TWO APPLICATIONS OF THE BIJECTION ON FIBONACCI SET PARTITIONS

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Abstract. Fibonacci partitions refer to the partitions of $\{1, 2, \ldots, n\}$ into blocks of nonconsecutive elements. The name was coined by Prodinger because there are as many nonconsecutive subsets of $\{1, 2, \ldots, n\}$ as the Fibonacci number F_{n+2} [*Fibonacci Quart.* **19** (1981), 463–465]. In this note we discuss an application of the bijection between Fibonacci partitions and standard partitions to a new formula for the number of partitions with no circular successions, that is, pairs of elements $a < b$ in a block satisfying $b - a \equiv 1 \pmod{n}$. Then we demonstrate an application of an extended form of the bijection.

1. Introduction

The number of sets $A \subseteq \{1, 2, \ldots, n\}$ satisfying

$$a, b \in A \implies |b - a| \geq 2 \tag{1.1}$$

is known to be the Fibonacci number F_{n+2} (see for example [1]):

$$F_1 = F_2 = 1, \ F_{n+2} = F_n + F_{n+1}, \ n > 0.$$

Based on this fact Prodinger [6] called any set of natural numbers A with property (1.1) a *Fibonacci set*. For example, $F_6 = 8$ enumerates the following Fibonacci subsets of $\{1, 2, 3, 4\}$:

$$\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, \{2, 4\}.$$

A partition of $[n] = \{1, 2, \ldots, n\}$ is a decomposition of $[n]$ into nonempty subsets called *blocks*. A partition into k-blocks is also called a k-partition and denoted by $H_1/H_2/\ldots/H_k$, where the blocks are arranged in standard order, that is, $\min(H_1) < \min(H_2) < \cdots < \min(H_k)$ with the elements in H_i in increasing order for all i.

A partition consisting of Fibonacci subsets is called a *Fibonacci partition* (a. k. a. nonconsecutive partition or 2-regular partition). For example, there are six 3-partitions of [4]:

12/3/4, 13/2/4, 1/23/4, 14/2/3, 1/24/3, 1/2/34,

of which three are Fibonacci partitions:

13/2/4, 14/2/3, 1/24/3.

The number of k-partitions of $[n]$ is the Stirling number of the second kind $S(n, k)$ which satisfies the recurrence relation:

$$S(n, k) = S(n-1, k-1) + k\,S(n-1, k), \quad S(0, 0) = 1, \quad S(n, 0) = S(0, n) = 1 \text{ for } n > 0. \tag{1.2}$$

The number of Fibonacci k-partitions of $[n]$ will be denoted by $f_2(n, k)$.

The bijection to be discussed in this paper is the one which Prodinger [6] used to prove the following identity.

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Theorem 1.1. *The number of Fibonacci k-partitions of $[n]$ is equal to the number of $(k-1)$-partitions of $[n-1]$:*

$$f_2(n,k) = S(n-1,k-1). \tag{1.3}$$

Other bijections have since appeared in [2] and [3]. Even though the bijection of Prodinger has the apparent weakness of not being applicable to d-regular partitions for $d > 2$, it possesses the unique feature of relying on the parity of strings of consecutive elements. (In a d-regular partition, every pair of elements a, b in a block satisfies $|a-b| \geq d$).

In Section 2 we describe the bijection asserting the equality of the sets enumerated by the left- and right-hand sides of (1.3). The sets will be denoted by $F_2(n,k)$ and $\Pi(n-1,k-1)$ respectively. Then in Section 3 we use the reverse construction of the bijection to obtain a formula for partitions without circular successions (defined below). Lastly, in Section 4 we highlight a possible extension of the bijection.

2. The bijection: $F_2(n,k) \longrightarrow \Pi(n-1,k-1)$

We begin with two key definitions.

Succession: a pair of elements (a,b) in one block of a partition of $[n]$ satisfying $|a-b| = 1$.
Succession string: any contiguous sequence of one or more consecutive integers in one block of a partition.

Thus the set $F_2(n,k)$ of Fibonacci partitions is precisely the set of k-partitions of $[n]$ which contain no successions.

The bijection runs as follows. If a partition $p \in \Pi(n-1,k-1)$ does not contain a succession, then insert the block $\{n\}$ to obtain a partition in $F_2(n,k)$. If p contains successions, then form a new block $H(n)$ to contain n and every ith term of each succession string of length t such that $i \equiv t+1 \pmod 2$. In other words, if t is odd, move elements in even positions to $H(n)$ and if t is even, move elements in odd positions.

Conversely if the block $H(n)$ containing n in a partition $q \in F_2(n,k)$ satisfies $|H(n)| = 1$, then delete $H(n)$; otherwise before deleting $H(n)$, put every $x < n$ into the block containing $x+1$. The resulting partition belongs to $\Pi(n-1,k-1)$.

For example, $135/24 \mapsto 135/24/6$ and $123/45 \mapsto 13/246/5$. □

The bijective map $F_2(n,k) \to \Pi(n-1,k-1)$ will be denoted by θ_2:

$$\theta_2 : F_2(n,k) \to \Pi(n-1,k-1). \tag{2.1}$$

3. New formula for partitions without circular successions

A *circular succession* is an ordered pair of elements (a,b) in one block of a partition of $[n]$ which satisfy $b - a \equiv 1 \pmod n$. In other words a circular succession is a succession or an occurrence of the pair $(n,1)$ in a block. For example the partition $1259/367/48$ contains three circular successions namely $(1,2), (9,1)$ and $(6,7)$.
The number of k-partitions of $[n]$ containing no circular successions is denoted by $c(n,k)$. Thus a partition enumerated by $c(n,k)$ is a Fibonacci partition which avoids the circular succession $(n,1)$.

In [4] the authors found the following formula using a direct construction of partitions enumerated by $f_2(n,k)$.

$$c(n,k) = \sum_{j=2}^{\lfloor \frac{n+2}{2} \rfloor} \binom{n-j}{j-2} \sum_{i=0}^{j-2} \binom{j-2}{i} S(n-j-i,k-2), \quad 2 \leq k \leq n, \tag{3.1}$$

with $c(n,1) = \delta_{1n}$.

In this section we obtain a different formula by considering the images of partitions enumerated by $f_2(n,k)$ under θ_2.

Proposition 3.1. *The number $\pi_t(n,k)$ of partitions $B_1/B_2/\cdots/B_k$ of $[n]$ in which $[t] \subseteq B_1$ and $t+1 \notin B_1$ is given by*

$$\pi_t(n,k) = S(n-t+1,k) - S(n-t,k).$$

Proof. First Proof: Partition $\{t+1,\ldots,n\}$ into k blocks and then put the elements of $[t]$ into any block except the block containing $t+1$. This gives $(k-1)S(n-t,k)$ partitions. The other partitions in which $B_1 = [t]$ are $S(n-t,k-1)$ in number. Hence we obtain

$$\pi_t(n,k) = S(n-t,k-1) + (k-1)S(n-t,k) = S(n-t+1,k) - S(n-t,k),$$

where the second equality follows from the recurrence (1.2).

Second Proof: The number of k-partitions of $[n]$ containing the string $[x]$, $x \geq 1$, is $S(n-(x-1),k)$, that is, obtain a k-partition of $\{x,\ldots,n\}$ and put the elements of $\{1,\ldots,x-1\}$ into the block containing x. Thus for a fixed $x = t$ the number of k-partitions of $[n]$ containing the string $[t]$ is $S(n-(t-1),k) - S(n-t,k)$. □

Let $\pi_{\text{odd}}(n,k)$ and $\pi_{\text{even}}(n,k)$ be the number of partitions in $\Pi(n,k)$ in which 1 belongs to a succession string of odd and even length respectively. Then $\pi_{\text{odd}}(n,k) = \sum_{i\geq 1}\pi_{2i-1}(n,k)$ and $\pi_{\text{even}}(n,k) = \sum_{i\geq 1}\pi_{2i}(n,k)$. Proposition 3.1 then implies the next result.

Corollary 3.2. *We have*

$$\pi_{odd}(n,k) = \sum_{i\geq 1}(S(n-2i+2,k) - S(n-2i+1,k)).$$

$$\pi_{even}(n,k) = \sum_{i\geq 1}(S(n-2i+1,k) - S(n-2i,k)).$$

Note that the sum $\pi_{\text{odd}}(n,k) + \pi_{\text{even}}(n,k)$ telescopes to $S(n,k)$.

Our new formula is stated below.

Theorem 3.3. *We have*

$$c(n,k) = \sum_{i=1}^{\lfloor \frac{n-k+2}{2} \rfloor} \left(S(n-2i+1,k-1) - S(n-2i,k-1)\right).$$

Proof. Let $C(n,k)$ denote the set of partitions enumerated by $c(n,k)$. It will suffice to show that $c(n,k) = |C(n,k)| = \pi_{\text{odd}}(n-1,k-1)$ so that the theorem follows from Corollary 3.2. A partition $q \in C(n,k) \subseteq F_2(n,k)$ avoids the circular succession $(n,1)$. This means that **1 was not moved** during the transition $\theta_2^{-1}(p) \mapsto q$, where $p \in \Pi(n-1,k-1)$ (see (2.1)). Since 1 occupies an odd position, only integers occupying even positions were moved from the string containing 1 during execution of the map $\theta_2^{-1} : \Pi(n-1,k-1) \to F_2(n,k)$. This implies that p is a partition in which 1 belongs to a succession string of odd length. Thus the image of the restriction of θ_2^{-1} to the set of such partitions p is precisely $C(n,k)$, i.e., $|C(n,k)| = \pi_{odd}(n-1,k-1)$. Hence the result. (See Table 1 for an illustration). □

This result leads to a corresponding single-sum formula for $c_r(n,k)$, the number of k-partitions of $[n]$ containing r circular successions. Since $c_r(n,k) = \binom{n}{r}c(n-r,k)$ [4, Theorem 3.2], we have:

$p = B_1/B_2 \in \Pi(4,2)$	$t,\ [t] \subseteq B_1$	$q = \theta_2^{-1}(p) \in F_2(5,3)$	$q \in C(5,3)$?
1/234	1	1/24/35	yes
12/34	2	135/2/4	no
13/24	1	13/24/5	yes
14/23	1	14/25/3	yes
123/4	3	13/25/4	yes
124/3	2	15/24/3	no
134/2	1	14/2/35	yes

TABLE 1. Illustration of the proof of Theorem 3.3

Corollary 3.4.

$$c_r(n,k) = \binom{n}{r} \sum_{i=1}^{\lfloor \frac{n-r-k+2}{2} \rfloor} \left(S(n-r-2i+1, k-1) - S(n-r-2i, k-1) \right).$$

3.1. **Bonus result - a combinatorial identity.** The proof of the following identity was requested in [4].

$$S(n,k) = \sum_{j=0}^{\lfloor \frac{n-2}{2} \rfloor} \sum_{t=0}^{j} \binom{n-2-j}{j} \binom{j}{t} S(n-2-j-t, k-1) + \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{t=0}^{j} \binom{n-1-j}{j} \binom{j}{t} S(n-1-j-t, k-1). \tag{3.2}$$

Subsequently intricate combinatorial proofs were provided by Shattuck [7] and Munagi [5].

However, by comparing with (3.1), we see that the right-hand side of (3.2) is equal to

$$c(n,k+1) + c(n+1,k+1)$$

which, from the relation $c(n,k) = \pi_{\text{odd}}(n-1,k-1)$, is equal to

$$\pi_{odd}(n-1,k) + \pi_{odd}(n,k).$$

Thus using Corollary 3.2, the three quantities in (3.2) are, correspondingly,

$$S(n,k) = \pi_{even}(n,k) + \pi_{odd}(n,k).$$

So the identity (3.2) is just a splitting of $S(n,k)$ into the numbers of k-partitions of $[n]$ in which 1 belongs to a succession string of even and odd length.

4. A bijection for partitions with circular successions

Here we introduce a circular succession version of the bijection θ_2.

Consider the sum $c(n) = \sum_k c(n,k)$, the number of partitions of $[n]$ containing no circular successions, and $B(n) = \sum_k S(n,k)$, the n^{th} Bell number.

Proposition 4.1. *The number of partitions of $[n+1]$ containing no circular succession is equal to the number of partitions of $[n]$ containing at least one circular succession:*

$$c(n+1) = B(n) - c(n). \tag{4.1}$$

Proof. Denote the enumerated sets by $C(n+1)$ and $BC(n)$.
We propose an extension of θ_2 to partitions with circular successions given by

$$\varphi_2 : C(n+1) \longrightarrow BC(n).$$

First define a bijective transformation α on $C(n+1)$ as follows: if $p \in C(n+1)$ such that $\{n+1\} \in p$ and $n \notin H(1)$, then $\alpha(p)$ is the partition obtained by replacing the blocks $H(1)$ and $\{n+1\}$ with the block $H(1) \cup \{n+1\}$, otherwise $\alpha(p) = p$.

Now define

$$\varphi_2 = \theta_2\alpha : \; p \longmapsto q.$$

Examples using some cases of $C(5) \longrightarrow BC(4)$:
(i) $\varphi_2(14/2/3/5) = \theta_2\alpha(14/2/3/5) = \theta_2(14/2/3/5) = 14/2/3.$
$\varphi_2(14/2/35) = \theta_2\alpha(14/2/35) = \theta_2(14/2/35) = 134/2.$
(ii) $\varphi_2(13/24/5) = \theta_2\alpha(13/24/5) = \theta_2(135/24) = 1234.$
$\varphi_2(13/2/4/5) = \theta_2\alpha(13/2/4/5) = \theta_2(135/2/4) = 12/34.$

Conversely, we have

$$\varphi_2^{-1} = \alpha^{-1}\theta_2^{-1} : \; q \longmapsto p,$$

where the effect of α^{-1} is to replace $H(1)$ with the blocks $H(1) \setminus \{n+1\}$ and $\{n+1\}$ whenever $n+1 \in H(1)$.

Examples:
(i) $\varphi_2^{-1}(14/2/3) = \alpha^{-1}\theta_2^{-1}(14/2/3) = \alpha^{-1}(14/2/3/5) = 14/2/3/5.$
$\varphi_2^{-1}(134/2) = \alpha^{-1}\theta_2^{-1}(134/2) = \alpha^{-1}(14/2/35) = 14/2/35.$
(ii) $\varphi_2^{-1}(1234) = \alpha^{-1}\theta_2^{-1}(1234) = \alpha^{-1}(135/24) = 13/24/5.$
$\varphi_2^{-1}(12/34) = \alpha^{-1}\theta_2^{-1}(12/34) = \alpha^{-1}(135/2/4) = 13/2/4/5.$ □

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