COUNTING ON HOSOYA'S TRIANGLE

ARTHUR T. BENJAMIN AND DANIELA ELIZONDO

ABSTRACT. We present combinatorial proofs of identities inspired by the Hosoya Triangle.

1. INTRODUCTION

Behold the Hosoya Triangle, first introduced by Haruo Hosoya in 1976 [11].

$$\begin{array}{ccccccccccccccccc}
&&&&&&&&1&&&&&&&&\\
&&&&&&&1&&1&&&&&&&\\
&&&&&&2&&1&&2&&&&&&\\
&&&&&3&&2&&2&&3&&&&&\\
&&&&5&&3&&4&&3&&5&&&&\\
&&&8&&5&&6&&6&&5&&8&&&\\
&&13&&8&&10&&9&&10&&8&&13&&\\
&21&&13&&16&&15&&15&&16&&13&&21&\\
34&&21&&26&&24&&25&&24&&26&&21&&34
\end{array}$$

FIGURE 1. The Hosoya triangle

All of the entries of the triangle are products of Fibonacci numbers. For $m \geq 0$ and $0 \leq n \leq m$, we define

$$H(m,n) = f_{m-n}f_n,$$

where $f_0 = f_1 = 1$. Equivalently, if we let $h(m,n) = H(m+n,n)$, then for $m,n \geq 0$, we have

$$h(m,n) = f_m f_n.$$

From this definition, some properties of the triangle are immediately revealed. For instance, for $m \geq 2$ and $n \geq 0$, $h(m,n) = h(m-1,n) + h(m-2,n)$. Likewise, for $n \geq 2$ and $m \geq 0$, $h(m,n) = h(m,n-1) + h(m,n-2)$. Thus for $m,n \geq 2$, every entry is the sum of the two entries diagonal to the left as well as to the right. Likewise, we immediately have what Hosoya calls The Amoeba identity that for $a,b,c,d \geq 0$, $h(a,b)h(c,d) = h(a,c)h(b,d)$.

Hosoya's Triangle and its variants have been studied extensively by Rigoberto Flórez and others [2, 3, 5, 7, 9, 10, 12]. The goal of this paper is to prove many of these identities with combinatorial tiling proofs in the spirit of Benjamin and Quinn [1]. Here, the fundamental combinatorial property is that f_n counts tilings of a $1 \times n$ strip of length n with squares (of length 1) and dominoes (of length 2). Thus $h(m,n) = f_m f_n$ counts tiling pairs (X,Y), where X has length m and Y has length n. Equivalently, $h(m,n) = H(m+n,n)$ counts tilings of length $m+n$ that are *breakable* at cell n, since there are f_n ways to tile cells 1 through n, then f_m ways to tile cells $n+1$ through $m+n$. By the same logic, it's also equal to the number of tilings of length $m+n$ that are breakable at cell m.

2. Combinatorial Proofs

In this section, we explore many simple properties of the Hosoya Triangle and give correspondingly simple combinatorial proofs. Let's begin with the property that the sum of two entries that are "vertically consecutive" sum to a Fibonacci number. For example, $H(4,1) + H(6,2) = 3 + 10 = 13 = f_6$. Specifically, for $m, n \geq 0$,

$$H(m+n,n) + H(m+n+2,n+1) = f_{m+n+2}.$$

The right side of the identity counts all tilings of length $m+n+2$. On the left, $H(m+n+2,n+1)$ counts those tilings of length $m+n+2$ that are breakable at cell $n+1$. The unbreakable tilings have a domino covering cells $n+1$ and $n+2$ and the remaining cells can be tiled $H(m+n,n)$ ways. Altogether there are $H(m+n,n) + H(m+n+2,n+1)$ tilings of length $m+n+2$.

Hosoya gave colorful names to many of the identities in his original paper. The next four identities pertained to "Magic Diamonds" involving four diagonally adjacent entries like $h(3,1) = 3$, $h(4,1) = 5$, $h(3,2) = 6$, and $h(4,2) = 10$. Summing these four numbers gives us the number directly below the diamond. (In our example, they sum to $24 = h(5,3)$.) In general, we have the following Downward Generation identity. For $m, n \geq 0$,

$$h(m-1,n-1) + h(m,n-1) + h(m-1,n) + h(m,n) = h(m+1,n+1).$$

The right side counts tiling pairs (X, Y) where X has length $m+1$ and Y has length $n+1$. The four terms on the left represent the four ways that these tiling pairs can end (dd, sd, ds, or ss). For example, the second summand counts those tiling pairs where X ends in a square and Y ends in a domino.

When the vertical elements of the diamond are added and the horizontal ones subtracted, we obtain the number above the diamond. That is, for $m, n \geq 0$,

$$h(m,n) + h(m-1,n-1) - h(m-1,n) - h(m,n-1) = h(m-2,n-2),$$

which we rearrange to read

$$h(m,n) = h(m-2,n-2) + h(m-1,n) + h(m,n-1) - h(m-1,n-1).$$

Here the left side counts tiling pairs (X, Y), where X has length m, and Y has length n. On the right, the first summand counts those where both end with a domino. The next three terms count those where X or Y ends with a square.

When we add the lower left pair and subtract the upper right pair, we get the number to the right of the diamond. That is, for $m, n \geq 0$,

$$h(m,n) - h(m-1,n) + h(m,n-1) - h(m-1,n-1) = h(m-2,n+1).$$

The first two terms count tiling pairs (X, Y) where X has length $m-2$ and Y has length n. The next two terms count tiling pairs (X, Y) where X has length $m-2$ and Y has length $n-1$. Together, they count tiling pairs (X, Y) where X has length $m-2$ and Y has length $n+1$, which is counted by the $h(m-2,n+1)$ term.

The same type of argument (which we leave as an exercise) generates the number to the left of the diamond: for $m, n \geq 0$,

$$h(m,n) - h(m,n-1) + h(m-1,n) - h(m-1,n-1) = h(m+1,n-2).$$

The next identity is named the "Crawling Crab" property by Hosoya. For $m \geq 2$, and $1 \leq n \leq m-1$,

$$H(m-1,n-1) + H(m,n) + H(m-1,n) = f_{m+1}.$$

The right side counts tilings of length $m+1$. Consider the tile that covers cell $n+1$. It can be covered by a square in $H(m,n)$ ways; it can be covered by the left end of a domino in

$H(m-1,n)$ ways and it can be covered by the right end of a domino in $H(m-1,n-1)$ ways, as enumerated by the left side.

Likewise, the "Inverted Crab" identity says that for $m \geq n \geq 0$,

$$H(m+1,n) - H(m,n) + H(m+1,n+1) = f_{m+1}.$$

Here we exploit the fact that a tiling cannot be unbreakable at two consecutive cells. The number of $(m+1)$-tilings that are breakable at cell n is $H(m+1,n)$; the number that are breakable at cell $n+1$ is $H(m+1,n+1)$. To avoid double-counting, we subtract the $H(m,n)$ tilings that have a square at cell $n+1$.

Our next theorems, from [8] are reminiscent of the familiar Hockey Stick theorem from Pascal's triangle, which states that $\sum_{i=k}^{n} \binom{n}{k} = \binom{n+1}{k+1}$ for natural numbers n, k. In Pascal's triangle, this identity is aptly named because the sum is on the "blade" of the hockey stick, and the terms of the sum form the "handle."

We will start with the Central Hockey Stick Theorem, obtained by partially summing the central numbers beginning with $H(0,0)$. For $n \geq 0$,

$$\sum_{k=0}^{n} h(k,k) = h(n,n+1).$$

We note that when this is expressed in terms of Fibonacci numbers, it becomes the familiar identity $\sum_{k=0}^{n} f_k^2 = f_n f_{n+1}$, and our tiling proof is essentially the same as the one given in [1]. The right side counts tiling pairs (X,Y) where X has length n (covering cells 1 through n and Y has length $n+1$ (covering cells 1 through $n+1$. For $0 \leq k \leq n$, how many tiling pairs where k is the *last* cell where both tilings are breakable? (We say (X,Y) has a *fault line* at k.) After cell k, there is exactly one way to tile X and Y to avoid a future fault line. (Specifically, if $n-k$ is even, then X has all dominos after cell k. Y has a square at cell $k+1$ and all dominoes after that. If $n-k$ is odd, then the situation is reversed.) Prior to cell k, (X,Y) can be tiled in $h(k,k)$ ways. (We note that X and Y are both considered to be breakable at cell 0.) All together, the number of (X,Y) pairs is $\sum_{k=0}^{n} h(k,k)$, as enumerated by the left side.

The closed form for the general Hockey Stick Theorem depends on the parity of the number of terms that are being added. Starting with the first entry of Row $m \geq 0$, we have,

$$\sum_{k=0}^{n} h(m+k,k) = \begin{cases} h(m+n,n+1) & n \text{ is even} \\ h(m+n+1,n) & n \text{ is odd.} \end{cases}$$

We note that when $m = 0$, both cases reduce to $h(n,n+1)$, as given in the Central Hockey Stick Theorem, and our proof is a generalization of the previous argument.

We begin with the case where n is even. The right side term, $h(m+n,n+1)$ counts tiling pairs (X,Y) where X has length $m+n$, and Y has length $n+1$. Here X covers cells 1 through $m+n$, but Y covers cells $m+1$ through $m+n+1$ as in Figure 2.

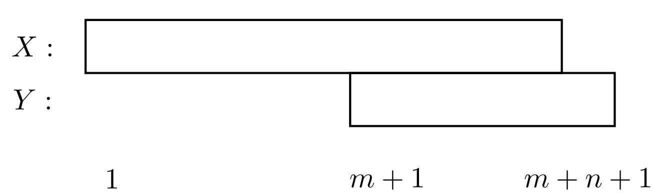

FIGURE 2. Finding the last fault when n is even

Note that since $n+1$ is odd, tiling Y cannot consist of all dominoes, so there must exist a fault line at cell $m+k$ for some $0 \le k \le n$. As in the previous proof, to the right of the fault line, there is exactly one way to complete the tiling so that no fault line occurs. Hence the number of (X, Y) pairs where the last fault line occurs at cell $m+k$ is $h(m+k, k)$. Summing over all possible values of k gives us the left side of our identity.

In the case where n is odd, the right side $h(m+n+1, n)$ counts tiling pairs (X, Y) where X has length $m+n+1$ and Y has length n. Here we let X cover cells 1 through $m+n+1$, and Y covers cells $m+1$ through $m+n$, as shown in Figure 3. Since n is odd, tiling Y cannot consist of all dominoes, so a fault must exist at some cell $m+k$ for some $0 \le k \le n$. As before, there is one way to place the tiles of X and Y after cell $m+k$, so there are $h(m+k, k)$ tiling pairs where the last fault occurs at cell $m+k$, and the proof is complete.

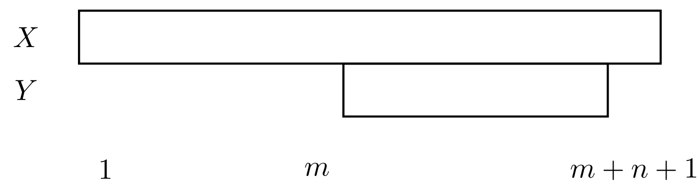

FIGURE 3. Finding the last fault in the odd case

We conclude this section with the following Lucas number identity from Hosoya's original paper. For $m \ge n \ge 0$,

$$5h(m, n) = L_{m+n+2} + (-1)^n L_{m-n}.$$

Although the identity is straightforward to prove by Binet's formula, the challenge was to find a combinatorial proof. As usual, $h(m, n)$ counts tiling pairs (X, Y) where X has length m and Y has length n. Recall from [1] that L_n counts *bracelets* of length n that can be tiled with squares and dominoes, where a domino is allowed to cover cells n and 1. (Such bracelets are called *out of phase*.) We denote out of phase bracelets by ending the tiling with $\overline{d}$. Thus, for example, the $L_4 = 7$ bracelets of length 4 can be described by $ssss$, ssd, sds, dss, dd, $ss\overline{d}$, and $d\overline{d}$, where the first 5 bracelets are in phase and the last 2 bracelets are out of phase. Note that there are f_n in phase bracelets of length n and f_{n-2} out of phase bracelets of length n, and therefore $L_n = f_n + f_{n-2}$.

To prove the identity combinatorially, we create an "almost 1-to-5 correspondence" between our tiling pairs and bracelets of length $m+n+2$. That is, for every tiling pair, we attempt to create five bracelets of length $m+n+2$, and create each $(m+n+2)$-bracelet exactly once. As our proof will show, when n is odd, we will always be able to create five bracelets from each tiling pair, and all $5h(m, n)$ bracelets will be distinct, but there will be L_{m-n} bracelets of size $m+n+2$ that are not created. On the other hand, when n is even, we will be able to create all bracelets of size $m+n+2$, but there will be L_{m-n} tiling pairs where we are able to create only four bracelets instead of five.

The first three bracelets B_1, B_2, B_3 are straightforward to construct. They are

$$B_1 = YXd,$$

$$B_2 = YXss,$$

$$B_3 = YX\overline{d}.$$

Note that these are bracelets of length $m+n+2$ that are breakable at cell n.

For the fourth bracelet, we will have three cases depending on how X and Y end. Suppose that X ends in a square, that is, $X = X's$ for some $(m-1)$-tiling X'. Then we can create

$$B_{4a} = YX'ds,$$

which is an $(m+n+2)$-bracelet that ends in ds and is breakable at cell n.

Note that a bracelet may end with either a square, a domino, or an out of phase domino. With braclets B_1, B_2, B_3 and B_{4a}, we have created all possible bracelets that are breakable at n. Thus, it remains for us to create the *unbreakable* bracelets, which have a domino covering cells n and $n+1$. With bracelet B_{4a}, we used the domain item where X ends in a square, so now suppose that X ends in a domino and Y ends in a square—that is, $X = X''d, Y = Y's$. We can then create an unbreakable bracelet that ends with sd, namely

$$B_{4b} = Y'dX''sd.$$

Lastly, if $X = X''d, Y = Y''d$, we create

$$B_{4c} = Y''sdX''s\overline{d}.$$

We move on to the fifth and final bracelet. We still need to create unbreakable bracelets that end in s, dd, or $d\overline{d}$. Also, since B_{4c} has a square at cell $n-1$, we also need to create an unbreakable bracelet that ends in $s\overline{d}$, but has a domino ending at cell $n-1$. To achieve these bracelets, we consider six cases, depending on the endings of X and Y.

If Y ends in a square, that is, $Y = Y's$, then we create

$$B_{5a} = Y'dXs,$$

which covers all unbreakable bracelets that end in s. If instead we have $Y = Y''d, X = X'''sd$, then we create

$$B_{5b} = Y''sdX'''d\overline{d}.$$

If $Y = Y''d, X = X'''ds$, then

$$B_{5c} = Y''sdX'''dd.$$

It remains for us to create unbreakable bracelets that end in $s\overline{d}$, $d\overline{d}$, or dd that have a domino ending at cell $n-1$. That is, they need to have two dominoes covering cells $n-2$ through $n+1$. At this point, we have yet to say what to do when Y ends with a domino and X ends in ss or dd.

Next, if $Y = Y'''sd, X = X''ss$, then we create bracelet

$$B_{5d} = Y'''ddX''s\overline{d},$$

which covers unbreakable bracelets that end in $s\overline{d}$, and have a domino ending at cell $n-1$. It remains to create unbreakable bracelets that end in $d\overline{d}$ or dd that have a domino ending at cell $n-1$.

To complete bracelets that end with $d\overline{d}$, we need to use a *tail swap*. Suppose that $Y = Y''d, X = X''''dd$. We arrange X and Y so that Y is on top of X and Y begins at cell $m-n$. That is, Y is an n-tiling with cells $m-n$ through $m-1$. We say that there is a *fault* at cell k if both X and Y are breakable at k. See Figure 4.

The parts of X and Y to the right of the last fault are called the *tails* of X and Y. If we swap the tails of X and Y, we obtain two new tilings: X^- tiles an $(m-1)-$tiling that ends in a domino, and Y^+ tiles an $(n+1)-$tiling that ends in two dominoes. This is illustrated in Figure 5.

Then, as shown in Figure 6,

$$B_{5e} = Y^+X^-\overline{d}.$$

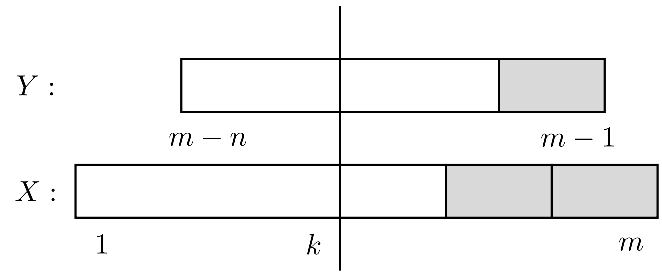

FIGURE 4. The X and Y tilings before the first tail swap.

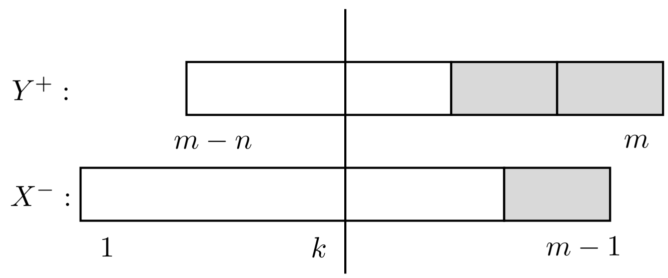

FIGURE 5. The X^- and Y^+ tilings after the first tail swap.

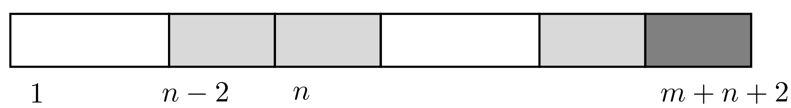

FIGURE 6. Bracelet B_{5e}

B_{5e} has length $m+n+2$, covers cells 1 through $m+n+2$, is unbreakable at n, ends in $d\overline{d}$, and has a domino ending at cell $n-1$. We have now constructed all bracelets with $\overline{d}$ endings.

We will use a similar tail swap to complete bracelets with dd endings. The only domain item remaining is $Y = Y''''dd, X = X''ss$. We replace the two final squares of X with a domino. We arrange X and Y as in the previous tail swap, except this time Y starts at cell $m-n+2$. We then glue an extra domino to the right of X. This is illustrated in Figure 7.

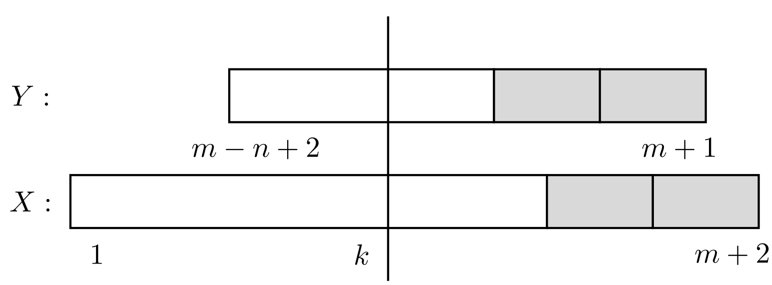

FIGURE 7. The X and Y tilings before the second tail swap.

The tail swap yields an $(n+1)$-tiling that ends in two dominos which we call Y^+ and an $(m+1)-$tiling that ends in two dominos which we call X^-, as seen in Figure 8.

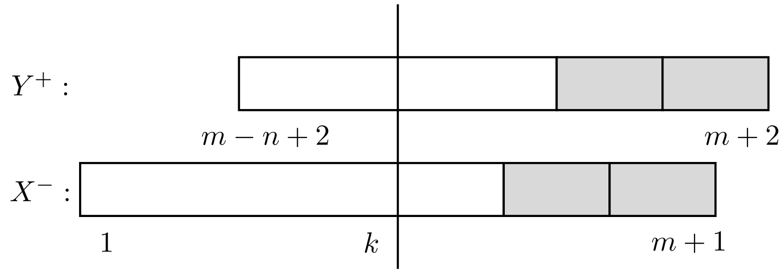

FIGURE 8. The X^- and Y^+ tilings after the second tail swap.

Then, the final bracelet is

$$B_{5f} = Y^+X^-.$$

Like B_{5e}, B_{5f} is a bracelet of length $m+n+2$, covers cells 1 through $m+n+2$, and has two dominoes covering cells $n-2$ through $n+1$. The only difference between the two bracelets is that B_{5f} ends in dd instead of $d\overline{d}$.

We now account for the error term $(-1)^n L_{m-n}$. We first consider the case where n is even. In this case, it is not always possible to tail swap. Specifically, the tail swap used to create bracelet B_{5e} is not possible whenever Y consists of exactly $\frac{n}{2}$ dominoes and X has no squares among its last $n+2$ cells. In such cases, there is no fault to enable a tail-swap. Since this requires fixing the last $n+2$ cells of X to be tiled with dominoes, then we fail to create B_{5e} exactly $f_{m-(n+2)}$ times.

Similarly, we cannot always perform the tail swap necessary to create bracelet B_{5f}. If Y again is all dominoes and X has no squares among its last n cells, then we cannot tail swap. We note that X needs one less domino to ensure the impossibility of tail swapping than in the previous swap because we glued an additional domino onto X to create B_{5f}. If X ends in $\frac{n}{2}$ dominoes, then X can be tiled in f_{m-n} ways. So, there are exactly f_{m-n} times we cannot perform the tail swap to create B_{5f}.

Summarizing our results, although we are able to construct all bracelets of length $m+n+2$, there are $f_{m-n-2} + f_{m-n} = L_{m-n}$ times that we are unable to create a fifth bracelet. Thus, when n is even, we have that $5f_mf_n - L_{m-n} = L_{m+n+2}$.

We now consider the case where n is odd. In this case, Y must contain at least one square and so tail swapping will always be possible. However, there are some $(m+n+2)$-bracelets that we fail to create. In both tail swapping constructions, the swap occurs at the last fault, which requires the existence of a square in X or Y.

For B_{5e}, if the square was in Y, then the tail swap necessarily puts a square among the last n cells of X^-. We are therefore not creating bracelets that end in $\overline{d}$ preceded by $\frac{n+1}{2}$ dominoes. If the square was in X, then the tail swap puts a square in the first n cells of Y^+. Since the square may come from X or Y, we are not creating bracelets that start with $\frac{n+1}{2}$ dominoes and end with $\overline{d}$ preceded by $\frac{n+1}{2}$ dominoes, of which there are exactly $f_{m+n+2-(n+1)-(n+1)-2} = f_{m-n-2}$.

For B_{5f}, the tail swap construction similarly puts a square in either the first n cells of Y^+ or the last n cells of X^-. Therefore, we are not creating bracelets that start with $\frac{n+1}{2}$ dominoes and end with $\frac{n+1}{2}$ dominoes, of which there are $f_{m+n+2-(n+1)-(n+1)} = f_{m-n}$.

All together, when n is odd, there are exactly $f_{m-n-2} + f_{m-n} = L_{m-n}$ bracelets that we fail to create. So, in this case, we have that $5f_mf_n + L_{m-n} = L_{m+n+2}$.

Putting the results of these two cases together, we get that $5h(m,n) = L_{m+n+2}+(-1)^n L_{m-n}$, as desired.

3. Further Work

The Hosoya triangle has many more properties than we have discussed. It would be interesting to see how many of those could also be proved by combinatorial arguments.

We briefly explore a variant of the Hosoya Triangle, described in [4] The triangle in Figure 9 is called the Determinant Hosoya Triangle. Like the Hosoya Triangle, it has Fibonacci numbers on the boundary: $d(m,0) = f_m$ and $d(0,n) = f_n$, for $m,n \geq 0$. The only initial condition that differs from the previous triangle is that $d(1,1) = 0$. (We note that [4] uses slightly shifted initial conditions.) The remaining points in the triangle satisfy the same two recurrences as before, namely that for $n \geq 2, d(m,n) = d(m,n-1) + d(m,n-2)$ and for $m \geq 2$, $d(m,n) = d(m-1,n) + d(m-2,n)$. See Figure 9.

1
1 1
2 0 2
3 1 1 3
5 1 3 1 5
8 2 4 4 2 8
13 3 7 5 7 3 13
21 5 11 9 9 11 5 21
34 8 18 14 16 14 18 8 34

Figure 9. The Determinant Hosoya triangle

It is straightforward to verify that for $m,n \geq 1$,

$$d(m,n) = h(m,n) - h(m-1,n-1) = f_m f_n - f_{m-1} f_{n-1},$$

which is the determinant of the matrix $\begin{vmatrix} f_m & f_{n-1} \\ f_n & f_{m-1} \end{vmatrix}$. Like the original Hosoya triangle, the determinant Hosoya triangle has many interesting patterns and properties, as explored in [2, 3].

The formula for $d(m,n)$ leads to two simple combinatorial interpretations. Specifically, $d(m,n)$ counts tiling pairs (X,Y) where X has length m, Y has length n, and they do not both end with squares. Equivalently, $d(m,n)$ counts tilings of length $m+n$ that are breakable at cell m, with the restriction that cells m and $m+1$ are not both occupied by squares. As initiated in [6], it would be interesting to see if these interpretations can yield simple combinatorial proofs of patterns found in this triangle.

References

[1] A. T. Benjamin and J. J. Quinn, *Proofs That Really Count: The Art of Combinatorial Proofs*, MAA Press, 2003.

[2] M. Blair, R. Florez, and A. Mukherjee, *Matrices in the Hosoya Triangle*, The Fibonacci Quarterly, **57.5** (2019), 15–28

[3] M. Blair, R. Florez, and A. Mukherjee, *Honeycombs in the Hosoya Triangle* Math Horizons, **29.3** (2022), 24–27

[4] M.Blair, R. Florez, A. Mukherjee, and J. L. Ramirez, *Matrices in the determinant Hosoya triangle,* The Fibonacci Quarterly, **57.5** (2020), 34–54.

[5] E. Czabarka, R. Florez, and L. Junes, *A Discrete Convolution on the Generalized Hosoya Triangle.* Journal of Integer Sequences, **18.1** (2015), 1–22.

[6] D. Elizondo, *Counting on the Hosoya Triangle*, Senior Thesis, Harvey Mudd College, 2021.

[7] R. Flórez, R. Higuita, and A. Mukherjee, *Alternating sums in the Hosoya polynomial triangle*, J. Integer Seq., **17** (2014), Article 14.9.5.
[8] R. Florez, R. Higuita, and A. Mukherjee, The Geometry of some Fibonacci Identities in the Hosoya Triangle, in preparation.
[9] R. Flórez, R. Higuita, and L. Junes, *GCD property of the generalized star of David in generalized Hosoya triangle*, J. Integer Seq., **17** (2014), Article 14.3.6.
[10] R. Flórez and L. Junes, *GCD properties in Hosoya's triangle*, Fibonacci Quart. **50** (2012), 163–174.
[11] H. Hosoya, *Fibonacci Triangle*, The Fibonacci Quarterly, **14** (1976), 173–178.
[12] T. Koshy, *Fibonacci and Lucas Numbers with Applications,* Wiley-Interscience, 2001.

MSC2020: 11B39, 05A19

Department of Mathematics, Harvey Mudd College, Claremont, CA, 91711
Email address: benjamin@hmc.edu

Department of Mathematics, Rutgers University, Piscataway, NJ, 08854
Email address: dae86@scarletmail.rutgers.edu