HIGHER ORDER BOUSTROPHEDON TRANSFORMS FOR CERTAIN WELL-KNOWN SEQUENCES

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Abstract. A review of the boustrophedon transform is presented and transforms of several familiar sequences are obtained. In addition higher transforms are also investigated. Representations of the transform will be given in terms of members of the original sequence using the Euler Up-Down number coefficients.

1. Introduction

The boustrophedon (or "ox-plowing") method of transforming a given sequence into a new one was introduced by Millar in [4] relating to a generalization of the Seidel-Entringer-Arnold method for calculating secant-tangent numbers [2]. Millar also related the story behind the terminology "double ox transform".

Transformed sequences found in [5] will be indicated by the appropriate OEIS numbers.

Beginning with a sequence $\{a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, \ldots\}$ the transformed sequence $\{b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, \ldots\}$ can be obtained from the recursion $T_{n,0} = a_0$, $(n \geq 0)$, $T_{n+1,k+1} = T_{n+1,k} + T_{n,n-k}$, $(n \geq k \geq 0)$; where $b_n = T_{n,n}$, $(n \geq 0)$ can be obtained from the zig-zag pattern constructed as shown in [4], or from the sum $b_n = \sum_{k=0}^{n} \binom{n}{k} a_k E_{n-k}$, where E_j is the jth Euler, or also known as the Up/Down number sequence, A000111: $\{1, 1, 1, 2, 5, 16, 61, 272, 1385, \ldots\}$ with exponential generating function $\sec x + \tan x$. In addition they can be computed using the zig-zag diagram indicated below.

Note that in [4] when applying the boustrophedon transform to the sequence $\{1, 0, 0, 0, 0, 0, 0, 0, 0, \ldots\}$, the integers $T_{n,k}$ are called the Entringer numbers. Also, the zig-zag triangle formed by the $T_{n,k}$ numbers was coined by Dumont [2] as the Seidel-Entringer-Arnold triangle.

Zig-zag diagrams for the boustrophedon transform occur, for example in [1, 4]. A short outline of the Seidel-Entringer-Arnold triangle is provided here.

$$\begin{array}{c}
b_0 \\
T_{1,0} \rightarrow b_1 \\
b_2 \leftarrow T_{2,1} \leftarrow T_{2,0} \\
T_{3,0} \rightarrow T_{3,1} \rightarrow T_{3,2} \rightarrow b_3 \\
b_4 \leftarrow T_{4,3} \leftarrow T_{4,2} \leftarrow T_{4,1} \leftarrow T_{4,0} \\
\cdots
\end{array}$$

In [4] the sequence $\{1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots\}$ was double oxed to first yield the sequence A000667: $\{1, 2, 4, 9, 24, 77, 294, 1309, 6664, \ldots\}$ and then purported to yield the sequence A000834: $\{1, 3, 9, 35, 177, 1123, 8569, 76355, 777697, \ldots\}$. But after careful derivations (independently by each author of this paper) using the above mentioned techniques, the double ox sequence was found to be $\{1, 3, 9, 29, 105, 433, 2029, 10709, 63025, \ldots\}$ which has no number in [5].

In addition the inverse transform [4],

$$a_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} b_k E_{n-k}, (n \geq 0),$$

using b_k in A000834 did not recover the sequence A000667, but recovered the sequence $\{1, 2, 4, 15, 72, 467, 3534, 31675, 321832, \ldots\}$, whereas the one found here did recover it. Also note that [4] is cited under the sequence A000834 in [5].

Since these two sequences were explored in [4] the double ox and triple ox transforms are included here:

For $\{1, 0, 0, 0, 0, 0, 0, 0, 0, 0, \ldots\}$
boustrophedon transform sequence $\{1, 1, 1, 2, 5, 16, 61, 272, 1385, \ldots\}$
double ox $\{1, 2, 4, 10, 32, 122, 544, 2770, 15872, \ldots\}$
triple ox $\{1, 3, 9, 30, 117, 528, 2709, 15600, 99657, \ldots\}$

For $\{1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots\}$
boustrophedon transform sequence $\{1, 2, 4, 9, 24, 77, 294, 1309, 6664, \ldots\}$
double ox $\{1, 3, 9, 29, 105, 433, 2029, 10709, 63025, \ldots\}$
triple ox $\{1, 4, 16, 67, 304, 1519, 8386, 51007, 340024, \ldots\}$

A considerable number of boustrophedon transformed sequences do appear in [5] and a selection, including some not included there, will be double and triple oxed in Section 3.

2. Boustrophedon and Higher "Ox" Transforms

As noted above, the Seidel-Entringer-Arnold triangle expanded from the sequence $\{1.0, 0, 0, 0, 0, 0, 0, 0, \ldots\}$ yields the boustrophedon transformed sequence $\{1, 1, 1, 2, 5, 16, 61, 272, 1385, \ldots\}$. From this the $\{b_n\}$ can be expressed in terms of the $\{a_n\}$ as follows

$$\begin{aligned}
b_0 &= a_0 \\
b_1 &= a_0 + a_1 \\
b_2 &= a_0 + 2a_1 + a_2 \\
b_3 &= 2a_0 + 3a_1 + 3a_2 + a_3 \\
b_4 &= 5a_0 + 8a_1 + 6a_2 + 4a_3 + a_4 \\
b_5 &= 16a_0 + 25a_1 + 20a_2 + 10a_3 + 5a_4 + a_5 \\
b_6 &= 61a_0 + 96a_1 + 75a_2 + 40a_3 + 15a_4 + 6a_5 + a_6 \\
b_7 &= 272a_0 + 427a_1 + 336a_2 + 175a_3 + 70a_4 + 21a_5 + 7a_6 + a_7 \\
b_8 &= 1385a_0 + 2176a_1 + 1708a_2 + 896a_3 + 350a_4 + 112a_5 + 28a_6 + 8a_7 + a_8.
\end{aligned}$$

Similarly the boustrophedon transform of $\{b_n\}$, that is to say the double ox transform of $\{a_n\}$, say $\{c_n\}$, can be expressed in terms of the original sequence $\{a_n\}$ as follows

$$\begin{aligned}
c_0 &= a_0 \\
c_1 &= 2a_0 + a_1 \\
c_2 &= 4a_0 + 4a_1 + a_2 \\
c_3 &= 10a_0 + 12a_1 + 6a_2 + a_3 \\
c_4 &= 32a_0 + 40a_1 + 24a_2 + 8a_3 + a_4 \\
c_5 &= 122a_0 + 160a_1 + 100a_2 + 40a_3 + 10a_4 + a_5
\end{aligned}$$

$$c_6 = 544a_0 + 732a_1 + 480a_2 + 200a_3 + 60a_4 + 12a_5 + a_6$$
$$c_7 = 2770a_0 + 3808a_1 + 2562a_2 + 1120a_3 + 350a_4 + 84a_5 + 14a_6 + a_7$$
$$c_8 = 15872a_0 + 22160a_1 + 15232a_2 + 6832a_3 + 2240a_4 + 560a_5 + 112a_6 + 16a_7 + a_8.$$

Likewise those for the triple ox sequence, $\{d_n\}$, can be expressed in terms of the $\{a_n\}$ as follows

$$d_0 = a_0$$
$$d_1 = 3a_0 + a_1$$
$$d_2 = 9a_0 + 6a_1 + a_2$$
$$d_3 = 30a_0 + 27a_1 + 9a_2 + a_3$$
$$d_4 = 117a_0 + 120a_1 + 54a_2 + 12a_3 + a_4$$
$$d_5 = 528a_0 + 585a_1 + 300a_2 + 90a_3 + 15a_4 + a_5$$
$$d_6 = 2709a_0 + 3168a_1 + 1755a_2 + 600a_3 + 135a_4 + 18a_5 + a_6$$
$$d_7 = 15600a_0 + 18963a_1 + 11088a_2 + 4095a_3 + 1050a_4 + 189a_5 + 21a_6 + a_7$$
$$d_8 = 99657a_0 + 124800a_1 + 75852a_2 + 29568a_3 + 8190a_4 + 1680a_5 + 252a_6 + 24a_7 + a_8.$$

3. Examples

The boustrophedon transform OEIS numbers are provided if they are available. Some have more than one occurring in The On-Line Encyclopedia of Integer Sequences site. Although the example given in this section could be obtained directly from the formulas in Section 2, it is interesting to note the special generalizations given in this section.

First consider the generalized Fibonacci numbers, followed by some specific well-known cases.

3.1. **Generalized Fibonacci Numbers:** $a_{n+2} = Pa_{n+1} + Qa_n$

$$a_0 = a_0$$
$$a_1 = a_1$$
$$a_2 = Pa_1 + Qa_0$$
$$a_3 = (P^2 + Q)a_1 + PQa_0$$
$$a_4 = (P^3 + 2PQ)a_1 + (P^2Q + Q^2)a_0$$
$$a_5 = (P^4 + 3P^2Q + Q^2)a_1 + (P^3Q + 2PQ^2)a_0$$
$$a_6 = (P^5 + 4P^3Q + 3PQ^2)a_1 + (P^4Q + 3P^2Q^2 + Q^3)a_0$$
$$a_7 = (P^6 + 5P^4Q + 6P^2Q^2 + Q^3)a_1 + (P^5Q + 4P^3Q^2 + 3PQ^3)a_0$$
$$a_8 = (P^7 + 6P^5Q + 10P^3Q^2 + 4PQ^3)a_1 + (P^6Q + 5P^4Q^2 + 6P^2Q^3 + Q^4)a_0.$$

So the boustrophedon transform sequence is found to be

$$b_0 = a_0$$
$$b_1 = a_1 + a_0$$
$$b_2 = (P + 2)a_1 + (1 + Q)a_0$$
$$b_3 = (P^2 + 3P + 3 + Q)a_1 + (2 + [P + 3]Q)a_0$$
$$b_4 = (P^3 + 4P^2 + 6P + 8 + [2P + 4]Q)a_1 + (5 + [P^2 + 4P + 6]Q + Q^2)a_0$$
$$b_5 = (P^4 + 5P^3 + 10P^2 + 20P + 25 + [3P^2 + 10P + 10]Q + Q^2)a_1 + (16 + [P^3 + 5P^2 + 10P + 20]Q + [2P + 5]Q^2)a_0$$
$$b_6 = (P^5 + 6P^4 + 15P^3 + 40P^2 + 75P + 96 + [4P^3 + 18P^2 + 30P + 40]Q + [3P + 6]Q^2)a_1 + (61 + [P^4 + 6P^3 + 15P^2 + 40P + 75]Q + [3P^2 + 12P + 15]Q^2 + Q^3)a_0$$

$$b_7 = (P^6+7P^5+21P^4+70P^3+175P^2+336P+427+[54P^4+28P^3+63P^2+140P +175]Q+[6P^2+21P+21]Q^2+Q^3)a_1+(272+[P^5+7P^4+21P^3+70P^2+175P +336]Q+[4P^3+21P^2+42P+70]Q^2+[3P+7]Q^3)a_0$$

$$b_8 = (P^7+8P^6+28P^5+112P^4+350P^3+896P^2+1708P+2176+[6P^5+40P^4+112P^3 +336P^2+700P+896]Q+[10P^3+48P^2+84P+112]Q^2+[4P+8]Q^3)a_1+(1385 +[P^6+8P^5+28P^4+112P^3+350P^2+896P+1708]Q+[5P^4+32P^3+84P^2 +224P+350]Q^2+[6P^2+24P+28]Q^3+Q^4)a_0$$

The double ox sequence is

$$c_0 = a_0$$
$$c_1 = a_1+2a_0$$
$$c_2 = (P+4)a_1+(4+Q)a_0$$
$$c_3 = (P^2+6P+12+Q)a_1+(10+[P+6]Q)a_0$$
$$c_4 = (P^3+8P^2+24P+40+[2P+8]Q)a_1+(32+[P^2+8P+24]Q+Q^2)a_0$$
$$c_5 = (P^4+10P^3+40P^2+100P+160+[3P^2+20P+40]Q+Q^2)a_1+(122+[P^3 +10P^2+40P+100]Q+[2P+10]Q^2)a_0$$
$$c_6 = (P^5+12P^4+60P^3+200P^2+480P+732+[4P^3+36P^2+120P+200]Q+[3P +12]Q^2)a_1+(544+[P^4+12P^3+60P^2+200P+480]Q+[3P^2+24P+60]Q^2+Q^3)a_0$$
$$c_7 = (P^6+14P^5+84P^4+350P^3+1120P^2+2562P+3808+[5P^4+56P^3+252P^2 +700P+1120]Q+[6P^2+42P+84]Q^2+Q^3)a_1+(2770+[2562+1120P+350P^2 +84P^3+14P^4+P^5]Q+[350+168P+42P^2+4P^3]Q^2+[14+3P]Q^3)a_0$$
$$c_8 = (P^7+16P^6+112P^5+560P^4+2240P^3+6832P^2+15232P+22160+[6P^5 +80P^4+448P^3+1680P^2+4480P+6832]Q+[10P^3+96P^2+336P+560]Q^2 +[4P+16]Q^3)a_1+(15872+[15232+6832P+2240P^2+560P^3+112P^4+16P^5 +P^6]Q+[2240+1120P+336P^2+64P^3+5P^4]Q^2+[112+48P+6P^2]Q^3+Q^4)a_0$$

The triple ox sequence is

$$d_0 = a_0$$
$$d_1 = a_1+3a_0$$
$$d_2 = (P+6)a_1+(9+Q)a_0$$
$$d_3 = (P^2+9P+27+Q)a_1+(30+[P+9]Q)a_0$$
$$d_4 = (P^3+12P^2+54P+120+[2P+12]Q)a_1+(117+[P^2+12P+54]Q+Q^2)a_0$$
$$d_5 = (P^4+15P^3+90P^2+300P+585+[3P^2+30P+90]Q+Q^2)a_1+(528+[P^3 +15P^2+90P+300]Q+[2P+15]Q^2)a_0$$
$$d_6 = (P^5+18P^4+135P^3+600P^2+1755P+3168+[4P^3+54P^2+270P+600]Q +[3P+18]Q^2)a_1+(2709+[P^4+18P^3+135P^2+600P+1755]Q+[3P^2+36P +135]Q^2+Q^3)a_0$$
$$d_7 = (P^6+21P^5+189P^4+1050P^3+4095P^2+11088P+18963+[5P^4+84P^3+567P^2 +2100P+4095]Q+[6P^2+63P+189]Q^2+Q^3)a_1+(15600+[1108+4095P+1950P^2 +189P^3+21P^4+P^5]Q+[1050+378P+63P^2+4P^3]Q^2+[21+3P]Q^3)a_0$$
$$d_8 = (P^7+24P^6+252P^5+1680P^4+8190P^3+29568P^2+75852P+124800+[6P^5 +120P^4+1008P^3+5040P^2+16380P+29568]Q+[10P^3+144P^2+756P+1690]Q^2 +[4P+24]Q^3)a_1+(99657+[75852+29568P+8190P^2+1680P^3+252P^4+24P^5 +P^6]Q+[8190+3360P+756P^2+96P^3+5P^4]Q^2+[252+72P+6P^2]Q^3+Q^4)a_0.$$

The following examples are special cases of the generalized Fibonacci numbers with coefficients and initial conditions $(P, Q; a_0, a_1)$ in the recurrence relations, respectively $(1, 1; 0, 1)$, $(1, 1; 2, 1)$, $(2, 1; 0, 1)$, $(2, 1; 2, 2)$, $(1, 2; 0, 1)$, $(1, 2; 2, 1)$.

3.1.1. *Fibonacci numbers:* $\{0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$
boustrophedon transform sequence A000738: $\{0, 1, 3, 8, 25, 85, 334, 1497, 7635, \ldots\}$
double oxed sequence $\{0, 1, 5, 20, 83, 375, 1860, 10205, 61701, \ldots\}$
triple oxed sequence $\{0, 1, 7, 38, 201, 1115, 6626, 42517, 295107, \ldots\}$

Using $\{1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots\}$ for the Fibonacci sequence yields:
boustrophedon transform sequence $\{1, 2, 5, 14, 42, 144, 563, 2526, 12877, \ldots\}$
double oxed sequence $\{1, 3, 10, 37, 149, 660, 3245, 17687, 106498, \ldots\}$
triple oxed sequence $\{1, 4, 17, 78, 386, 2066, 12019, 76080, 523069, \ldots\}$

3.1.2. *Lucas numbers,* A000032: $\{2, 1, 3, 4, 7, 11, 18, 29, 47, \ldots\}$
boustrophedon transform sequence $\{2, 3, 7, 20, 59, 203, 792, 3555, 18119, \ldots\}$
double oxed sequence $\{2, 5, 15, 54, 215, 945, 4630, 25169, 151295, \ldots\}$
triple oxed sequence $\{2, 7, 27, 118, 571, 3017, 17412, 109643, 751031, \ldots\}$

3.1.3. *Pell numbers,* A000129: $\{0, 1, 2, 5, 12, 29, 70, 169, 408, \ldots\}$
boustrophedon transform sequence $\{0, 1, 4, 14, 52, 204, 870, 4082, 21240, \ldots\}$
double oxed sequence $\{0, 1, 6, 29, 140, 709, 3830, 22317, 140856, \ldots\}$
triple oxed sequence $\{0, 1, 8, 50, 300, 1844, 11890, 81334, 593448, \ldots\}$

3.1.4. *Pell-Lucas numbers,* A002203: $\{2, 2, 6, 14, 34, 82, 198, 478, 1157, \ldots\}$
boustrophedon transform sequence $\{2, 4, 12, 42, 152, 594, 2524, 11830, 61520, \ldots\}$
double oxed sequence $\{2, 6, 22, 94, 434, 2146, 11454, 66246, 416162, \ldots\}$
triple oxed sequence $\{2, 8, 36, 152, 1000, 5878, 36948, 248818, 1796720, \ldots\}$

3.1.5. *Jacobsthal numbers,* A001045: $\{0, 1, 1, 3, 5, 11, 21, 43, 85, \ldots\}$
boustrophedon transform sequence $\{0, 1, 3, 9, 31, 111, 453, 2059, 10571, \ldots\}$
double oxed sequence $\{0, 1, 5, 21, 93, 441, 2265, 12741, 78373, \ldots\}$
triple oxed sequence $\{0, 1, 7, 39, 215, 1241, 7617, 50149, 355195, \ldots\}$

3.1.6. *Jacobsthal-Lucas numbers,* A014551: $\{2, 1, 5, 7, 17, 31, 65, 127, 257, \ldots\}$
boustrophedon transform sequence $\{2, 3, 9, 29, 93, 343, 1379, 6299, 32273, \ldots\}$
double oxed sequence $\{2, 5, 17, 69, 297, 1385, 7077, 39589, 242897, \ldots\}$
triple oxed sequence $\{2, 7, 29, 139, 725, 4057, 24479, 159469, 1121345, \ldots\}$

3.2. **General polygonal numbers:**

$$P_n^{(r)} = \frac{n[(r-2)n - (r-4)]}{2}.$$

Although the polygonal numbers require $n \geq 3$, consider the sequence beginning with $n = 0$. Then

$$\begin{array}{lll} P_0 = 0 & P_3 = 3(r-1) & P_6 = 3(5r-8) \\ P_1 = 1 & P_4 = 2(3r-4) & P_7 = 7(3r-5) \\ P_2 = r & P_5 = 5(2r-3) & P_8 = 4(7r-12)\ldots. \end{array}$$

So the boustrophedon transform sequence is found to be

$$\begin{array}{lll} b_0 = 0 & b_3 = 6r & b_6 = 6(60r-43) \\ b_1 = 1 & b_4 = 12(2r-1) & b_7 = 147(11r-8) \\ b_2 = r+2 & b_5 = 30(3r-2) & b_8 = 56(147r-107)\ldots. \end{array}$$

Note that if $r = 3$ this becomes $\{0, 1, 5, 18, 60, 210, 822, 3675, 18704, \ldots\}$ which agrees with the boustrophedon transform sequence obtained from the third representation of the triangle numbers below. Similarly if $r = 5$ this yields boustrophedon transform sequence $\{0, 1, 7, 30, 108, 390, 1542, 6909, 35168, \ldots\}$ which again, agrees with the transformed pentagonal sequence obtained below.

The double ox sequence is

$$\begin{array}{lll} c_0 = 0 & c_3 = 9(r+1) & c_6 = 1575r - 552 \\ c_1 = 1 & c_4 = 2(27r+4) & c_7 = 7(1299r - 569) \\ c_2 = r+4 & c_5 = 5(58r - 11) & c_8 = 4(14203r - 6988)\ldots \end{array}$$

and the triple ox sequence is

$$\begin{array}{lll} d_0 = 0 & d_3 = 12(r+2) & d_6 = 6(760r - 1) \\ d_1 = 1 & d_4 = 4(24r+19) & d_7 = 49(651r - 104) \\ d_2 = r+6 & d_5 = 10(67r + 18) & d_8 = 8(29351r - 7695)\ldots \end{array}$$

A few examples follow.

3.2.1. *Triangular numbers:* $(r = 3)$

Obtained from $\{1, 1, 3, 6, 10, 15, 21, 28, 36, \ldots\}$
boustrophedon transform sequence, A000718: $\{1, 2, 6, 20, 65, 226, 883, 3947, 20089, \ldots\}$
double oxed sequence $\{1, 3, 11, 46, 202, 937, 4717, 26066, 158356, \ldots\}$
triple oxed sequence $\{1, 4, 18, 90, 481, 2718, 16383, 106201, 742521, \ldots\}$

Obtained from $\{1, 3, 6, 10, 15, 21, 28, 36, 45, \ldots\}$
boustrophedon transform sequence $\{1, 4, 13, 39, 120, 407, 1578, 7042, 35840, \ldots\}$
double oxed sequence $\{1, 5, 22, 92, 391, 1773, 8800, 48208, 291181, \ldots\}$
triple oxed sequence $\{1, 6, 33, 175, 936, 5229, 31174, 200310, 1390944, \ldots\}$

Obtained from $\{0, 1, 3, 6, 10, 15, 21, 28, 36, \ldots\}$
boustrophedon transform sequence $\{0, 1, 5, 18, 60, 210, 822, 3675, 18704, \ldots\}$
double oxed sequence $\{0, 1, 7, 36, 170, 815, 4173, 23296, 14284, \ldots\}$
triple oxed sequence $\{0, 1, 9, 60, 364, 2190, 13674, 90601, 642864, \ldots\}$

3.2.2. *Square numbers:* $(r = 4)$

Obtained from $\{1, 1, 4, 9, 16, 25, 36, 49, 64, \ldots\}$
boustrophedon transform sequence, A000697: $\{1, 2, 7, 26, 89, 316, 1243, 5564, 28321, \ldots\}$
double oxed sequence $\{1, 3, 12, 55, 256, 1227, 6292, 35159, 215168, \ldots\}$
triple oxed sequence $\{1, 4, 19, 102, 577, 3388, 20943, 138100, 977329, \ldots\}$

Obtained from $\{1, 4, 9, 16, 25, 36, 49, 64, 81, \ldots\}$
boustrophedon transform sequence, A000745: $\{1, 5, 18, 57, 180, 617, 2400, 10717, 54544, \ldots\}$
double oxed sequence $\{1, 6, 29, 128, 561, 2588, 12973, 71504, 433665, \ldots\}$
triple oxed sequence $\{1, 7, 42, 235, 1300, 7419, 44848, 290911, 2033808, \ldots\}$

3.2.3. *Pentagonal numbers:* $(r = 5)$

Obtained from A000326: $\{0, 1, 5, 12, 22, 35, 51, 70, 92, \ldots\}$
boustrophedon transform sequence $\{0, 1, 7, 30, 108, 390, 1542, 6909, 35168, \ldots\}$
double oxed sequence $\{0, 1, 9, 54, 278, 1395, 7323, 41482, 256108, \ldots\}$
triple oxed sequence $\{0, 1, 11, 84, 556, 3530, 22794, 154399, 1112480, \ldots\}$

The interested reader may want to investigate the transforms of other polygonal as well as pyramidal numbers.

4. Additional Sequences Appearing in the OEIS

4.0.1. *Bell numbers,* A000110: $\{1, 1, 2, 5, 15, 52, 203, 877, 4140, \ldots\}$
boustrophedon transform sequence, A000764: $\{1, 2, 5, 16, 60, 258, 1247, 6686, 38371, \ldots\}$
double oxed sequence $\{1, 3, 10, 39, 175, 884, 4963, 30639, 206284, \ldots\}$
triple oxed sequence $\{1, 4, 17, 80, 420, 2440, 15551, 107932, 810555, \ldots\}$

4.0.2. *Catalan numbers,* A000108: $\{1, 1, 2, 5, 14, 42, 132, 429, 1430, \ldots\}$,
also listed that way in the Catalan number table in [3], p.106.
boustrophedon transform sequence $\{1, 2, 5, 16, 59, 243, 1101, 5461, 29619, \ldots\}$
double oxed sequence $\{1, 3, 10, 39, 174, 864, 4712, 28007, 180614, \ldots\}$
triple oxed sequence $\{1, 4, 17, 80, 419, 2415, 15165, 103053, 754211, \ldots\}$

Entry A000736 uses $\{1, 1, 1, 2, 5, 14, 42, 132, 429, \ldots\}$ to derive
boustrophedon transform sequence, A000736: $\{1, 2, 4, 10, 32, 120, 513, 2455, 13040, \ldots\}$
double oxed sequence $\{1, 3, 9, 30, 117, 526, 2666, 15026, 93213, \ldots\}$
triple oxed sequence $\{1, 4, 16, 68, 320, 1682, 9801, 62751, 438096, \ldots\}$

They also appear in [1] as $\{1, 2, 5, 14, 42, 132, 429, 1430, 4862, \ldots\}$
as well as in the Catalan number table in [3], p.391.
boustrophedon transform sequence $\{1, 3, 10, 37, 149, 648, 3039, 15401, 84619, \ldots\}$
double oxed sequence $\{1, 4, 17, 78, 386, 2054, 11741, 72100, 475790, \ldots\}$
triple oxed sequence $\{1, 5, 26, 143, 837, 5220, 34695, 245783, 1855499, \ldots\}$

4.0.3. *Odd numbers:* $\{1, 3, 5, 7, 9, 11, 13, 15, 17, \ldots\}$
boustrophedon transform sequence, A000754: $\{1, 4, 12, 33, 96, 317, 1218, 5427, 27626, \ldots\}$
double oxed sequence $\{1, 5, 21, 83, 337, 1483, 7225, 39117, 234403, \ldots\}$
triple oxed sequence $\{1, 6, 32, 163, 840, 4559, 26614, 168413, 1156186, \ldots\}$

4.0.4. *Padovan numbers:* $\{1, 1, 1, 2, 2, 3, 4, 5, 7, \ldots\}$
boustrophedon transform sequence $\{1, 2, 4, 10, 29, 94, 364, 1621, 8256, \ldots\}$
double oxed sequence $\{1, 3, 9, 30, 114, 485, 2316, 12393, 73623, \ldots\}$
triple oxed sequence $\{1, 4, 16, 68, 317, 1626, 9160, 56597, 382000, \ldots\}$

4.0.5. *Perrin numbers:* $\{3, 0, 2, 3, 2, 5, 5, 7, 10, \ldots\}$
boustrophedon transform sequence $\{3, 3, 5, 15, 41, 133, 518, 2300, 11725, \ldots\}$
double oxed sequence $\{3, 6, 14, 45, 170, 711, 3377, 17991, 106538, \ldots\}$
triple oxed sequence $\{3, 9, 29, 111, 497, 2489, 13802, 84418, 565597, \ldots\}$

4.0.6. *Powers of 2:* $\{1, 1, 2, 4, 8, 16, 32, 64, 128, \ldots\}$
boustrophedon transform sequence, A000734: $\{1, 2, 5, 15, 49, 177, 715, 3255, 16689, \ldots\}$
double oxed sequence $\{1, 3, 10, 38, 160, 738, 3740, 20838, 127440, \ldots\}$
triple oxed sequence $\{1, 4, 17, 79, 401, 2709, 13187, 85279, 596561, \ldots\}$

4.0.7. *Prime numbers,* (includes 1): $\{1, 2, 3, 5, 7, 11, 13, 17, 19, \ldots\}$
boustrophedon transform sequence, A000732: $\{1, 3, 8, 22, 66, 222, 862, 3838, 19542, \ldots\}$
double oxed sequence $\{1, 4, 5, 15, 57, 231, 823, 3813, 23045, 141235, \ldots\}$
triple oxed sequence $\{1, 5, 24, 116, 586, 2964, 16066, 100184, 692166, \ldots\}$

(Excluding 1): $\{2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots\}$
boustrophedon transform sequence $\{2, 5, 13, 35, 103, 345, 1325, 5911, 30067, \ldots\}$
double oxed sequence $\{2, 7, 25, 93, 371, 1627, 7917, 42813, 256359, \ldots\}$
triple oxed sequence $\{2, 9, 41, 193, 959, 5119, 29633, 186577, 1276643, \ldots\}$

4.0.8. *Tribonacci numbers:* $\{0, 1, 1, 2, 4, 7, 13, 24, 44, \ldots\}$
boustrophedon transform sequence $\{0, 1, 3, 8, 26, 92, 366, 1655, 8460, \ldots\}$
double oxed sequence $\{0, 1, 5, 20, 84, 387, 1949, 10804, 65820, \ldots\}$
triple oxed sequence $\{0, 1, 7, 38, 202, 1132, 6802, 44061, 308204, \ldots\}$

5. Concluding Comments

The variations of the sequences in [5] were usually motivated by the applications which suggested them. The interested reader might find it instructive to read [4] and use the zig-zag pattern obtained from the recursion formula given in the introduction of this paper as well as the identities developed here to obtain boustrophedon, ox, and higher order ox transforms of any sequences discovered in other applications.

Investigation into possible additional applications of the sequences addressed here but not included in [5] might prove to be a productive exercise. Attempts were made to find recurrence relations for some of the transformed sequences in this paper but a quick perusal of any of these "ox" transforms indicates that no such results are suggested.

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