PERFECT BALANCING NUMBERS

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Abstract. Perfect numbers are scarce, only 48 are known, and when the search is restricted to a specified sequence, the possibility of their adequate presence still reduces. The objective of this paper is to show that the only perfect number in the balancing sequence is 6.

1. Introduction

A perfect number is natural numbers which is equal to the sum of its proper positive divisors. These numbers are very scarce and to date only 48 numbers have been found, the largest one contains 34850340 digits. The infinitude of these numbers is yet to be established. Surprisingly, all the known perfect numbers are even although many properties and conjectures about odd perfect numbers are available in the literature. All the even perfect numbers are of the form $2^{n-1}(2^n - 1)$ where $2^n - 1$ is a prime (popularly known as Mersenne prime). It is well-known that any odd perfect number, if exists, is very large. A recent result by Ochem and Rao [7] ascertains that odd perfect numbers must be greater than 10^{1500}. Euler proved that every odd perfect number is of the form $p^{4\alpha+1}x^2$, where p is a prime of the form $4n + 1$. Neilson [8] verified that every odd perfect number has at least 9 prime factors.

The search of perfect numbers in various number sequences has been a motivating job for mathematicians. In 1994, McDaniel [5] proved that the only triangular number in the Pell sequence is 1, which is a clear indication that there is no even perfect number in this sequence. In [4], Luca established the absence of perfect numbers in the Fibonacci and Lucas sequences. The absence of even perfect numbers in the associated Pell sequence follows from the paper [12] by Prasad and Rao. The objective of this work is to explore all perfect numbers in the balancing sequence.

As defined by Behera and Panda [1], a natural number n is a balancing number if $1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)$ for some natural number r, which is the balancer corresponding to n. As a consequence of the definition, a natural number $n > 1$ is a balancing number if and only if n^2 is a triangular number, or equivalently, $8n^2 + 1$ is a perfect square. The nth balancing number is denoted by B_n, and $C_n = \sqrt{8B_n^2 + 1}$ is called the nth Lucas-balancing number [13, p. 25]. Customarily, 1 is accepted as the first balancing number, that is, $B_1 = 1$. Panda in [9] proved that the Lucas-balancing numbers are associated with balancing numbers in the way Lucas numbers are associated with Fibonacci numbers. The identities $B_{2n} = 2B_nC_n$ and $C_{2n} = C_n^2 + 8B_n^2$ (see [9]) resembling respectively $F_{2n} = F_nL_n$ and $L_{2n} = \frac{1}{2}(F_n^2 + 5L_n^2)$ will prove their usefulness in the next section.

2. Even Perfect Balancing Numbers

This section is devoted to the search of even perfect numbers in the balancing sequence. Since the balancing numbers are alternately odd and even and all the known perfect numbers are even, we first focus our attention on even balancing numbers. The second balancing number $B_2 = 6$ is perfect and using Microsoft Mathematics, we verified that there is no perfect number in the list of the first fifty even balancing numbers. Thus, a natural question: "Is there any other even perfect number in the balancing sequence?" In this section, this question is answered in the negative.

Throughout the paper p denotes a prime and (a, b) denotes the greatest common divisor of a and b.

To prove that 6 is the only even perfect balancing number, we need the following lemma which deals with an important divisibility property of balancing numbers.

Lemma 2.1. *For any natural numbers r and n, $2^r|B_n$ if and only if $2^r|n$.*

Proof. We will show that if $2^r|n$ then $2^r|B_n$. If $2^r|n$ then $n = 2^r k$ for some natural number k. By virtue of the identity $B_{2n} = 2B_nC_n$, it follows that

$$B_n = B_{2^r k} = 2B_{2^{r-1}k}C_{2^{r-1}k} = \cdots = 2^r B_kC_kC_{2k}C_{4k}\cdots C_{2^{r-1}k}.$$

Thus, $2^r|B_n$. Conversely, assume that $2^r|B_n$. To prove that $2^r|n$, we use mathematical induction on r. Observe that if $2|B_n$ then certainly n is even, since a balancing number is even or odd according as its index is even or odd. Thus, the assertion is true for $r = 1$. Assume that the assertion is true for $r = k$ and suppose that $2^{k+1}|B_n$. Then once again n is even say $n = 2l$ and $B_n = 2B_lC_l$. Further, $2^{k+1}|B_n$ implies $2^k|B_lC_l$. Since, by definition, the Lucas-balancing numbers are odd, it follows that $2^k|B_l$. By the inductive assumption, this implies that $2^k|l$ and finally, $2^{k+1}|2l = n$. This ends the proof. □

We are now in a position to prove the main result of this section.

Theorem 2.2. *B_{2n} is perfect if and only if $n = 1$, that is, 6 is the only even perfect balancing number.*

Proof. It is well-known that every even perfect number N is of the form $N = 2^{p-1}(2^p-1)$ where (2^p-1) is prime. Indeed, then p is also a prime [2]. If $p = 2$ then $N = 6$ is a perfect number as well as a balancing number. Now let $p \geq 3$ and assume to the contrary that $N = 2^{p-1}(2^p-1)$ is a balancing number say $2^{p-1}(2^p-1) = B_{2n}$ for some natural number n. Since $p \geq 3$, $4|B_{2n}$. By virtue of Lemma 2.1, $4|2n$ and hence $2|n$, say $n = 2k$ and then, $B_{2n} = B_{4k} = 4B_kC_kC_{2k}$. Thus, $4B_kC_kC_{2k} = 2^{p-1}(2^p-1)$. Since C_k and C_{2k} are odd, $(C_kC_{2k}, 2^{p-1}) = 1$. Therefore, $C_kC_{2k}|(2^p-1)$. Since $C_{2k} = C_k^2 + 8B_k^2$, it follows that $C_{2k} > C_k \geq C_1 = 3$. Thus, C_k and C_{2k} are distinct natural numbers and each of them is more than 1. This implies that 2^p-1 is not a prime, which is a contradiction. Thus, the only perfect number in the even balancing numbers is 6. Since $B_{2n} = 6$ only when $n = 1$, the proof is complete. □

3. Balancing Numbers of the Form px^2

Identification of terms of the form kx^2 in well-known sequences have been considered by many authors. In Section 1, we have already discussed the importance of these type of numbers in connection with odd perfect numbers. Robbins [14, 15] explored terms of the form px^2 in Fibonacci and Pell sequences while McDaniel [6] studied terms of the form kx^2 in these sequences. We will prove that there is only one balancing number of the form px^2 which is not a perfect number.

The Pell sequence is defined as $P_1 = 1, P_2 = 2$ and $P_{n+1} = 2P_n + P_{n-1}$ for $n \geq 2$, while the associated Pell sequence is defined as $Q_1 = 1$, $Q_2 = 3$, and $Q_{n+1} = 2Q_n + Q_{n-1}$ for $n \geq 2$. The importance of Pell and associated Pell sequences lies in the fact that the nth convergent of $\sqrt{2}$ expressed as a continued fraction is $\frac{Q_n}{P_n}$. Further, a crucial relationship of these two sequences with the balancing sequence needed to prove an important result of this section is $B_n = P_n Q_n$ [10, p.46].

To prove that there is only one balancing number of the form px^2, we need the following two lemmas. The first lemma ascertains the presence of only one square term in the associated Pell sequence.

Lemma 3.1. *The only square term in the associated Pell sequence is* $Q_1 = 1$.

Proof. It is known that for any odd order associated Pell number $k, 2(k^2+1)$ is a perfect square. Assume that k is a square say $k = x^2$ and let $2(k^2+1) = z^2$. We thus have the Diophantine equation

$$2(x^4 + 1) = z^2. \tag{3.1}$$

Since the left-hand side is even, z is also even, say $z = 2y$ and (3.1) reduces to

$$x^4 + 1 = 2y^2. \tag{3.2}$$

But (3.2) has no solution other than $x = y = 1$ [11, p. 133]. Thus, the only square term in the associated Pell sequence is $Q_1 = 1$ corresponding to $k = 1$. □

The following lemma due to Ljunggren [3] caters to all square terms in the Pell sequence.

Lemma 3.2. *The only square terms in Pell sequence are* $P_1 = 1$ *and* $P_7 = 169$.

Now, we are in a position to prove the main result of this section.

Theorem 3.3. *The only balancing number of the form* px^2 *is* $B_7 = 40391$.

Proof. Let N be a balancing number of the form px^2. So it must have the prime factorization $N = p\, p_1^{2a_1} p_2^{2a_2} \cdots p_m^{2a_m}$ for some natural number m. Since N is a balancing number, $N = B_n$ for some n. Further, $B_n = P_n Q_n$ and it is well-known that $(P_n, Q_n) = 1$ for each n. Thus, p is a factor of either P_n or Q_n. If $p|P_n$, then Q_n is a perfect square and by virtue of Lemma 3.1, this happens if $n = 1$ and consequently $N = 1$ which is not of the form px^2. On the other hand, if $p|Q_n$, then P_n is a perfect square and by Lemma 3.2, this is possible only if $n = 1$ or 7. In the former case $N = 1$, which is not of the form px^2 and in the latter case $N = 40391 = 239 \cdot 13^2$. □

As mentioned earlier, Ochem and Rao [7] proved that an odd perfect number, if exists, must be greater than 10^{1500}. Many great number theorists believe that odd perfect numbers do not exist. So how can one expect such a number in the balancing sequence? The following theorem ascertains their absence.

Theorem 3.4. *There is no odd perfect balancing number.*

Proof. Let N is an odd perfect number. Hence, it has the canonical decomposition $N = p^{4s+1} p_1^{2a_1} p_2^{2a_2} \cdots p_m^{2a_m}$ where $p \equiv 1 \pmod 4$ and $m \geq 9$ [8]. By virtue of Theorem 3.3, $B_7 = 40391 = 239 \cdot 13^2$ is the only balancing numbers of the form px^2. Since the prime $239 \not\equiv 1 \pmod 4$, 40391 is not a perfect number. Hence, no odd balancing number is perfect. □

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References

[1] A. Behera and G. K. Panda, *On the square roots of triangular numbers*, The Fibonacci Quarterly, **37.2** (1999), 98–105.
[2] T. Koshy, *Elementary Number Theory with Applications* Second ed., Academic Press, New York, 2007.
[3] W. Ljunggren, *Zur Theorie der Gleichung* $x^2 + 1 = Dy^4$, Avh. Norske, Vid. Akad. Oslo, **1.5** (1942), 5–27.
[4] F. Luca, *Perfect Fibonacci and Lucas numbers*, Rend. Cir. Mat. Palermo (Series II) XLIX, (2000), 313–318.
[5] W. L. McDaniel, *Triangular numbers in the Pell sequence*, The Fibonacci Quarterly, **34.2** (1996), 105–107.
[6] W. L. McDaniel, *On Fibonacci and Pell numbers of the form* kx^2*, (almost all terms has 4r+1 prime factor)*, The Fibonacci Quarterly, **40.1** (2002), 41–42.
[7] P. Ochem and M. Rao, *Odd perfect numbers are greater than* 10^{1500}, Math. Comp., **81.279** (2012), 1869–1877.
[8] P. N. Pace, *Odd perfect numbers have at least nine distinct prime factors*, Math. Comp., **76.260** (2007), 2109–2126.
[9] G. K. Panda, *Some fascinating properties of balancing numbers*, Cong. Numerantium, **194** (2009), 185–189.
[10] G. K. Panda and P. K. Ray, *Some links of balancing and cobalancing numbers with Pell and associated Pell numbers*, Bull. Inst. Math., Acad. Sinica (New Series), **6.1** (2011), 41–72.
[11] G. K. Panda, *Arithmetic progression of squares and solvability of the Diophantine equation* $8x^4 + 1 = y^2$, East-West J. Math., **1.2** (2012), 131–137.
[12] V. S. R. Prasad and B. S. Rao, *Triangular numbers in the associated Pell sequence and Diophantine equations* $x^2(x+1)^2 = 8y^2 \pm 4$, Indian J. Pure Appl. Math., **33.11** (2002), 1643–1648.
[13] P. K. Ray, *Balancing and cobalancing numbers*, Ph.D. Thesis, National Institute of Technology, Rourkela, 2009.
[14] N. Robbins, *On Fibonacci numbers of the form* px^2*, where* p *is prime*, The Fibonacci Quarterly, **21.4** (1983), 266–271.
[15] N. Robbins, *On Pell numbers of the form* px^2*, where* p *is prime*, The Fibonacci Quarterly, **22.4** (1984), 340–348.

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