduate student), Albert Stadler, Andrés Ventas, Dan Weiner, Yunyong Zhang, and the proposer.

An Inequality with an Improvement

B-1334 Proposed by Toyesh Prakash Sharma (student), Agra College, Agra, India.
(Vol. 61.3, August 2023)

For all integers $n \geq 3$, show that

$$\frac{(F_{n+1}-1)^2}{(F_n-1)(L_n-1)} + \frac{(F_n-1)^2}{(F_{n-1}-1)(L_{n-1}-1)} < F_{2n+1}.$$

Solution by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

For $n = 3$, the problem is incorrect, because $F_2 - 1 = 0$. Next, using $F_k - 1 \geq 1$ and $L_k - 1 > 1$ for $k \geq 3$, we obtain, for $n \geq 4$,

$$\frac{(F_{n+1}-1)^2}{(F_n-1)(L_n-1)} + \frac{(F_n-1)^2}{(F_{n-1}-1)(L_{n-1}-1)} < F_{n+1}^2 + F_n^2 = F_{2n+1}.$$

We can prove a stronger result using the inequalities

$$F_k - 1 < F_{k-2} + F_k - 1 = L_{k-1} - 1$$

and

$$F_k - 1 \leq F_k - 1 + F_{k-3} - 1 = 2(F_{k-1} - 1)$$

for $k \geq 3$. We find, for $n \geq 4$,

$$\frac{(F_{n+1}-1)^2}{(F_n-1)(L_n-1)} + \frac{(F_n-1)^2}{(F_{n-1}-1)(L_{n-1}-1)} < 2 + 2 = 4.$$

Also solved by Thomas Achammer, Michel Bataille, Brian D. Beasley, Dmitry Fleischman, Robert Frontczak, G. C. Greubel, Ralph P. Grimaldi, Won Kyun Jeong, Wei-Kai Lai, Hideyuki Ohtsuka, Ángel Plaza, Albert Stadler, Caitlyn Tyson (undergraduate), Andrés Ventas, and the proposer.

A Finite Product of Generalized Fibonacci Numbers

B-1335 Proposed by Michel Bataille, Rouen, France.
(Vol. 61.3, August 2023)

Let the sequence $\{G_n\}_{n\geq 0}$ be defined by arbitrary $G_0, G_1 \in \mathbb{N}$, and the recurrence $G_{n+1} = G_n + G_{n-1}$ for any integer $n \geq 1$. If m and n are integers such that $m \geq 1$ and $n \geq 0$, prove that

$$\prod_{k=1}^{m} \frac{G_n + G_{n+2^{k+1}}}{G_{n+2^k}}$$

is a Fibonacci number.

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Each factor in the product is equal to a Lucas number. Explicitly we have

$$\frac{G_n + G_{n+2^{k+1}}}{G_{n+2^k}} = L_{2^k}. \tag{2}$$

The Binet's formula for G_n takes the form of $G_p = A\alpha^p + B\beta^p$ for some constants A and B, the values of which depend on the initial values. Then

$$\begin{aligned} G_{n+2^k} L_{2^k} &= \left(A\alpha^{n+2^k} + B\beta^{n+2^k}\right)\left(\alpha^{2^k} + \beta^{2^k}\right) \\ &= A\alpha^{n+2^{k+1}} + B\beta^{n+2^{k+1}} + A\alpha^n + B\beta^n \\ &= G_{n+2^{k+1}} + G_n. \end{aligned}$$

This proves our assertion.

Note that for $k = 1$, $L_2 = 3 = F_4$. With the identity $F_n L_n = F_{2n}$, we obtain

$$\prod_{k=1}^{m} \frac{G_n + G_{n+2^{k+1}}}{G_{n+2^k}} = \prod_{k=1}^{m} L_{2^k} = F_4 \prod_{k=2}^{m} L_{2^k} = F_{2^{m+1}}.$$

Editor's Notes: It is clear that (2) is the key to the solution. Ohtsuka derived it by applying Identity 10(a) from [2, page 176]:

$$G_{s+t} + (-1)^t G_{s-t} = G_s L_t.$$

Smith noted that (2) can be obtained from a modified version of Problem 20 from [1, page 114].

References

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons, Inc., New York, NY, 2001.
[2] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section*, Dover, New York, NY, 2008.

Also solved by Thomas Achammer, the Eagle Problem Solvers (Georgia Southern University), I. V. Fedak, Dmitry Fleischman, Robert Frontczak, G. C. Greubel, Won Kyun Jeong, Hideyuki Ohtsuka, Raphael Schumacher (graduate student), Jason L. Smith, Albert Stadler, David Terr, Yunyong Zhang, and the proposer.

Correction: Ell Torek's name was misspelled in the list of solvers of Problem B-1325 in the February issue. The section editor apologizes to Ell and the readers for the mistakes he made in the past two issues.