ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.

Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by May 15, 2022. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1296 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

For integers $s > r \geq 0$, evaluate

$$\prod_{n=1}^{\infty} \left(1 + \frac{L_{2^n r}}{L_{2^n s}}\right).$$

B-1297 Proposed by D. M. Bătineţu-Giurgui, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.

For integers $m > 1$ and $n \geq 1$, prove that

(A) $\displaystyle\sum_{k=1}^{n}(1+F_k)^{2(m+1)} \geq \frac{(m+1)^{2(m+1)}}{m^{2m}}\, F_n F_{n+1}$

(B) $\displaystyle\sum_{k=1}^{n}(1+L_k)^{2(m+1)} \geq \frac{(m+1)^{2(m+1)}}{m^{2m}}\, \big(L_n L_{n+1} - 2\big)$

B-1298 **Proposed by Diego Rattaggi, Realgymnasium Rämibühl, Zürich, Switzerland.**

For any positive integer k, prove that

$$\sum_{n=0}^{\infty} \frac{1}{F_{2kn+k+1}^2 + F_k^2} = \frac{1}{\alpha F_{2k}}.$$

B-1299 **Proposed by Toyesh Prakash Sharma (high school student), St. C. F. Anrews School, Agra, India.**

Let n be a positive even integer. Prove that

$$\exp\left(\sum_{k=1}^{n} F_{k-1}F_{k+1}\right) > \prod_{k=1}^{n} \big[(F_k)!\,(F_k+1)!\big].$$

B-1300 **Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.**

Let the sequence $\{a_n\}_{n\geq 0}$ be defined by $a_0 = 1$, $a_1 = 3$, and $a_{n+2} = a_{n+1}\big(5a_n^2+2\big)$. Evaluate $\sum_{n=0}^{\infty} \frac{1}{a_n}$.

SOLUTIONS

The Sixth Oldie from the Vault

B-408 **Proposed by Lawrence Somer, Washington, D.C.**
(Vol. 17.3, October 1979)

Let $d \in \{2, 3, \dots\}$ and $G_n = F_{dn}/F_n$. Let p be an odd prime and $z = z(p)$ be the least positive integer n with $F_n \equiv 0 \pmod{p}$. For $d = 2$ and $z(p)$ an even integer $2k$, it was shown in B-386 that

$$F_{n+1}G_{n+k} \equiv F_nG_{n+k+1} \pmod{p}.$$

Establish a generalization for $d \geq 2$.

Solution by Raphael Schumacher (graduate student), ETH Zurich, Switzerland.

We will prove the following generalization.

Theorem. *Let $d \in \{1, 2, 3, \dots\}$ and $G_n = F_{dn}/F_n$. Let p be an odd prime and $z = z(p)$ be the least positive integer n with $F_n \equiv 0 \pmod{p}$. If $z(p) = dk$ is an integer divisible by d, then*

$$F_{(d-1)(n+1)}G_{n+k} + C_{k,d}(n) \equiv F_{(d-1)n}G_{n+k+1} \pmod{p},$$

where

$$C_{k,d}(n) = F_{(d-1)n}E(n+k+1) - F_{(d-1)(n+1)}E(n+k) + (-1)^{(d-1)n}F_{d-1}E(k),$$

with $E(m) = G_m - L_{(d-1)m}$.

In the case of $d = 2$, since $G_m = L_m$, we have $E(m) = 0$; hence $C_{k,d}(n) = 0$, which leads to the result stated in Problem B-386. The following proof is inspired by Paul S. Bruckman's solution to Problem B-386 [1].

Proof. The theorem is true when $d = 1$, so we may assume $d \geq 2$. Using the identity $F_{-t} = (-1)^{t+1}F_t$ and the product formula $F_sL_t = F_{s+t} + (-1)^tF_{s-t}$, we find

$$F_{(d-1)(n+1)}L_{(d-1)(n+k)} - F_{(d-1)n}L_{(d-1)(n+k+1)} = (-1)^{(d-1)n}F_{d-1}L_{(d-1)k}.$$

We see that

$$F_{(d-1)(n+1)}G_{n+k} - F_{(d-1)n}G_{n+k+1} + C_{k,d}(n) = (-1)^{(d-1)n}F_{d-1}G_k.$$

Take note that

$$F_{z(p)} = F_{dk} = F_kG_k \equiv 0 \pmod{p}.$$

Because $z(p) = dk \geq 2k$ is the least positive integer n with the property that $F_n \equiv 0 \pmod{p}$, we know $F_k \not\equiv 0 \pmod{p}$. Therefore, we conclude that

$$G_k \equiv 0 \pmod{p}.$$

This implies that

$$F_{(d-1)(n+1)}G_{n+k} - F_{(d-1)n}G_{n+k+1} + C_{k,d}(n) \equiv 0 \pmod{p},$$

which is equivalent to the congruence stated in the theorem.

Editor's Note: The proposer posed the problem as an open problem. No other solutions were received besides the one featured above. The section editor invites the readers to continue the pursuit of other possible generalizations.

References

[1] P. S. Bruckman, *Solution to Problem B-386*, The Fibonacci Quarterly, **17.3** (1979), 284.

Catalan Reduces It to a Telescopic Sum

B-1276 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 58.3, November 2020)

Prove that

$$\sum_{n=1}^{\infty} \frac{F_{2n}}{(L_n^2 - 5)^2} = \frac{1}{3}.$$

Solution by Brian Bradie, Christopher Newport University, Newport News, VA.

Using the Catalan's identity $L_{k-r}L_{k+r} - L_k^2 = 5(-1)^{k-r}F_r^2$, we deduce that

$$L_k^2 - 5 = \begin{cases} L_{k-2}L_{k+2} & \text{if } k \text{ is odd,} \\ L_{k-1}L_{k+1} & \text{if } k \text{ is even.} \end{cases}$$

Thus,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{F_{2n}}{(L_n^2-5)^2} &= \sum_{n=1}^{\infty}\frac{F_{4n-2}}{(L_{2n-1}^2-5)^2}+\sum_{n=1}^{\infty}\frac{F_{4n}}{(L_{2n}^2-5)^2}\\ &= \sum_{n=1}^{\infty}\frac{F_{4n-2}}{L_{2n-3}^2L_{2n+1}^3}+\sum_{n=1}^{\infty}\frac{F_{4n}}{L_{2n-1}^2L_{2n+1}^2}.\end{aligned}$$

Now,

$$\begin{aligned}L_{2n+1}^2-L_{2n-3}^2 &= (L_{2n+1}+L_{2n-3})(L_{2n+1}-L_{2n-3})\\ &= (L_{2n}+2L_{2n-1}-L_{2n-2})(L_{2n}+L_{2n-2})\\ &= (3L_{2n-1})\,(5F_{2n-1})\\ &= 15F_{4n-2},\end{aligned}$$

and

$$L_{2n+1}^2-L_{2n-1}^2=(L_{2n+1}+L_{2n-1})(L_{2n+1}-L_{2n-1})=(5F_{2n})\,L_{2n}=5F_{4n}.$$

Hence,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{F_{4n-2}}{L_{2n-3}^2L_{2n+1}^2} &= \sum_{n=1}^{\infty}\frac{F_{4n-2}}{L_{2n+1}^2-L_{2n-3}^2}\left(\frac{1}{L_{2n-3}^2}-\frac{1}{L_{2n+1}^2}\right)\\ &= \frac{1}{15}\sum_{n=1}^{\infty}\left(\frac{1}{L_{2n-3}^2}-\frac{1}{L_{2n+1}^2}\right)=\frac{1}{15}\left(\frac{1}{L_{-1}^2}+\frac{1}{L_1^2}\right)=\frac{2}{15},\end{aligned}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{F_{4n}}{L_{2n-1}^2L_{2n+1}^2} &= \sum_{n=1}^{\infty}\frac{F_{4n}}{L_{2n+1}^2-L_{2n-1}^2}\left(\frac{1}{L_{2n-1}^2}-\frac{1}{L_{2n+1}^2}\right)\\ &= \frac{1}{5}\sum_{n=1}^{\infty}\left(\frac{1}{L_{2n-1}^2}-\frac{1}{L_{2n+1}^2}\right)=\frac{1}{5}\cdot\frac{1}{L_1^2}=\frac{1}{5}.\end{aligned}$$

Finally,

$$\sum_{n=1}^{\infty}\frac{F_{2n}}{(L_n^2-5)^2}=\frac{2}{15}+\frac{1}{5}=\frac{1}{3}.$$

Also solved by Thomas Achammer, Michel Bataille, Steve Edwards, I. V. Fedak, Robert Frontczak, Ángel Plaza, Raphael Schumacher (graduate student), J. N. Senadheera, Albert Stadler, and the proposer.

Apply a Generalized Inequality

B-1277 Proposed by Ivan V. Fedak, Precarpathian National University, Ivano-Frankivsk, Ukraine.
(Vol. 58.4, November 2020)

For all positive integers n, prove that

$$\frac{F_{n-1}^2}{2F_{n+2}}\le\sqrt{\frac{F_{2n+1}}{2}}-\sqrt{\sum_{k=1}^{n}F_k^2}\le\frac{F_{n-1}^2}{F_{n+2}}.$$

Solution by Hideyuki Ohtsuka, Saitama, Japan.

For any positive real numbers s and t, we have (because $2st \le s^2 + t^2$)

$$s + t = \sqrt{(s+t)^2} = \sqrt{s^2 + 2st + t^2} \le \sqrt{2s^2 + 2t^2},$$

and

$$2(s+t) = \sqrt{4(s+t)^2} = \sqrt{4s^2 + 8st + 4t^2} > \sqrt{2s^2 + 2t^2}.$$

Thus, we have

$$\frac{1}{2} < \frac{s+t}{\sqrt{2s^2+2t^2}} \le 1.$$

If, in addition, $s - t \ge 0$, then

$$\frac{s-t}{2} < \frac{s^2-t^2}{\sqrt{2s^2+2t^2}} \le s - t;$$

that is,

$$\frac{s^2-t^2}{\sqrt{2s^2+2t^2}} \le s - t \le \frac{2(s^2-t^2)}{\sqrt{2s^2+2t^2}}.$$

Let

$$s = \sqrt{\frac{F_{2n+1}}{2}} = \sqrt{\frac{F_{n+1}^2 + F_n^2}{2}}, \qquad \text{and} \qquad t = \sqrt{\sum_{k=1}^{n} F_k^2} = \sqrt{F_n F_{n+1}}.$$

We have $s - t \ge 0$, since

$$s = \sqrt{\frac{F_{n+1}^2 + F_n^2}{2}} = \sqrt{\frac{(F_{n+1} - F_n)^2 + 2F_{n+1}F_n}{2}} \ge \sqrt{F_{n+1}F_n} = t.$$

We also find

$$\frac{s^2-t^2}{\sqrt{2s^2+2t^2}} = \frac{\frac{F_{n+1}^2+F_n^2}{2} - F_nF_{n+1}}{\sqrt{F_{n+1}^2 + F_n^2 + 2F_nF_{n+1}}} = \frac{(F_{n+1}-F_n)^2}{2\sqrt{(F_{n+1}+F_n)^2}} = \frac{F_{n-1}^2}{2F_{n+2}}.$$

Therefore, we obtain the desired inequality.

Also solved by Michel Bataille, Brian Bradie, Kenny B. Davenport, Steve Edwards, Dmitry Fleischman, Ángel Plaza, Raphael Schumacher (graduate student), Albert Stadler, and the proposer.

<u>A Simple Closed-Form Expression</u>

<u>B-1278</u> Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.
(Vol. 58.3, November 2020)

Show that the finite product

$$\prod_{k=0}^{n} \frac{F_{k+2}^2 + 2F_{k+1}F_{k+2}}{L_kF_{k+2} + (-1)^{k+1}}$$

is divisible by L_{n+2} for each integer $n \ge 0$.

Solution by Steve Edwards, Roswell, GA.

We will show that the product equals $F_{n+2}L_{n+2}$. Using the defining recursion for the Fibonacci numbers and the identity $F_{n-1}+F_{n+1}=L_n$, we have for the numerator of the k^{th} factor

$$\begin{aligned} F_{k+2}^2+2F_{k+1}F_{k+2} &= F_{k+2}(F_{k+2}+2F_{k+1}) \\ &= F_{k+2}(F_{k+2}+F_{k+1}+F_{k+1}) \\ &= F_{k+2}(F_{k+3}+F_{k+1}) \\ &= F_{k+2}L_{k+2}. \end{aligned}$$

Next, note that the identity $L_kF_{k+2}-F_{k+1}L_{k+1}=(-1)^k$ can be verified via the Binet formulas. This means the k^{th} denominator equals $F_{k+1}L_{k+1}$. This gives

$$\prod_{k=0}^{n}\frac{F_{k+2}^2+2F_{k+1}F_{k+2}}{L_kF_{k+2}+(-1)^{k+1}}=\prod_{k=0}^{n}\frac{F_{k+2}L_{k+2}}{F_{k+1}L_{k+1}}=\frac{F_2L_2}{F_1L_1}\cdot\frac{F_3L_3}{F_2L_2}\cdots\frac{F_{n+2}L_{n+2}}{F_{n+1}L_{n+1}}=F_{n+2}L_{n+2},$$

as desired.

Also solved by Michel Bataille, Brian D. Beasley, Brian Bradie, Alejandro Cardona Castrillón (undergraduate), Charles K. Cook, Kenny B. Davenport, I. V. Fedak, Dmitry Fleischman, Wei-Kai Lai, Hideyuki Ohtsuka, Ángel Plaza, Raphael Schumacher (graduate student), J. N. Senadheera, Albert Stadler, and the proposer.

An Unusual Generalization

B-1279 Proposed by Pridon Davlianidze, Tbilisi, Republic of Georgia. (Vol. 58.3, November 2020)

Prove that

(A) $\displaystyle\prod_{n=1}^{\infty}\left(1+\frac{1}{F_{2n}F_{2n+1}}\right)=\alpha,$

(B) $\displaystyle\prod_{n=1}^{\infty}\left(1-\frac{1}{F_{2n-1}F_{2n+2}}\right)=\frac{1}{\alpha}.$

Solution by J. N. Senadheera, The Open University of Sri Lanka, Sri Lanka.

For $x\neq 0$, define $U_k=x^k-\frac{(-1)^k}{x^k}$ and $V_k=x^k+\frac{(-1)^k}{x^k}$. Observe that $U_0=0$ and $V_0=2$. Also $U_{-k}=(-1)^{k+1}U_k$ and $V_{-k}=(-1)^kV_k$. Furthermore, direct calculation yields the product formula

$$U_sU_t=V_{s+t}-(-1)^tV_{s-t}.$$

In particular, we obtain

$$\begin{aligned} U_{2k}U_{2k+1} &= V_{4k+1}-V_1, \\ U_{2k-1}U_{2k+2} &= V_{4k+1}+V_3. \end{aligned}$$

Thus,

$$U_{2k-1}U_{2k+2}=V_1+V_3+U_{2k}U_{2k+1}.$$

This implies that

$$1+\frac{V_1+V_3}{U_{2k}U_{2k+1}} = \frac{U_{2k-1}U_{2k+2}}{U_{2k}U_{2k+1}},$$
$$1-\frac{V_1+V_3}{U_{2k-1}U_{2k+2}} = \frac{U_{2k}U_{2k+1}}{U_{2k-1}U_{2k+2}}.$$

Hence, by the telescoping method,

$$\prod_{k=1}^{m}\left(1+\frac{V_1+V_3}{U_{2k}U_{2k+1}}\right) = \frac{U_1U_{2m+2}}{U_2U_{2m+1}},$$
$$\prod_{k=1}^{m}\left(1-\frac{V_1+V_3}{U_{2k-1}U_{2k+2}}\right) = \frac{U_2U_{2m+1}}{U_1U_{2m+2}}.$$

When $x=\alpha$, we find $U_k=\sqrt{5}\,F_k$, $V_1=L_1=1$, and $V_3=L_3=4$. Substituting these into the results above, we obtain

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{F_{2n}F_{2n+1}}\right)=\lim_{m\to\infty}\prod_{k=1}^{m}\left(1+\frac{1}{F_{2k}F_{2k+1}}\right)=\lim_{m\to\infty}\frac{F_{2m+2}}{F_{2m+1}}=\alpha,$$

and

$$\prod_{n=1}^{\infty}\left(1-\frac{1}{F_{2n-1}F_{2n+2}}\right)=\lim_{m\to\infty}\prod_{k=1}^{m}\left(1-\frac{1}{F_{2k-1}F_{2k+2}}\right)=\lim_{m\to\infty}\frac{F_{2m+1}}{F_{2m+2}}=\frac{1}{\alpha}.$$

Editor's Note: The two sequences $\{U_n\}_{n\geq 0}$ and $\{V_n\}_{n\geq 0}$ defined in the solution are associated with the characteristic polynomial $(q-x)\left(q+\frac{1}{x}\right)=q^2-\left(x-\frac{1}{x}\right)q-1$. However, they are not exactly the usual Fibonacci-Lucas pair that one would expect. The Binet formula would have yielded $U_n=\left(x^n-\frac{(-1)^n}{x^n}\right)/\left(x+\frac{1}{x}\right)$. The corresponding product formula would have been $U_sU_t=\left(V_{s+t}-(-1)^tV_{s-t}\right)/\left(x+\frac{1}{x}\right)$. Nevertheless, the definitions the solver used lead to a somewhat simpler product formula (with the absence of the denominator $x+\frac{1}{x}$), which may be regarded as an advantage.

Also solved by Thomas Achammer, Michel Bataille, Brian D. Beasley, Brian Bradie, Alejandro Cardona Castrillón (undergraduate), Charles K. Cook, Kenny B. Davenport, Steve Edwards, I. V. Fedak, Dmitry Fleischman, Robert Frontczak, Hideyuki Ohtsuka, Ángel Plaza, Raphael Schumacher (graduate student), Albert Stadler, David Terr, and the proposer.

<u>An Alternating Sum of the Squares of Tetranacci Numbers</u>

<u>B-1280</u> Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 58.4, November 2020)

The Tetranacci numbers T_n satisfy

$$T_n=T_{n-1}+T_{n-2}+T_{n-3}+T_{n-4}, \qquad \text{for } n\geq 3,$$

with $T_{-1}=T_0=0$ and $T_1=T_2=1$. Find a closed form expression for the sum $\displaystyle\sum_{k=1}^{n}(-1)^kT_k^2$.

Solution by Raphael Schumacher (graduate student), ETH Zurich, Switzerland.

It holds for all $k \in \mathbb{N}$ by the Tetranacci recurrence that

$$\begin{aligned}(-1)^k T_k^2 &= (-1)^k T_k \left(T_{k+2} - T_{k+1} - T_{k-1} - T_{k-2}\right)\\ &= (-1)^k \left(T_k T_{k+2} - T_k T_{k+1} - T_{k-1} T_k - T_{k-2} T_k\right)\\ &= (-1)^k \left(T_k T_{k+2} - T_k T_{k+1} - T_{k-1} T_{k+1} + T_{k-1} T_{k+1} - T_{k-1} T_k - T_{k-2} T_k\right)\\ &= (-1)^k \left(T_k T_{k+2} - T_k T_{k+1} - T_{k-1} T_{k+1}\right)\\ &\quad - (-1)^{k-1} \left(T_{k-1} T_{k+1} - T_{k-1} T_k - T_{k-2} T_k\right).\end{aligned}$$

Using this result, we obtain by telescoping that

$$\begin{aligned}\sum_{k=1}^{n} (-1)^k T_k^2 &= \sum_{k=1}^{n} \Big[(-1)^k \left(T_k T_{k+2} - T_k T_{k+1} - T_{k-1} T_{k+1}\right)\\ &\quad - (-1)^{k-1} \left(T_{k-1} T_{k+1} - T_{k-1} T_k - T_{k-2} T_k\right)\Big]\\ &= (-1)^n \left(T_n T_{n+2} - T_n T_{n+1} - T_{n-1} T_{n+1}\right),\end{aligned}$$

because $T_0 T_2 - T_0 T_1 - T_{-1} T_1 = 0$.

Editor's Notes: In a late submission, Tuenter derived a generalization for the sequence μ_n that satisfies the recurrence

$$\mu_n = \mu_{n-1} + \mu_{n-2} + \mu_{n-3} + \mu_{n-4} \qquad n \geq 3,$$

with arbitrary initial conditions μ_{-1}, μ_0, μ_1, and μ_2. Define

$$a_n = \mu_n \mu_{n+2} - \mu_n \mu_{n+1} - \mu_{n-1} \mu_{n+1}.$$

Using an argument similar to the solution featured above, it is easy to show that $\mu_k^2 = a_k + a_{k-1}$, which led Tuenter to draw the conclusion $\sum_{k=1}^{n} (-1)^k \mu_k^2 = (-1)^n a_n - a_0$.

Also solved by Kenny B. Davenport, I. V. Fedak, Albert Stadler, Hans J. H. Tuenter, and the proposer.

Belated Acknowledgment: Kenny B. Davenport also solved Problem B-1267.

Corrections: In the list of solvers in Problems B-1266 – B-1270, Senadheera's name was misspelled as Senadherra, and in Problem B-415, Paulso should be Paluso. The section editor apologizes for the errors in misspelling the last names of the solvers.