TOPOGRAPHS; CONWAY AND OTHERWISE

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Dedicated to the memory of John H. Conway, 1937–2020

Abstract. A planar embedding of the three-regular tree results in a hyperbolic tiling of the plane. A topograph is this figure along with numbers (or a generalization of numbers) assigned to to these tiles such that a certain local rule is satisfied. Conway topographs, the first topographs to have been studied, are closely related to the theory of quadratic forms. Changing the type of "number" or the local rule gives rise to other types of topograph. We study several of these and their relationship with various areas of mathematics.

1. Introduction

A topograph is an arrangement of numbers on the faces of an embedding of the 3-regular tree that satisfies a given local rule. As such, they are higher dimensional analogues of recursively defined sequences. In particular, we embed the 3-regular tree (there is only one) in the plane as illustrated here:

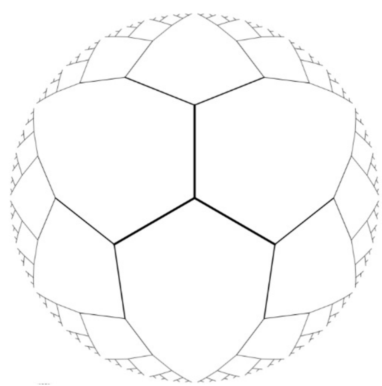


Figure 1. The Geometric Topograph

To each edge, we may name regions where N, S, E, W as illustrated in Figure 2 and assign values according to some formula of the form $F(N, S) = G(E, W)$. For consistency, this must be equivalent to $F(S, N) = G(W, E)$. Further, for three initial adjacent values to inductively define all values in the topograph, we also assume that $F(\cdot, S)$ is 1-1 for every S. In most of our examples, the function F will be the same for all edges but, in a couple of examples, we allow F and G to vary according to the "color" of the central edge.

Conway topographs — topographs with rule $N+S = 2(E+W)$ — are, by now, well studied. They are a main feature of a book by J.H. Conway [2] and they also feature prominently in a book by Weissman [12] as well as a very recent book by Hatcher [4]. We cannot add much to their thorough expositions but we do give an apparently novel application to Fermat's two-square theorem.

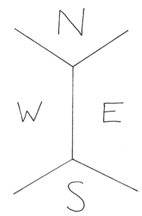


Figure 2. At a generic edge

In later sections, we consider Conway topographs with vectors as well as topographs with entries that are elements of a group and topographs with different rules. In total, a great many areas of mathematics find some expression through topographs.

2. Conway Topographs

Conway topographs obey the rule

$$N + S = 2(E + W).$$

With 3 (adjacent) cells numbered, the rest are determined. For example, starting with 1,2,3, we get the topograph [1,2,3].

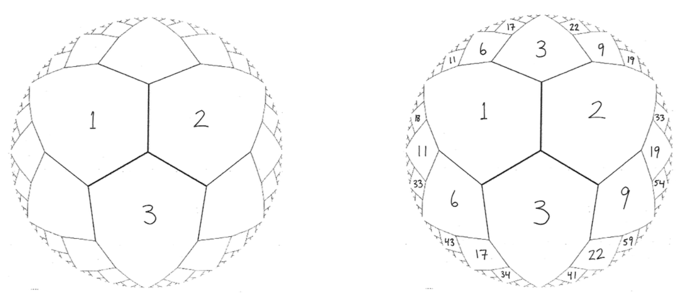


Figure 3. The Conway topograph [1,2,3]

As there is no special "center" of a topograph, we consider, for example, that [1,2,3], [2,3,9], [11,6,33] are really the same. Is there a formula satisfied by all such triples?

2.1. The Discriminant.

$N + S = 2(E + W)$ implies $N - E - W = E + W - S$ and so

$$(N - E - W)^2 = (S - E - W)^2.$$

That is, $N^2 + E^2 + W^2 - 2NE - 2NW + 2EW = S^2 + E^2 + W^2 - 2SE - 2SW + 2EW$ and subtracting $4EW$ from both sides gives

$$D := N^2 + E^2 + W^2 - 2NE - 2NW - 2EW = S^2 + E^2 + W^2 - 2SE - 2SW - 2EW$$

This quantity D is the same for any triple of adjacent values (in any order) from a Conway topograph. We call D the *discriminant* of the the topograph. Alternatively we may define

$$D(x,y,z) := x^2 + y^2 + z^2 - 2xy - 2xz - 2yz = (z-x-y)^2 - 4xy. \tag{1}$$

For example, $[1,2,3]$ has discriminant $D(1,2,3) = (3-1-2)^2 - 4\cdot 1\cdot 2 = -8$ and the basis topograph $[0,0,1]$ has discriminant $D(0,0,1) = 1$.

The discriminant gives some immediate information. For example, its sign determines sign changes within the topograph: if $D := (N-E-W)^2 - 4EW < 0$, then $EW > 0$ and this holds for *all* adjacent E, W and so the numbers never change sign.

If a cell of a topograph is 0, then there is a triple $\{a,b,0\}$ and so $D = a^2+b^2-2(ab) = (a-b)^2$, a square. Therefore, if D is not a square, then there is no cell numbered 0.

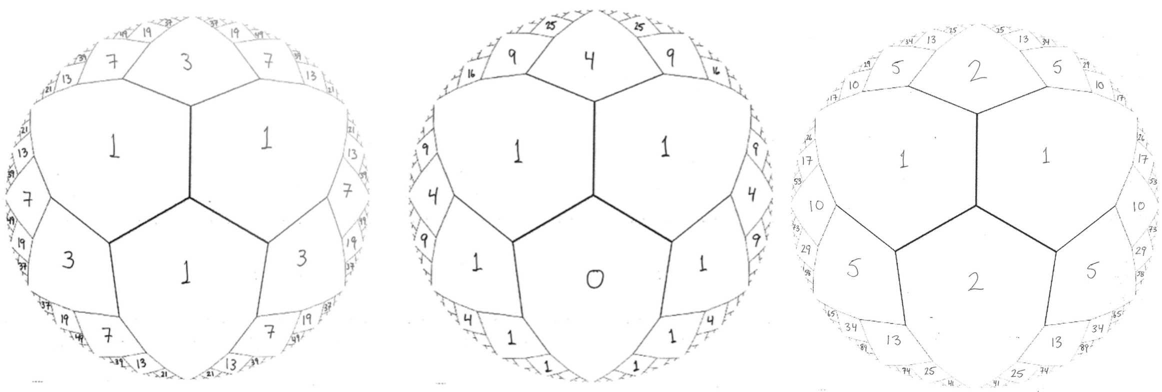


FIGURE 4. $[1,1,1], D=-3; [1,1,0], D=0; [1,2,1], D=-4$

2.2. **The Basis Topograph.** By the "linearity" of $N+S = 2(E+W)$, topographs may be added or multiplied by a scalar. Hence a "basis" topograph [0,0,1] is of particular interest.

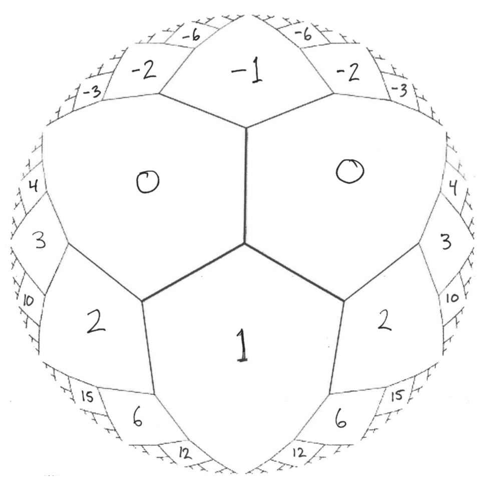


FIGURE 5. The basis topograph [0,0,1]

Every triple in the Conway topograph [0,0,1] is of the form $\pm\{b_{2n}, b_{2n+1}, b_{2n+2}\}$ where b_n is an analogue of Stern's sequence – see [10]:

$$0, 0, 1, 0, 2, 1, 2, 0, 3, 2, 6, 1, 6, 2, 3, 0, 4, 3, 10, 2, 15, 6, 12, 1, ...$$

defined by

$$b_1 = 0, b_{2n} = b_n, b_{2n+1} = b_n \oplus b_{n+1}$$

where

$$x \oplus y = x + y + \sqrt{1 + 4xy}.$$

2.3. **Fermat's Two-Square Theorem.** *"The 'topograph' ... makes the entire theory of binary quadratic forms so easy that we no longer need to think or prove theorems about these forms — just look!"* – J.H. Conway. To illustrate this, we give a sketch of a proof of Fermat's two-square theorem which states that every prime congruent to 1 modulo 4 is a sum of two squares. When D is a prime (and thus positive and non square), the numbers in the topograph take on positive and negative values (but never 0). The edges between values of different sign form a single infinite path, the "river". Since $D = n^2 - 4ab$ can be satisfied for only finitely many pairs (a, b) with different sign, the values along the river must repeat.

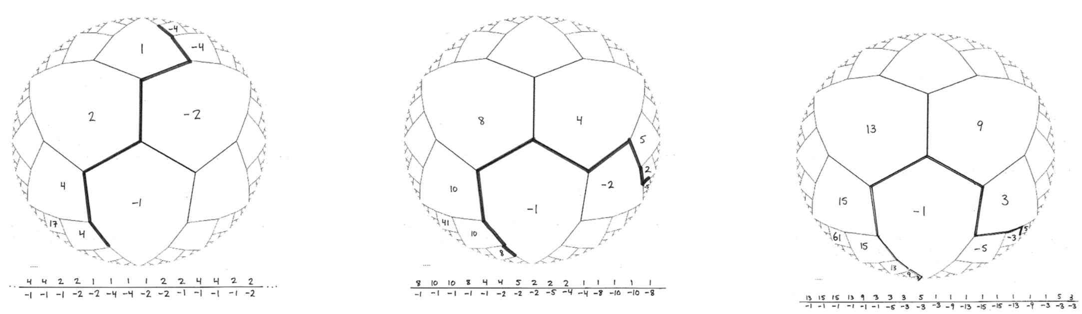


FIGURE 6. Three rivers with $D = 17, 41$ and 61

Theorem 2.1 (Fermat)**.** *Every prime p satisfying $p \equiv 1 \pmod 4$ is the sum of two integral squares.*

Proof. Given p, let $n := (p-1)/4$. The topograph $[-1, n, n]$ then has discriminant p. For every edge in the river between values a, b, let's consider the unordered pair $\{|a|, |b|\}$. Consecutive edges along the river have pairs $\{|a|, |b|\}, \{|a|, |c|\}$ for some a, b, c and, in particular, consecutive pairs differ in at most one place. Since p is prime, the only situation where $|b| = |c|$ is when $|a| = 1$ and $|b| = |c| = n$. The pairs $\{|a|, |b|\}$ are symmetric on either side of $\{n, 1\}$,$\{n, 1\}$ and the number of distinct such pairs is finite (because $ab < 0$ and $p = (c - a - b)^2 - 4ab$ for some c, ab is bounded). Since consecutive pairs differ in at most one place, there must be a pair of the form $\{m, m\}$. In this case, $p = c^2 + 4m^2$ for some c and the result is shown. □

For $D = 61, 41, 17$, we have, respectively, the rivers with unordered pairs $\{|a|, |b|\}$:

$$\ldots, \{1,13\}, \{1,15\}, \{1,15\}, \{1,13\}, \{1,9\}, \{1,3\}, \{3,5\}, \{3,3\}, \{3,5\}, \{1,3\}, \{1,9\}, \{1,13\}, \ldots$$
$$\ldots, \{1,8\}, \{1,10\}, \{1,10\}, \{1,8\}, \{1,4\}, \{2,4\}, \{2,5\}, \{2,2\}, \{2,5\}, \{2,4\}, \{1,4\}, \{1,8\}, \ldots$$
$$\ldots, \{1,4\}, \{1,4\}, \{1,2\}, \{2,2\}, \{1,2\}, \{1,4\}, \{1,4\}, \{1,2\}, \{2,2\}, \{1,2\}, \{1,4\}, \{1,4\}, \ldots$$

3. Vector Topographs

One may use vectors instead of numbers in a Conway topograph.

3.1. **The 3-vector Topograph.** Consider the Conway topograph with the three basis vectors of $\mathbb{R}^3$ as initial values (Fig. 7). These vectors may be parameterized by pairs of coprime integers.

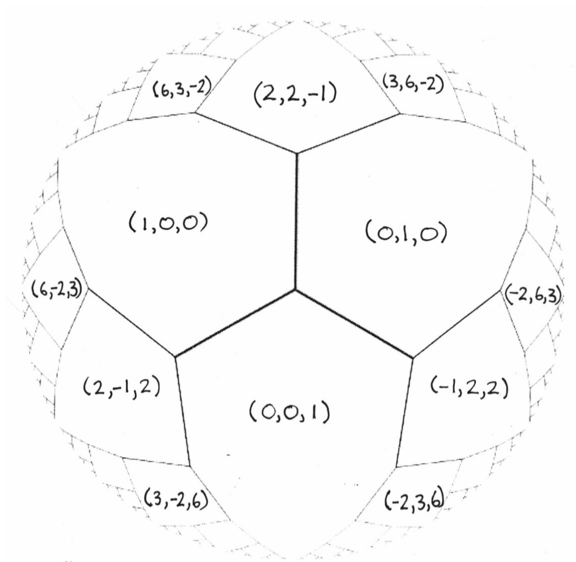


Figure 7. The 3-vector Topograph

Theorem 3.1. *Every vector in the 3-vector topograph is of the form*

$$(n(n-m), nm, m(m-n))$$

for some pair m, n of relatively prime integers.

Proof. Let $f(m,n) := (n(n-m), nm, m(m-n))$. If we have $E = f(m,n), W = f(u,v)$ and $S = f(m-u, n-v)$, then based on the fact that for all x, y, z, w,

$$2(xy + zw) = (x-z)(y-w) + (x+z)(y+w),$$

$N = f(m+u, n+v)$. By an induction argument, all the vectors in the 3-vector topograph are of the form $f(m,n)$ for some coprime m, n. □

Some consequences; the proofs of these are left to the reader. We let (a, b, c) denote a generic vector in the topograph.

- $ab+ac+bc = 0$ and, in fact, every relatively prime solution of $ab+ac+bc = 0$ appears, up to sign, as a vector in this topograph.

- $a+b, a+c, b+c$ and $-abc$ are all perfect squares.
- Let $x \oplus y = \frac{xy}{x+y}$ so that $\oplus$ is "resistor addition", the resistance of two resistors in parallel. $a \oplus b \oplus c = \infty$. We note that $(\overline{\mathbb{R}} - \{0\}, \oplus)$ is an abelian group with identity ∞ – see [7].
- The three vectors $(a, b, c), (b, c, a), (c, a, b)$ are mutually orthogonal and each has length $a + b + c$.
- (a, b, c) lies on the cone defined by those vectors of angle $\arccos(1/\sqrt{3})$ with the vector (1,1,1) (i.e., the cone containing the basis vectors).
- Any matrix whose three rows (or columns) are vectors mutually adjacent in the topograph has determinant ± 1.
- Every Conway topograph is this one "dotted" by some fixed vector.
- $(a, b, c) = (n^2 - nm, nm, m^2 - nm)$ for some relatively prime m, n and so the values of a Conway topograph $[A, B, C]$ are

$$(A, B, C) \cdot (n^2 - nm, nm, m^2 - nm) = An^2 + (B - A - C)mn + Cm^2.$$

Hence the numbers in a Conway topograph [A,B,C] are all the primitive values of the quadratic form

$$Q(m, n) = An^2 + (B - A - C)mn + Cm^2.$$

3.2. **A Pythagorean Topograph.** Note that for all m, n,

$$(n^2 - 2mn)^2 + (2m^2 - 2mn)^2 = (n^2 - 2mn + 2m^2)^2.$$

Since every (a,b,c) in the 3-vector topograph is of the form $(n(n-m), nm, m(m-n))$, $(a - b, 2c, a + b + 2c)$ is a Pythagorean triple. This implies that the 3-vector Conway topograph starting with (1,0,1), (-1,0,1), (0,2,2) contains only Pythagorean triples.

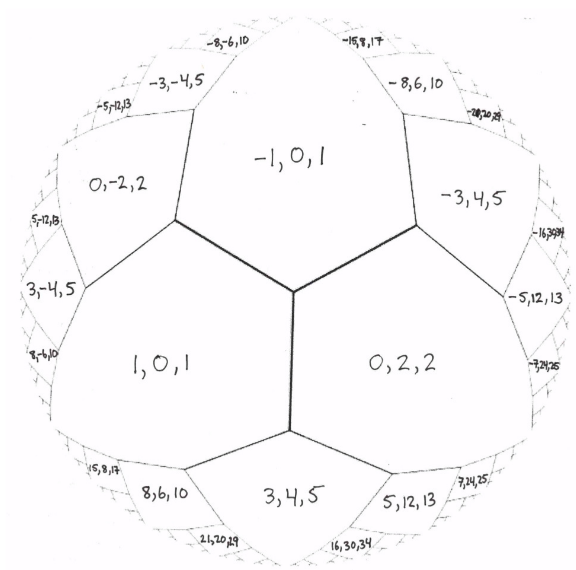
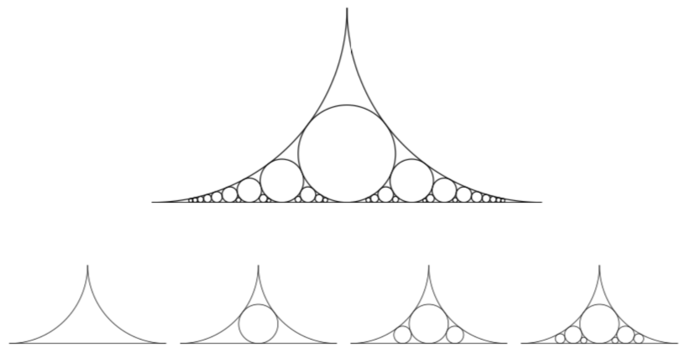


FIGURE 8. A Pythagorean Topograph

3.3. **Ford Circles and a 2-vector Topograph.** The Ford circles are generated iteratively as illustrated here (starting with circles tangent to the real line at 0 and 1):



The Ford circles may also be parameterized in two different ways. See [6] and [9]. Let $C(x, r)$ denote the circle with center (x, r) and radius r. First, the set of Ford circles may be parameterized by the rational numbers:

$$\left\{C\left(\frac{a}{b}, \frac{1}{2b^2}\right) : \frac{a}{b} \text{ is in lowest terms}\right\}$$

and, secondly, by solutions to $ab + ac + bc = 0$:

$$\left\{C\left(\frac{b}{a+b}, \frac{1}{2(a+b)}\right) : ab + ac + bc = 0, \gcd(a, b, c) = 1\right\}.$$

If we let $[a, b]$ denote $C\left(\frac{b}{a+b}, \frac{1}{2(a+b)}\right)$, then the connection between Ford circles and the Conway topograph of 2-vectors is clear.

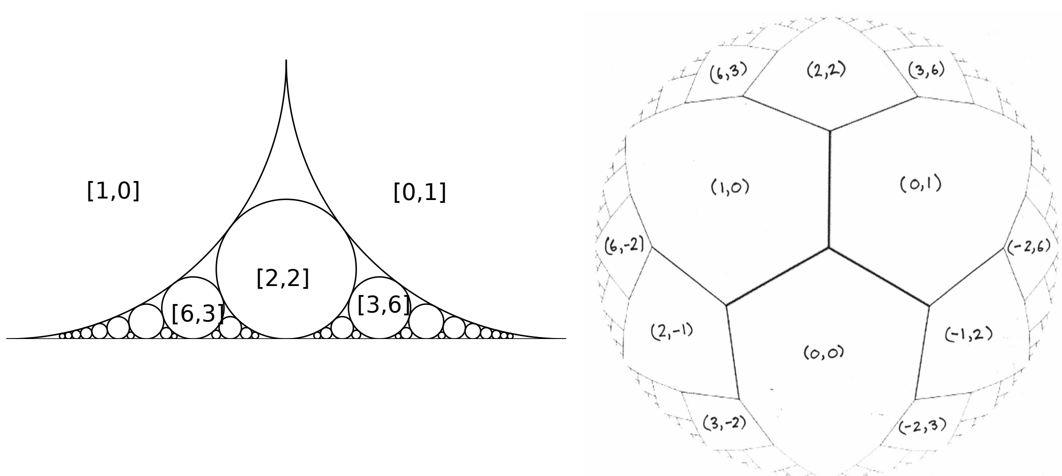


FIGURE 9. Ford Circles and a 2-vector Conway topograph

4. The Rational Topograph

If we replace each (a, b) in the two-vector topograph above, with the number $b/(a+b)$ we get a topograph with rational entries. This topograph can also be created from scratch: Let $\overline{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$. Every element of $\overline{\mathbb{Q}}$ can be represented as $\frac{a}{b}$ in exactly two ways where a, b are relatively prime integers. For example, $0 = \frac{0}{1} = \frac{0}{-1}$, $\infty = \frac{1}{0} = \frac{-1}{0}$, and $-\frac{2}{3} = \frac{2}{-3} = \frac{-2}{3}$. We may then define mediant addition and subtraction. Let

$$\frac{a}{b} \sqcap \frac{c}{d} = \frac{a+c}{b+d}, \frac{a}{b} \sqcup \frac{c}{d} = \frac{a-c}{b-d}$$

and define topograph with rule

$$\{N, S\} = \{E \sqcap W, E \sqcup W\}.$$

We note that even though the definition of $\sqcup$ and $\sqcap$ is ambiguous with respect to sign – e.g., $\frac{1}{1} \sqcup \frac{-2}{3} \neq \frac{1}{1} \sqcup \frac{2}{-3}$ – the rule still makes sense and allows for the construction of the topograph.

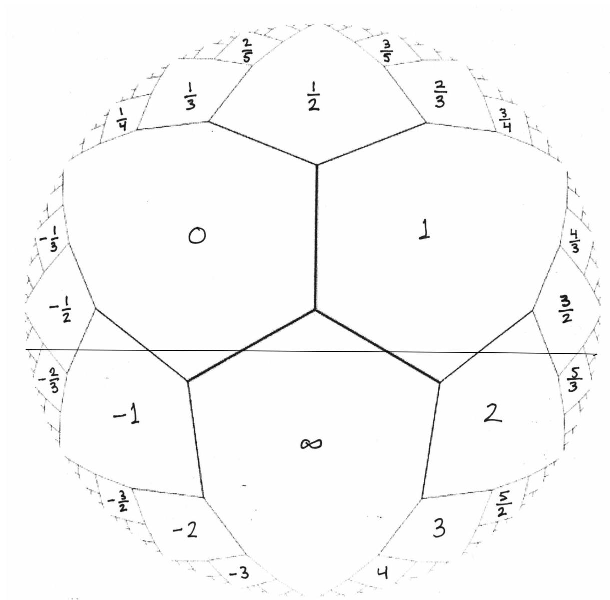


Figure 10. A Rational Topograph with a Chord Between ϕ and $\overline{\phi}$

The boundary of this topograph is naturally associated to the real number line with a one point compactification. We identify points on the boundary with numbers. Some interesting facts:

- Every rational appears exactly once.
- Any points horizontally across from each other sum to 1.
- Any points on a line with slope $-\sqrt{3}$ multiply to 1.
- Any points on a line with slope $\sqrt{3}$ "resistor multiply" to 1 (i.e., $xy/(x+y) = 1$).
- The chord connecting conjugates $\overline{\phi}$ and ϕ goes through *all* of the ratios of consecutive Fibonacci numbers $\ldots, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, \ldots$:

$$\ldots, \frac{-3}{5}, \frac{2}{-3}, \frac{-1}{2}, \frac{1}{-1}, \frac{0}{1}, \frac{1}{0}, \frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \ldots$$

The ratios of consecutive Fibonacci numbers are continued fraction convergents to ϕ and are thus "best rational approximations" to ϕ.

5. Curvature Topographs

The curvature of a circle is the reciprocal of its radius. For example, the curvature of the Ford circle tangent to a/b is $2b^2$. As noted in Section 3.3, the Ford circles are in one to one correspondence with the faces of a 2-vector topograph and so the Conway topograph [1,0,1] is, up to multiplication by 2, the same as the assignment of curvatures of Ford circles.

Consider a topograph generated by the rule

$$N + S = 2(E + W - 1).$$

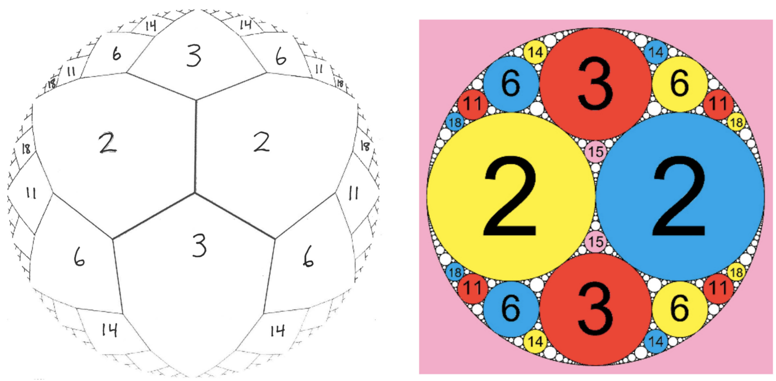


Figure 11. A Topograph and an Apollonian Circle Packing with Curvatures

Here we have an Apollonian circle packing with curvatures noted (the curvature of the outermost circle is considered -1 since it has radius 1 and its interior contains the point at infinity).

Note the connection: the circles tangent to the circle with curvature -1 along with their curvatures are in one-to-one correspondence with the faces of the topograph with rule $N+S = 2(E + W - 1)$. The general fact is that in any Apollonian circle packing, the curvatures of all the circles that are tangent to a given circle of radius k fill out a topograph with rule

$$N + S = 2(E + W + k).$$

6. Periodic Topographs

A sequence is periodic if, for some P, $x_{n+P} = x_n$. In this case, the sequence is the *lift* of a function on the edges of a finite cycle up to its covering space, the 2-regular tree. Alternatively, the sequence induces a covering of a finite cycle by the 2-regular tree.

A sequence is a "1-dimensional" topograph; namely a numbering of cells in a 2-regular tree. Hence a topograph is, in some sense, a higher dimensional sequence. What then does periodicity look like in this situation? Here are some possibilities:

- The topograph is a lift of a numbering of faces in a finite planar graph.
- The topograph is a proper lift of a numbering of faces in an infinite planar graph.
- The numbering of faces along any given face is a periodic sequence.
- Along some infinite path, the numbering of faces forms a periodic sequence. (E.g., along a river)

We have seen the last case in terms of rivers in Conway topographs with $D > 0$. We shall see examples of the others.

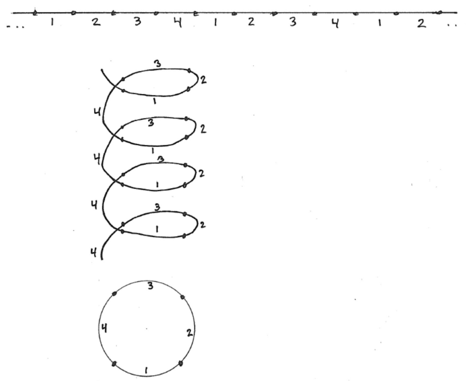


FIGURE 12. A covering induced by a periodic sequence

Our original interest in topographs was this: Every cubic planar graph is a (planar) covering by its universal cover – the geometric topograph of Figure 1. Just as a periodic sequence defines a numbered cycle it lives on, any "periodic" topograph defines a numbered – or colored – cubic planar graph that it lives on. Hence coloring problems involving cubic planar graphs (e.g., the 4-color theorem) might be understood in terms of periodic topographs. More specifically, what rules and/or generalizations of numbers will "fold" a geometric topograph into a given planar graph? Do these rules and numbers show four-colorability?

A Conway topograph makes sense in any setting where $N+S=2(E+W)$ makes sense; for example, if the integers are replaced by an abelian group like $\mathbb{Z}/n\mathbb{Z}$. In the following examples, we consider this situation when $n=2,3,4,5,6$.

6.1. **Modulo 2.** Consider $N+S=2(E+W)$ but modulo 2. In this case, $N+S=0$ so $N=-S=S$. For a,b,c taking values in $\mathbb{Z}/2\mathbb{Z}$, we have a lift of a $\mathbb{Z}/2\mathbb{Z}$-valued function on the graph G with three faces, three edges, and two vertices. We can think of a,b,c as colors and a three face-coloring of G then lifts to a periodic coloring of the geometric topograph.

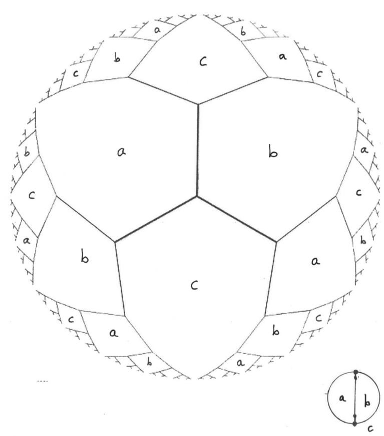


FIGURE 13. Mod 2 and its cover

6.2. **Modulo 3 or 4.** Consider $N+S=2(E+W)$ but modulo 3. In this case, $N+S+E+W=0$. This corresponds to a covering of a tetrahedron.

Modulo 4, the topograph requires 6 variables and we get a covering of a cube.

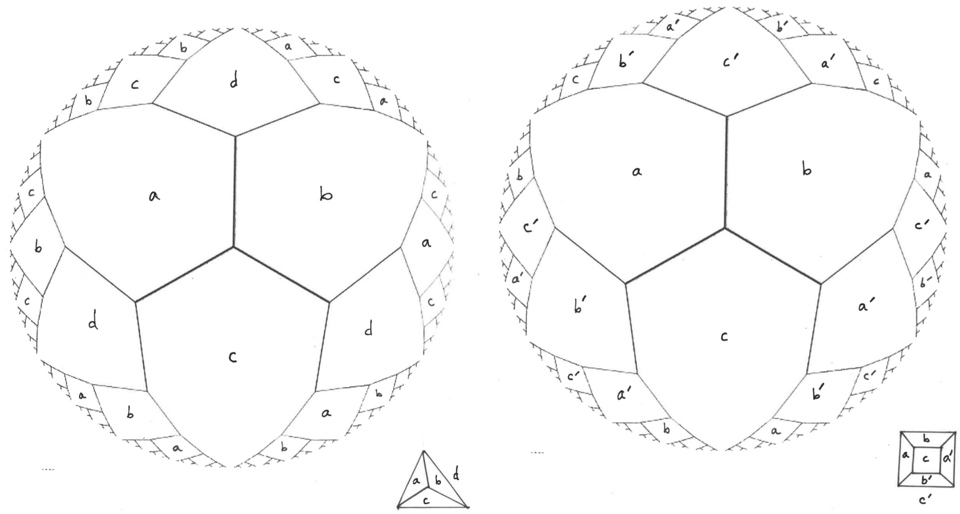


Figure 14. Mod 3, covers a tetrahedron; Mod 4, covers a cube

6.3. **Modulo 5 and 6.** Modulo 5, the rule $N+S=2(E+W)$ requires twelve variables and the topograph covers a dodecahedron.

After exhausting the three-regular Platonic solids, one might expect in the mod 6 case a covering of the infinite hexagonal lattice. This is indeed the case but, it turns out, the hexagonal lattice covers a finite graph. To see this, consider $N+S=2(E+W)$ with elements in $\mathbb{Z}_2 \times \mathbb{Z}_3$ instead of in $\mathbb{Z}_6$. We have a direct sum of the variables in the Mod 2 and Mod 3 cases; we have a planar cover of a dodecahedron but each face is a hexagon and it tiles a two-holed torus.

6.4. **The Hexagonal Lattice.** Here we are back with the integers but with the rule

$$N+S=E+W.$$

The [1,2,4] topograph with this new rule is shown.

Notice that numbers around any face appear to have period 6. This is indeed the case: On the left side of Figure 16, one sees the pattern around a face. It shows that a more appropriate topograph for this rule is the hexagonal lattice or, alternatively, every topograph with the $N+S=E+W$ rule must be the lift from a hexagonal lattice.

The general pattern is that every row in any of the three possible directions contains an arithmetic progression.

7. The Tropical Lyness Topograph

7.1. **Lyness Equation.** The Lyness equation is

$$x_{n+1}=\frac{x_n+\alpha}{x_{n-1}}.$$

Solutions are "almost" periodic; this can be made visible by considering points (x_n, x_{n+1}). In Figure 18, we see some such points when $\alpha = 0.8$. Further, the line segments show the trajectory $(x_n, x_{n+1}) \mapsto (x_{n+1}, x_n) \mapsto (x_{n+1}, x_{n+2})$.

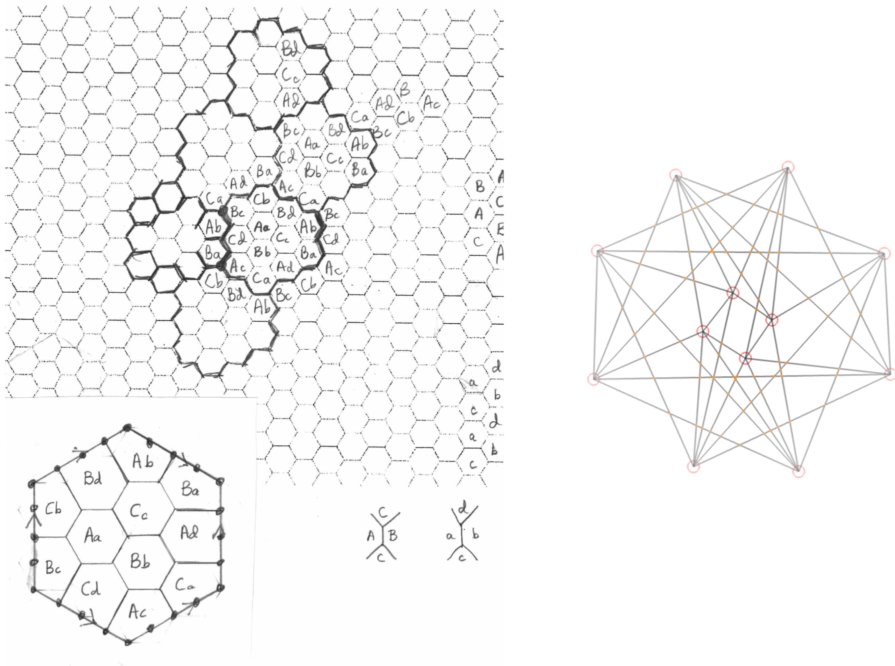


FIGURE 15. The direct sum of the Mod 2 and Mod 3 cases and the dual graph

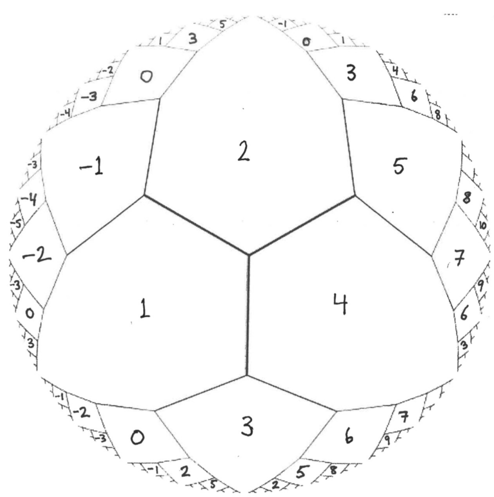


FIGURE 16. $N + S = E + W$ topograph

It turns out that every solution of the Lyness equation satisfies $F(x_{n-1}, x_n) = F(x_n, x_{n+1})$ where

$$F(x, y) := \frac{(x + y + \alpha)(x + 1)(y + 1)}{xy}.$$

That is, the points (x_n, x_{n+1}) lie on the level curve of F that contains the initial point (x_0, x_1).

It turns out that when $\alpha = 0$, *every* solution has period 6:

$$a, b, \frac{b}{a}, \frac{1}{a}, \frac{1}{b}, \frac{a}{b}, a, b, \ldots$$

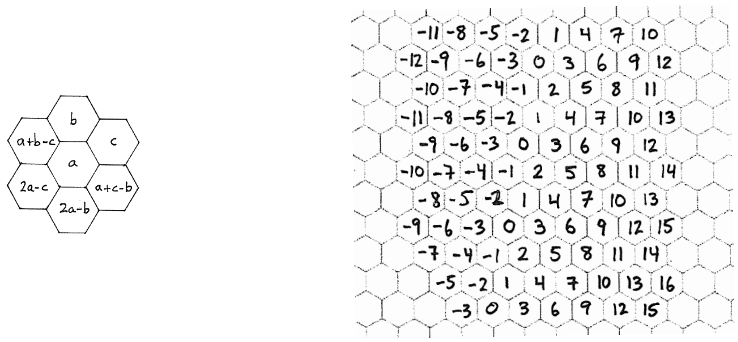


FIGURE 17. Around a generic face; the $[1, 2, 4]$ case

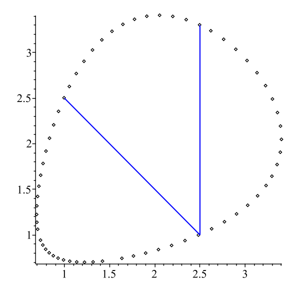


FIGURE 18. Some points (x_n, x_{n+1}) when $\alpha = 0.8$

Similarly, when $\alpha = 1$, *every* solution has period 5:

$$a, b, \frac{b+1}{a}, \frac{a+b+1}{ab}, \frac{a+1}{b}, a, b, \ldots$$

In these two cases, when α is 0 or 1, we say that the equation is globally periodic. The equation is not globally periodic for any other α. See [13] for an exposition of why this is true.

A topograph that would incorporate the Lyness equation would obey the rule

$$NS = E + W.$$

We do not include that here since the set of values would necessarily contain irrational numbers with unbounded complexity.

7.2. **Tropical Lyness Equation.** Instead, we look at a "tropical" version. Tropical mathematics is had by replacing addition by either min or max ($\wedge$ or $\vee$) and multiplication by addition. The tropical Lyness equation is then

$$x_{n+1} = (x_n \wedge \alpha) - x_{n-1}.$$

Solutions are always bounded so, when x_0, x_1, α are integers, the solution is periodic though the period depends on those values. See [11] for details. For example, if $x_0 = 2, x_1 = 7, \alpha = 4$

then the solution has period 39:

$$2, 7, 2, -5, -7, -2, 5, 6, -1, -7, -6, 1, 7, 3, -4, -7, -3, 4, 7, 0, -7, -7, 0,$$
$$7, 4, -3, -7, -4, 3, 7, 1, -6, -7, -1, 6, 5, -2, -7, -5, \ldots$$

7.3. **Tropical Lyness Topograph.** A topograph that incorporates the tropical Lyness equation has rule

$$N + S = E \wedge W.$$

This is then periodic around every face (since, around any face, we have a sequence satisfying a tropical Lyness equation). Is it a lift of a function on a finite graph?

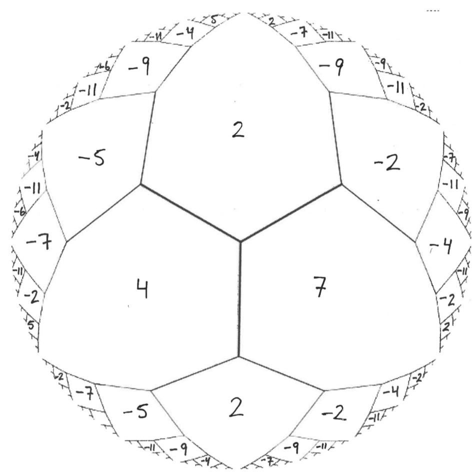


Figure 19. A tropical Lyness topograph

8. The Fibonacci Topograph

For another example of how a topograph can incorporate a given recursive defined sequence, consider a topograph with rule

$$N - S = E - W.$$

A sequence of values going counterclockwise around any face numbered 0 satisfies the Fibonacci recurrence $x_{n+1} = x_n + x_{n-1}$ and so the topograph [1,1,0] with this rule has the Fibonacci sequence $\ldots, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, \ldots$ going around faces numbered 0.

More generally, going counterclockwise around a face labelled c, $x_{n+1} - x_{n-1} = x_n - c$ and thus has the closed formula, with consecutive initial values x_0, x_1,

$$x_n = x_0 F_{n-1} + x_1 F_n - cF_{n+1} + c.$$

9. The Stern Topograph

Consider the topograph defined by the rule

$$N + S = 2(E \vee W).$$

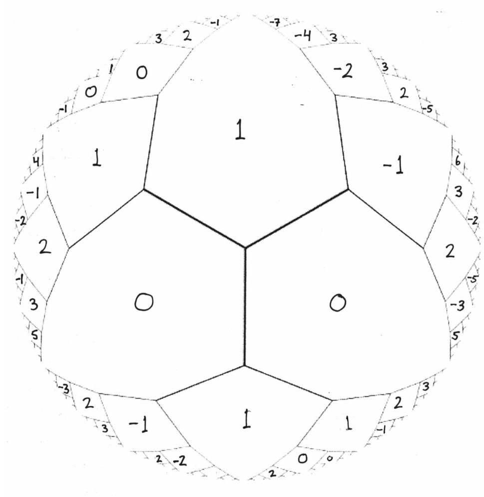


FIGURE 20. Fibonacci topograph

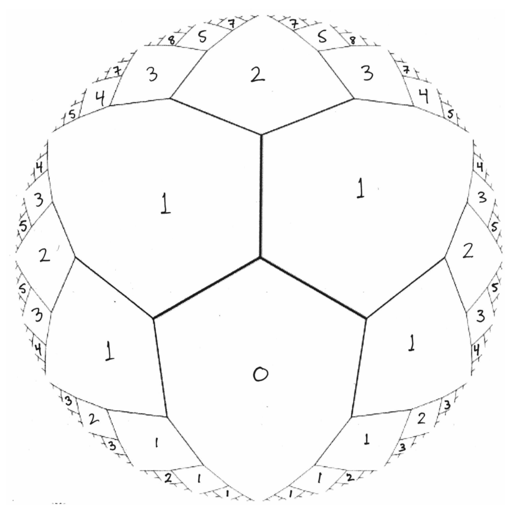


FIGURE 21. Stern Topograph

Note the presence of Stern's diatomic array.

1																1
1								2								1
1				3				2				3				1
1		4		3		5		2		5		3		4		1
1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	1
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

and sequence $a_n = 0, 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7, 3, 8, 5, \ldots$ $[a_{2n} = a_n,\ a_{2n+1} = a_n + a_{n+1}]$.

Every triple is of the form $\{a_{2n}, a_{2n+1}, a_{2n+2}\}$. Note the parallel with the Conway topograph [0,0,1] of Section 2; the sequence (b_n) there replaced by (a_n) here. See [8] about Stern's sequence and [10] about the relationship between the sequences (a_n) and (b_n).

10. The Markov Topograph

Consider a topograph with rule

$$NS = E^2 + W^2,$$

a de-tropicalized version of the rule $N + S = 2(E \vee W)$ of Section 9.

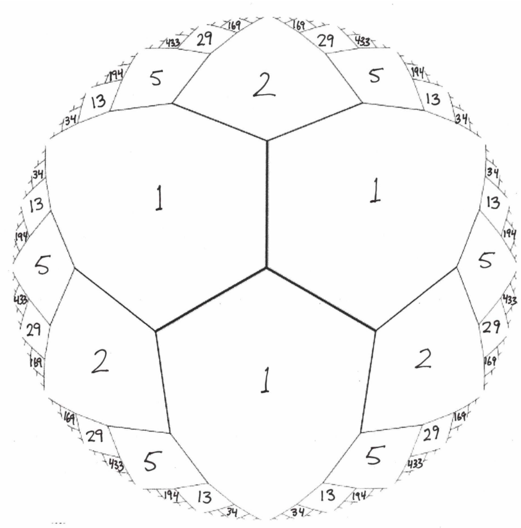


Figure 22. The Markov Topograph

It is, in the [1,1,1] case, equivalent to $N + S = 3EW$. Some properties:

- The entries are all positive integers.
- The numbers around faces labelled 1 are the odd indexed Fibonacci numbers.
- The numbers around faces labelled 2, namely $1, 5, 29, 169, 985, 5741, \ldots$, are the k's satisfying $2k^2 - 1 = \square$ (OEIS A001653). They are the odd indexed Pell numbers.
- Generally, around each face labelled N, $x_{n+1} = 3Nx_n - x_{n-1}$.
- All Markov triples (integer solutions of $x^2 + y^2 + z^2 = 3xyz$) are here.
- The Unicity Conjecture (posed by Frobenius in 1913 and still open) can be expressed as: Every number has unique "parents" (the two smaller numbers on adjacent faces).
- There is no non-trivally distinct example; the only relatively prime integer valued examples are of the form $[\pm 1, \pm 1, \pm 1]$ in which case the signs propagate like the Mod 2 example of Subsection 6.1.

11. The Rascal Topograph, Frieze Patterns, and the Hosoya Triangle

This section gives three examples of how one can make topographs with more than one rule. They all are on the hexagonal lattice (which, of course, can be lifted to the geometric topograph).

11.1. The Rascal Triangle.

In the College Math. Journal of Nov. 2010, three middle school students published a paper on the "Rascal Triangle" [1]. Apparently they were asked to continue the first 3 rows of Pascal's triangle and came up with the triangle in Figure 23.

The rule they used was $N + S = E + W + 1$ on the hexagonal lattice (which is equivalent to $NS = EW + 1$). To make a topograph out of this, we must use a different rule, namely $N + S = E + W - 1$ for the two directions *other* than vertical.

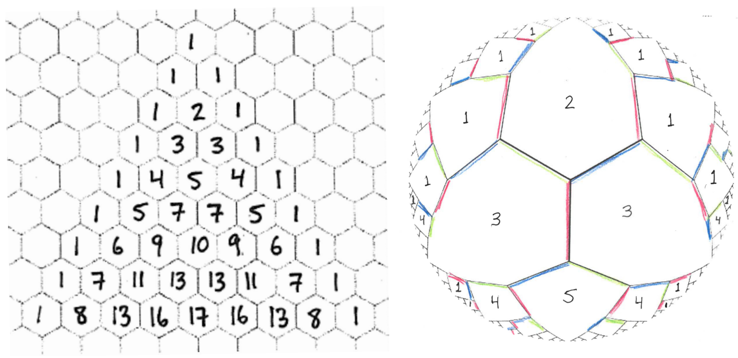


FIGURE 23. The Rascal Triangle and the Rascal Topograph

For a proper topograph, we assigned directions via a Tait coloring of edges (here, arbitrarily, 'RGB' clockwise at each vertex) and have $N+S = E+W+1$ for red edges and $N+S = E+W-1$ for green and blue edges.

11.2. **Frieze Patterns.** Conway and Coxeter [3] studied "frieze patterns" which obey the rule

$$NS = EW - 1$$

on the vertical edges of a hexagonal lattice (but not on the other edges). They all start with a row of ones followed by a row composed of a periodic sequence of positive integers. It is not immediately clear if this always results in an integer valued array or if it ends with a row of ones.

Given a triangulation of a polygon, consider the periodic sequence defined by counting the number of faces meeting each vertex as one goes around the polygon. If that forms the second row, then the pattern is always integer valued and it always ends with a row of ones. Furthermore, all such Frieze patterns arise this way.

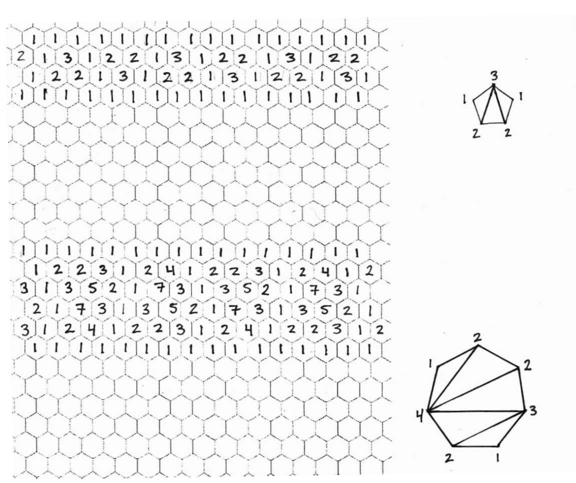


FIGURE 24. Frieze patterns

11.3. **The Hosoya Triangle.** Hosoya introduced a "Fibonacci triangle" in 1976 [5] and it has, since then, elicited a steady flow of research.

It begins

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 2 & & 1 & & 2 & & \\ & 3 & & 2 & & 2 & & 3 & \\ 5 & & 3 & & 4 & & 3 & & 5 \end{array}$$

The nth row is of the form

$$F_1F_n, F_2F_{n-1}, \ldots, F_nF_1.$$

It is a Pascal type triangle which, when put in the hexagonal lattice, as was done for Frieze patterns, it obeys the rule

$$NS = EW$$

for vertical edges. Thus, starting with row $\{F_nF_{-n} : n \in \mathbb{Z}\}$ followed by row $\{F_{n+1}F_{-n} : n \in \mathbb{Z}\}$ allows an extension to the entire lattice.

12. Conclusion

Some questions

- Is there an analogue of discriminant for topographs with rule $N - S = E - W$?
- Is it possible to understand Gauss's composition of quadratic forms using topographs?
- Is it possible to cover all finite cubic graphs with a suitable rule and/or suitable choice of starting values?
- What is a good way to define generating functions for topographs?
- What is the relation between divisibility sequences and topographs with rule $N - S = c \cdot (E - W)$?

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