THE FIBONACCI WORD AS A 2-ADIC NUMBER AND ITS CONTINUED FRACTION

PETER G. ANDERSON

Abstract. The infinite Fibonacci word, $\ldots 0110110101101$, considered as a 2-adic integer, is the limit of fixed points of a Fibonacci-like recursively defined sequence of linear functions. These fixed points, and their limit, have "remarkable continued fractions" of the form $-\frac{2^0}{1+}\,\frac{2^1}{1+}\,\frac{2^1}{1+}\,\frac{2^2}{1+}\,\frac{2^3}{1+}\,\cdots\,\frac{2^{F_n}}{1+}\,\cdots$. Previous work showed the Fibonacci word $1011010110110\ldots$ as a traditional number (in the Euclidean metric) between 0 and 1 (i.e., preceded by "0.") has continued fraction $\frac{1}{2^0+}\,\frac{1}{2^1+}\,\frac{1}{2^1+}\,\frac{1}{2^2+}\,\frac{1}{2^3+}\,\cdots\,\frac{1}{2^{F_n}+}\,\cdots$.

1. Introduction

We treat the Fibonacci words as numbers written in binary in two number fields, the usual field using the Euclidean metric ($\lim_{n\to\infty} |2^{-n}| = 0$) and the 2-adic field (using $\lim_{n\to\infty} |2^n|_2 = 0$).

We use functions of two variables to construct sequences in several different monoids in a Fibonacci-like manner: $X_{n+1} = X_n \heartsuit X_{n-1}$, where X_1 and X_2 are given explicitly. The operators $\heartsuit$ we use are addition "+" on integers (the usual Fibonacci numbers), concatenation "," on character strings (the Fibonacci words), and function composition "$\circ$." In the latter two cases, $\heartsuit$ is not commutative, and we will use the two orders of composition in the contexts of the two fields.

2. The 2-adic case

The *Fibonacci words* are strings over the alphabet $\{0, 1\}$ defined by

$$v_1 = 0, \quad v_2 = 1, \quad v_{n+1} = v_{n-1}, v_n. \tag{1}$$

The lengths of these words are $|v_n| = F_n$, and, considered as binary numbers,

$$v_{n+1} = 2^{F_n} v_{n-1} + v_n. \tag{2}$$

The words $v_1, \ldots, v_8$ are

0, 1, 01, 101, 01101, 10101101, 0110110101101, 101011010110110101101.

For $n \geq 2$, v_n is a suffix of v_{n+1}, so, as binary numbers, they converge to a limit in the 2-adic numbers: $v = \lim_{n\to\infty} v_n$. Label the individual letters (i.e., the 0s and 1s) in v from right to left, so $a_1 = 1$, $a_2 = 0$, $a_3 = 1$, $a_4 = 1$, $a_5 = 1$, etc. The subscript positions for which $a_n = 1$ is the sequence $1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, \ldots$ which is sequence A000201 in the On-line Encyclopedia of Integer Sequences [7], the "lower Wythoff sequence (a Beatty sequence): $a_n = \lfloor n\phi \rfloor$, where $\phi = (1 + \sqrt{(5)})/2$." See also [6]. See [2] for an extensive discussion of v_n, v, and the w_n, and w of Section 3.

So, in other words, we have

$$v = \sum_{n=1}^{\infty} 2^{\lfloor n\phi \rfloor}. \tag{3}$$

Define a sequence of functions $S_n : \mathbb{Q} \to \mathbb{Q}$ via a Fibonacci-like recurrence:

$$S_1(x) = 2x, \quad S_2(x) = 2x + 1, \quad S_{n+1}(x) = S_n \circ S_{n-1}(x) = S_n(S_{n-1}(x)).$$

These functions have the form

$$S_n(x) = a_n x + b_n.$$

Lemma 2.1. *The coefficients a_n and b_n satisfy $a_n = 2^{F_n}$ and $b_n = v_n$.*

Proof. $S_{n+1}(x) = S_n \circ S_{n-1}(x) = a_n(a_{n-1}x + b_{n-1}) + b_n = a_n a_{n-1} x + a_n b_{n-1} + b_n$. Thus,

$$a_1 = 2, \quad a_2 = 2, \quad a_{n+1} = a_{n-1} a_n, \text{ for } n > 1$$

giving $a_n = 2^{F_n}$, and

$$b_1 = 0, \quad b_2 = 1, \quad b_{n+1} = a_n b_{n-1} + b_n, \text{ for } n > 1$$

giving $b_n = v_n$ (see Eq. (2)). □

The fixed point, f_n, of S_n is

$$f_n = \frac{b_n}{1 - a_n} = \frac{v_n}{1 - 2^{F_n}}. \tag{4}$$

Expressed as a 2-adic number, f_n is the infinitely repeating pattern $f_n = \overline{v}_n.0$.

Lemma 2.2. *For $n > 1$, the fixed points of S_n satisfy*

$$f_{n+1} = \frac{v_n + 2^{F_n} v_{n-1}}{(1 - 2^{F_n}) + 2^{F_n}(1 - 2^{F_{n-1}})}. \tag{5}$$

Proof. The denominator of Eq. 5 evaluates to

$$1 - 2^{F_n} + 2^{F_n} - 2^{F_{n-1}} 2^{F_n},$$

which simplifies as

$$1 - 2^{F_{n+1}},$$

which agrees with the denominator of Eq. 4. □

Theorem 2.3. *For $n \geq 2$, the fixed points have continued fractions*

$$f_n = \overline{v}_n = -\frac{2^0}{1+}\,\frac{2^1}{1+}\,\frac{2^1}{1+}\,\frac{2^2}{1+}\,\frac{2^3}{1+}\,\cdots\,\frac{2^{F_{n-1}}}{1}.$$

Proof. The basis of an inductive proof is: $f_2 = \overline{1}.0 = -\frac{2^0}{1}$ (this is -1 in $\mathbb{Z}_2$) and $f_3 = \overline{01}.0 = -\frac{2^0}{1+}\frac{2^1}{1}$ ($-\frac{1}{3}$ in $\mathbb{Z}_2$). Lemma 2.2 provides the inductive step. □

Theorem 2.4. *The infinite Fibonacci word as a 2-adic integer has continued fraction*

$$v = -\frac{2^0}{1+}\,\frac{2^1}{1+}\,\frac{2^1}{1+}\,\frac{2^2}{1+}\,\frac{2^3}{1+}\,\cdots\,\frac{2^{F_n}}{1}\,\cdots.$$

Proof. Clearly, $v = \lim_{n\to\infty} f_n$, and the result follows. □

3. Previous results—the Euclidean field

Here is a sketch of the previous results dealing with ordinary (Euclidean metric) numbers in the interval $(0,1)$ [1, 3, 4, 5]. Details of the proofs are essentially the same as in Section 2. Note that the order of combining two previous items in the recursive parts of the definitions of the Fibonacci words w_n and the functions T_n is the opposite of their analogs in Section 2.

The *Fibonacci words*, w_n, are strings over $\{0,1\}$ defined by

$$w_1 = 0, \quad w_2 = 1, \quad w_{n+1} = w_n, w_{n-1}.$$

As binary integers,

$$w_{n+1} = 2^{F_{n-1}} w_n + w_{n-1}.$$

The words $w_1, \ldots, w_8$ are

$$0,\ 1,\ 10,\ 101,\ 10110,\ 10110101,\ 1011010110110,\ 101101011011010110101.$$

It is easy to see inductively that w_n is the reversal of v_n. For $n \geq 2$, w_n is a prefix of w_{n+1}, so, as binary numbers in the interval $(0,1)$, there is a limit, $0.w = \lim_{n\to\infty} 0.w_n$. (Using the *Unix*™ tool **bc** gives 0.7097167968750 as the decimal equivalent of the binary number $0.w_8 = 0.101101011011010110101$.

In contrast to Section 2, Eq. 3, we have

$$w = \sum_{n=1}^{\infty} \frac{1}{2^{\lfloor n\phi \rfloor}}. \tag{6}$$

Define a sequence of functions $T_n : \mathbb{R} \to \mathbb{R}$ by

$$T_1(x) = \frac{x}{2}, \quad T_2(x) = \frac{x+1}{2}, \quad T_{n+1}(x) = T_n \circ T_{n-1}(x) = T_n(T_{n-1}(x)).$$

These functions have the form:

$$T_n(x) = \frac{x + c_n}{d_n}.$$

Lemma 3.1. *The coefficients satisfy $c_n = w_n$ and $d_n = 2^{F_n}$.*

The fixed point, g_n, of T_n is

$$g_n = \frac{c_n}{d_n - 1} = \frac{w_n}{2^{F_n} - 1} = 0.\overline{w}_n.$$

Lemma 3.2. *For $n > 1$, the fixed points of T_n satisfy*

$$g_{n+1} = \frac{2^{F_{n-1}} w_n + w_{n-1}}{2^{F_{n-1}}(2^{F_n} - 1) + (2^{F_{n-1}} - 1)}.$$

Theorem 3.3. *For $n \geq 2$, the fixed points have continued fractions*

$$g_n = 0.\overline{w}_n = \frac{1}{2^0+}\ \frac{1}{2^1+}\ \frac{1}{2^1+}\ \frac{1}{2^2+}\ \frac{1}{2^3+}\ \cdots\ \frac{1}{2^{F_{n-1}}}.$$

Theorem 3.4. *The infinite Fibonacci word has continued fraction*

$$w = \frac{1}{2^0+}\ \frac{1}{2^1+}\ \frac{1}{2^1+}\ \frac{1}{2^2+}\ \frac{1}{2^3+}\ \cdots\ \frac{1}{2^{F_n}+}\ \cdots.$$

References

[1] W. W. Adams and J. L. Davison, A remarkable class of continued fractions, *Proc. Amer. Math. Soc.* **65** (1977), 194–198.
[2] J.-P. Allouche, Some extremalities of the binary Fibonacci sequence, *Fibonacci Quart.* **55** (2017), no. 5, 1-8.
[3] J.-P. Allouche and J. Shallit, *Automatic Sequences: Theory, Applications, Generalizations*, Cambridge University Press, 2003.
[4] P. G. Anderson, Notes & extensions for a remarkable continued fraction, *Fibonacci Quart.* **55** (2017), no. 5, 9–14.
[5] P. G. Anderson, T. C. Brown, and P. J.-S. Shiue, A simple proof of a remarkable continued fraction identity, *Proc. Amer. Math. Soc.* **123** (1995), no. 7, 2005–2009.
[6] J. L. Davison, A series and its associated continued fraction, *Proc. Amer. Math. Soc.* **63** (1977), 29–32.
[7] OEIS Foundation Inc. (2016), The On-Line Encyclopedia of Integer Sequences, `http://oeis.org`.

MSC2020: 11B39, 11A55

Computer Science Department, Rochester Institute of Technology, Rochester, NY 14531
E-mail address: `pga@cs.rit.edu`