ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to Robert Frontczak, LBBW, Am Hauptbahnhof 2, 70173 Stuttgart, Germany, or by email at the address robert.frontczak@lbbw.de. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-931 Proposed by Ángel Plaza, Gran Canaria, Spain

For any positive integer k, the k-Fibonacci numbers $F_{k,n}$ and the k-Lucas numbers $L_{k,n}$ satisfy the recurrence relation $u_{n+2} = ku_{n+1} + u_n$ for $n \geq 0$, with initial values $F_{k,0} = 0$, $F_{k,1} = 1$, and $L_{k,0} = 2$, $L_{k,1} = k$.

(A) Find

$$\lim_{k\to\infty} \frac{1}{k} \sum_{n=1}^{\infty} \zeta(2n+1) \frac{F_{k,2n}}{(k^2+4)^n}.$$

(B) Prove that

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)F_{k,2n}}{(k^2+4)^n} = 1 + \sum_{n=1}^{\infty} \frac{\zeta(2n)L_{k,2n}}{(k^2+4)^n} = \frac{k\pi}{2\sqrt{k^2+4}} \tan\left(\frac{k\pi}{2\sqrt{k^2+4}}\right).$$

H-932 Proposed by Michel Bataille, Rouen, France

Let m, n be positive integers with m even and $n^2 \equiv 1 \pmod 6$. Prove that

$$\frac{L_{mn}-1}{L_m-1} = \varepsilon_0 + \sum_{j=1}^{n-1} \varepsilon_j L_{mj}$$

for some $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1} \in \{0, 1, -1\}$.

H-933 Proposed by Hideyuki Ohtsuka, Saitama, Japan

For positive integers r and n, let

$$P_r(n) = \frac{1}{\alpha^{2^{n+r}-2^n} F_{2^n} F_{2^{n+1}} \cdots F_{2^{n+r-1}}}.$$

Prove that

$$\sum_{n=1}^{\infty} P_r(n) \left(\frac{1}{F_{2^n}} + \frac{1}{F_{2^{n+1}}} + \cdots + \frac{1}{F_{2^{n+r}}}\right) = -\sqrt{5}\beta P_r(1).$$

H-934 Proposed by D. M. Bătineţu-Giurgiu, Bucharest, and Neculai Stanciu, Buzău, Romania

For $n \geq 1$, show that

$$m^m \sum_{k=1}^{n} (1+L_{2k-1})^{m+1} \geq (m+1)^{m+1} \left(L_{2n+2}-2\right) \quad \text{for all}\, m \in [1,\infty).$$

H-935 Proposed by the editor

Let $(B_n)_{n\geq 0}$ be the Bernoulli numbers defined by

$$\frac{z}{e^z-1} = \sum_{n\geq 0} B_n \frac{z^n}{n!}, \qquad |z| < 2\pi.$$

For $n \geq 0$, prove the identity

$$\sum_{j=0}^{n} \binom{2n+1}{2j+1} 5^{n-j} \left(2^{1-2n+2j}-1\right) B_{2n-2j} F_{2j+1} = \frac{2n+1}{4^n}.$$

SOLUTIONS

H-901 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 60, No. 3, August 2022)

Prove that

$$(i) \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{L_{2F_{n+1}}+L_{2F_n}} = \frac{1}{10} \qquad \text{and} \quad (ii) \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{L_{2L_{n+1}}+L_{2L_n}} = \frac{1}{15}.$$

Solution by Albert Stadler, Herrliberg, Switzerland

We note that

$$\begin{aligned}
\frac{1}{L_{2F_{n+1}}+L_{2F_n}} &= \frac{1}{\alpha^{2F_{n+1}}+\alpha^{-2F_{n+1}}+\alpha^{2F_n}+\alpha^{-2F_n}} \\
&= \frac{1}{\left(\alpha^{F_{n+1}+F_n}+\alpha^{-F_{n+1}-F_n}\right)\left(\alpha^{F_{n+1}-F_n}+\alpha^{-F_{n+1}+F_n}\right)} \\
&= \frac{1}{\left(\alpha^{F_{n+2}}+\alpha^{-F_{n+2}}\right)\left(\alpha^{F_{n-1}}+\alpha^{-F_{n-1}}\right)} \\
&= \frac{1}{\left(\alpha^{F_{n+2}}+\alpha^{-F_{n+2}}\right)\left(\alpha^{F_{n+2}}+\alpha^{-F_{n+2}}+\alpha^{F_{n-1}}+\alpha^{-F_{n-1}}\right)} \\
&\quad + \frac{1}{\left(\alpha^{F_{n-1}}+\alpha^{-F_{n-1}}\right)\left(\alpha^{F_{n+2}}+\alpha^{-F_{n+2}}+\alpha^{F_{n-1}}+\alpha^{-F_{n-1}}\right)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{1}{L_{2F_{n+1}}+L_{2F_n}}\\
=\;&\frac{1}{\left(\alpha^{F_{n+2}}+\alpha^{-F_{n+2}}\right)\left(\alpha^{\frac{F_{n+2}+F_{n-1}}{2}}+\alpha^{-\frac{F_{n+2}+F_{n-1}}{2}}\right)\left(\alpha^{\frac{F_{n+2}-F_{n-1}}{2}}+\alpha^{-\frac{F_{n+2}-F_{n-1}}{2}}\right)}\\
&+\frac{1}{\left(\alpha^{F_{n-1}}+\alpha^{-F_{n-1}}\right)\left(\alpha^{\frac{F_{n+2}+F_{n-1}}{2}}+\alpha^{-\frac{F_{n+2}+F_{n-1}}{2}}\right)\left(\alpha^{\frac{F_{n+2}-F_{n-1}}{2}}+\alpha^{-\frac{F_{n+2}-F_{n-1}}{2}}\right)}\\
=\;&\frac{1}{\left(\alpha^{F_{n+2}}+\alpha^{-F_{n+2}}\right)\left(\alpha^{F_{n+1}}+\alpha^{-F_{n+1}}\right)\left(\alpha^{F_n}+\alpha^{-F_n}\right)}\\
&+\frac{1}{\left(\alpha^{F_{n+1}}+\alpha^{-F_{n+1}}\right)\left(\alpha^{F_n}+\alpha^{-F_n}\right)\left(\alpha^{F_{n-1}}+\alpha^{-F_{n-1}}\right)}
\end{aligned}$$

and similarly,

$$\begin{aligned}
\frac{1}{L_{2L_{n+1}}+L_{2L_n}} &= \frac{1}{\left(\alpha^{L_{n+2}}+\alpha^{-L_{n+2}}\right)\left(\alpha^{L_{n+1}}+\alpha^{-L_{n+1}}\right)\left(\alpha^{L_n}+\alpha^{-L_n}\right)}\\
&\quad+\frac{1}{\left(\alpha^{L_{n+1}}+\alpha^{-L_{n+1}}\right)\left(\alpha^{L_n}+\alpha^{-L_n}\right)\left(\alpha^{L_{n-1}}+\alpha^{-L_{n-1}}\right)}.
\end{aligned}$$

This gives

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{(-1)^n}{L_{2F_{n+1}}+L_{2F_n}} &= \frac{1}{\left(\alpha^{F_1}+\alpha^{-F_1}\right)\left(\alpha^{F_0}+\alpha^{-F_0}\right)\left(\alpha^{F_{-1}}+\alpha^{-F_{-1}}\right)}\\
&= \frac{1}{2(\alpha^1+\alpha^{-1})^2}=\frac{1}{10},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{(-1)^n}{L_{2L_{n+1}}+L_{2L_n}} &= \frac{1}{\left(\alpha^{L_1}+\alpha^{-L_1}\right)\left(\alpha^{L_0}+\alpha^{-L_0}\right)\left(\alpha^{L_{-1}}+\alpha^{-L_{-1}}\right)}\\
&= \frac{1}{\left(\alpha^1+\alpha^{-1}\right)^2\left(\alpha^2+\alpha^{-2}\right)}=\frac{1}{15}.
\end{aligned}$$

Also solved by Dmitry Fleischman, David Terr, and the proposer.

H-902 Proposed by I. V. Fedak, Ivano-Frankivsk, Ukraine (Vol. 60, No. 3, August 2022)

For all positive integers n, prove that

$$2L_nL_{n+2}\left(\sqrt[L_n]{2}-\sqrt[L_{n+2}]{2}\right)\left(\sqrt[L_n]{6}-\sqrt[L_{n+2}]{6}\right)<L_{n+1}\left(\sqrt[L_n]{12}-\sqrt[L_{n+2}]{12}\right).$$

Solution by Michel Bataille, Rouen, France

Let $a=\ln 2$, $b=\ln 6$, and let $m=\frac{L_{n+1}}{L_nL_{n+2}}=\frac{1}{L_n}-\frac{1}{L_{n+2}}$. The inequality can be written as

$$(e^{a/L_n}-e^{a/L_{n+2}})(e^{b/L_n}-e^{b/L_{n+2}})<\frac{m}{2}(e^{(a+b)/L_n}-e^{(a+b)/L_{n+2}}),$$

that is,

$$(1-e^{-ma})(1-e^{-mb})<\frac{m}{2}(1-e^{-m(a+b)}). \tag{1}$$

Now, let $f(x) = \ln(\phi(x))$, where ϕ is the function defined on $[0, \infty)$ by $\phi(0) = 1$ and $\phi(x) = \frac{1-e^{-x}}{x}$ for positive x. The function f satisfies $f(0) = 0$ and is convex on $[0, \infty)$ (see a proof at the end). From Petrovic's inequality (see reference [1] or [2]), we have $f(ma) + f(mb) \leq f(ma + mb)$, hence $\phi(ma)\phi(mb) \leq \phi(m(a + b))$. This provides the inequality

$$(1 - e^{-ma})(1 - e^{-mb}) \leq \frac{abm}{a + b}(1 - e^{-m(a+b)})$$

and (1) follows because $\frac{ab}{a+b} = \frac{(\ln 2)(\ln 6)}{\ln 12} < \frac{1}{2}$.

Proof of the convexity of f. For positive x, we have

$$f'(x) = \frac{\phi'(x)}{\phi(x)} = \frac{xe^{-x} + e^{-x} - 1}{x - xe^{-x}}, \qquad f''(x) = \frac{N(x)}{(x - xe^{-x})^2},$$

where $N(x) = e^{-2x} - 2e^{-x} + 1 - x^2e^{-x}$. We have $N'(x) = e^{-x}U(x)$, where $U(x) = x^2 - 2x + 2 - 2e^{-x}$. From $U'(x) = 2(x - 1 + e^{-x}) > 0$ we deduce that $U(x) > U(0) = 0$, hence $N'(x) > 0$, and therefore $N(x) > N(0) = 0$. Thus, $f''(x) > 0$ for $x > 0$ and f is convex on $(0, \infty)$. In addition, because f is continuous on $[0, \infty)$, f is convex on $[0, \infty)$.

References

[1] D. S. Mitrinovic, *Analytic Inequalities*, Springer, 1970, p. 22.
[2] M. Bataille, *Focus on no. 41*, Crux Mathematicorum, 46.5 (2020), 222.

Also solved by Dmitry Fleischman, Ángel Plaza, Albert Stadler, and the proposer.

<u>H-903</u> Proposed by Robert Frontczak, Stuttgart, Germany (Vol. 60, No. 3, August 2022)

Prove that the Diophantine equations

$$3^nL_n + 4^n = 5^m \qquad \text{and} \qquad 3^n + 4^nL_n = 5^m$$

have no solutions in positive integers n and m.

Solution by David Terr, Coronado, CA

Let $a_n = 3^nL_n + 4^n$ and $b_n = 3^n + 4^nL_n$. For a solution of either of the above Diophantine equations to exist, either a_n or b_n must be divisible by 5. The following table lists all possibilities of 3^n, 4^n, L_n, a_n, and b_n modulo 5, corresponding to all possible values of n modulo 4. Here, $[x]$ denotes the value of x modulo 5.

n mod 4	$[3^n]$	$[4^n]$	$[L_n]$	$[a_n]$	$[b_n]$
0	1	1	2	3	3
1	3	4	1	2	2
2	4	1	3	3	2
3	2	4	4	2	3

Clearly, neither a_n nor b_n is ever divisible by 5, so there are no positive integer solutions to either of the given equations.

Also solved by Michel Bataille, Hideyuki Ohtsuka, Raphael Schumacher, Jason L. Smith, Albert Stadler, Andrés Ventas, and the proposer.

H-904 Proposed by Robert Frontczak, Stuttgart, Germany (Vol. 60, No. 3, August 2022)

Show that the following identities are valid for each even integer $m \geq 2$:

$$\sum_{n=1}^{\infty} \frac{L_{2mn} - L_{2n}}{(L_{2n} + L_{2m})(L_{2mn} + L_{2m})} = \frac{m-1}{\sqrt{5}F_{2m}}$$

and

$$\sum_{n=1}^{\infty} \frac{L_{2mn} + L_{2n} + 2L_{2m}}{(L_{2n} + L_{2m})(L_{2mn} + L_{2m})} = \frac{m+1}{\sqrt{5}F_{2m}} - \frac{1}{L_{2m}+2}.$$

Solution by Brian Bradie, Newport News, VA

Let $m \geq 2$ be an even integer, and let

$$S_1 = \sum_{n=1}^{\infty} \frac{1}{L_{2n} + L_{2m}} \quad \text{and} \quad S_2 = \sum_{n=1}^{\infty} \frac{1}{L_{2mn} + L_{2m}}.$$

Using the Binet form for the Lucas numbers,

$$\begin{aligned}
S_1 &= \sum_{n=1}^{\infty} \frac{1}{\alpha^{2n} + \frac{1}{\alpha^{2n}} + \alpha^{2m} + \frac{1}{\alpha^{2m}}} = \sum_{n=1}^{\infty} \frac{\alpha^{2n}}{\alpha^{4n} + \left(\alpha^{2m} + \frac{1}{\alpha^{2m}}\right)\alpha^{2n} + 1} \\
&= \sum_{n=1}^{\infty} \frac{\alpha^{2n}}{(\alpha^{2n} + \alpha^{2m})(\alpha^{2n} + \frac{1}{\alpha^{2m}})} \\
&= \frac{1}{\sqrt{5}F_{2m}} \sum_{n=1}^{\infty} \left(\frac{\alpha^{2m}}{\alpha^{2n} + \alpha^{2m}} - \frac{\frac{1}{\alpha^{2m}}}{\alpha^{2n} + \frac{1}{\alpha^{2m}}} \right).
\end{aligned}$$

Note that this last sum telescopes after $2m$ terms. Thus,

$$\begin{aligned}
S_1 &= \frac{1}{\sqrt{5}F_{2m}} \sum_{n=1}^{2m} \frac{\alpha^{2m}}{\alpha^{2n} + \alpha^{2m}} \\
&= \frac{1}{\sqrt{5}F_{2m}} \left(\frac{1}{2} + \frac{\alpha^{2m}}{\alpha^{4m} + \alpha^{2m}} + \sum_{n=1}^{m-1} \left(\frac{\alpha^{2m}}{\alpha^{2n} + \alpha^{2m}} + \frac{\alpha^{2m}}{\alpha^{4m-2n} + \alpha^{2m}} \right) \right) \\
&= \frac{1}{\sqrt{5}F_{2m}} \left(m + \frac{\alpha^{2m}}{\alpha^{4m} + \alpha^{2m}} - \frac{1}{2} \right) \\
&= \frac{m}{\sqrt{5}F_{2m}} + \frac{1}{2\sqrt{5}F_{2m}} \cdot \frac{\alpha^{2m} - \alpha^{4m}}{\alpha^{2m} + \alpha^{4m}} \\
&= \frac{m}{\sqrt{5}F_{2m}} - \frac{\sqrt{5}F_m}{2\sqrt{5}F_{2m}L_m} = \frac{m}{\sqrt{5}F_{2m}} - \frac{1}{2L_m^2}.
\end{aligned}$$

Because m is an even integer, $L_m^2 = L_{2m} + 2$. Therefore,

$$S_1 = \frac{m}{\sqrt{5}F_{2m}} - \frac{1}{2(L_{2m}+2)}.$$

Next,

$$\begin{aligned} S_2 &= \sum_{n=1}^{\infty} \frac{1}{\alpha^{2mn} + \frac{1}{\alpha^{2mn}} + \alpha^{2m} + \frac{1}{\alpha^{2m}}} = \sum_{n=1}^{\infty} \frac{\alpha^{2mn}}{\alpha^{4mn} + \left(\alpha^{2m} + \frac{1}{\alpha^{2m}}\right)\alpha^{2mn} + 1} \\ &= \sum_{n=1}^{\infty} \frac{\alpha^{2mn}}{(\alpha^{2mn} + \alpha^{2m})(\alpha^{2mn} + \frac{1}{\alpha^{2m}})} \\ &= \frac{1}{\sqrt{5}F_{2m}} \sum_{n=1}^{\infty} \left(\frac{\alpha^{2m}}{\alpha^{2mn} + \alpha^{2m}} - \frac{\frac{1}{\alpha^{2m}}}{\alpha^{2mn} + \frac{1}{\alpha^{2m}}} \right), \end{aligned}$$

which telescopes after two terms. Thus,

$$S_2 = \frac{1}{\sqrt{5}F_{2m}} \left(\frac{1}{2} + \frac{\alpha^{2m}}{\alpha^{4m} + \alpha^{2m}} \right) = \frac{1}{\sqrt{5}F_{2m}} - \frac{1}{2(L_{2m} + 2)}.$$

Finally,

$$\sum_{n=1}^{\infty} \frac{L_{2mn} - L_{2n}}{(L_{2n} + L_{2m})(L_{2mn} + L_{2m})} = S_1 - S_2 = \frac{m-1}{\sqrt{5}F_{2m}}$$

and

$$\sum_{n=1}^{\infty} \frac{L_{2mn} + L_{2n} + 2L_{2m}}{(L_{2n} + L_{2m})(L_{2mn} + L_{2m})} = S_1 + S_2 = \frac{m+1}{\sqrt{5}F_{2m}} - \frac{1}{L_{2m} + 2}.$$

Also solved by Michel Bataille, Dmitry Fleischman, Won Kyun Jeong, Hideyuki Ohtsuka, Albert Stadler, Yunyong Zhang, and the proposer.

H-905 Proposed by Kai Wang, Henderson, NV
(Vol. 60, No. 3, August 2022)

Prove the following identities:

(i) $\dfrac{\pi^2}{16} = \displaystyle\sum_{i=0}^{\infty} \left(\arctan \frac{1}{F_{4i-1}} \arctan \frac{3}{F_{4i+1}} - \arctan \frac{3}{F_{4i+1}} \arctan \frac{1}{F_{4i+3}} \right)$, where $F_{-1} = 1$.

(ii) $\dfrac{\pi}{4} = \displaystyle\sum_{i=0}^{\infty} (-1)^i \arctan \frac{3}{F_{4i+1}}$.

Solution by Ángel Plaza, Gran Canaria, Spain

Note that for $i \geq 0$, we have

$$\tan\left[\arctan \frac{1}{F_{4i-1}} - \arctan \frac{3}{F_{4i+1}}\right] = \frac{\frac{1}{F_{4i-1}} - \frac{3}{F_{4i+1}}}{1 + \frac{1}{F_{4i-1}} \cdot \frac{3}{F_{4i+1}}} = \frac{F_{4i+1} - 3F_{4i-1}}{F_{4i-1}F_{4i+1} + 3} = \frac{-F_{4i-3}}{F_{4i}^2 + 4} = \frac{-1}{F_{4i+3}}.$$

Therefore, $\arctan \frac{3}{F_{4i+1}} = \arctan \frac{1}{F_{4i-1}} + \arctan \frac{1}{F_{4i+3}}$, and so

$$\text{(ii)} \sum_{i=0}^{\infty} (-1)^i \arctan \frac{3}{F_{4i+1}} = \sum_{i=0}^{\infty} (-1)^i \left(\arctan \frac{1}{F_{4i-1}} + \arctan \frac{1}{F_{4i+3}} \right)$$
$$= \arctan \frac{1}{F_{-1}} - \lim_{i\to\infty} \arctan \frac{1}{F_{4i+3}}$$
$$= \arctan 1 = \frac{\pi}{4}.$$

$$\text{(i)} \sum_{i=0}^{\infty} \left(\arctan \frac{1}{F_{4i-1}} \arctan \frac{3}{F_{4i+1}} - \arctan \frac{3}{F_{4i+1}} \arctan \frac{1}{F_{4i+3}} \right)$$
$$= \sum_{i=0}^{\infty} \left(\arctan^2 \frac{1}{F_{4i-1}} - \arctan^2 \frac{1}{F_{4i+3}} \right)$$
$$= \arctan^2 \frac{1}{F_{-1}} - \lim_{i\to\infty} \arctan^2 \frac{1}{F_{4i+3}} = \frac{\pi^2}{16}.$$

Also solved by Michel Bataille, Brian Bradie, Dmitry Fleischman, Hideyuki Ohtsuka, Albert Stadler, Andrés Ventas, Yunyong Zhang, and the proposer.

Late Acknowledgement. Jesús Jiménez, San Diego, CA, solved **H-871**.
He showed the following generalization: If $p > 2$ is a prime number and the sequence $(U_n)_{n=0}^{\infty}$ is defined by the linear recurrence relation $U_{n+2} = a_1 U_{n+1} + a_0 U_n$, where $U_0 = 0$, $U_1 = 1$, and a_0, a_1 are integers satisfying $a_1^2 - 4a_0 \neq 0$, $a_0 \not\equiv 0 \bmod p$, $a_1^2 - 4a_0 \equiv 0 \bmod p$, and the characteristic polynomial $f(x) = x^2 - a_1 x - a_0$ is irreducible in $\mathbb{F}_2[x]$, then

$$\frac{U_n U_{2pn} - U_{2n} U_{pn}}{p U_n^2} \equiv 0 \bmod 2p.$$