ENUMERATING DISTINCT CHESSBOARD TILINGS

DARYL DEFORD

Abstract. Counting the number of distinct colorings of various discrete objects, via Burnside's Lemma and Pólya Counting, is a traditional problem in combinatorics. Motivated by a method for proving upper bounds on the order of the minimal recurrence relation satisfied by a set of tiling instances, we address a related problem in a more general setting. Given an $m \times n$ chessboard and a fixed set of (possibly colored) tiles, how many distinct tilings exist, up to symmetry?

More specifically, we are interested in the sequences formed by counting the number of distinct tilings of boards of size $(m \times 1), (m \times 2), (m \times 3) \ldots$, for a fixed set of tiles and some natural number m. We present explicit results and closed forms for several well known classes of tiling problems as well as a general result showing that all such sequences satisfy some linear, homogeneous, constant–coefficient recurrence relation. Additionally, we give a characterization of all $1 \times n$ distinct tiling problems in terms of the generalized Fibonacci tilings.

1. Introduction

1.1. **Background.** Enumerating the number of ways to cover a rectangular chessboard with a fixed set of tiles is a motivating problem for many interesting recurrence relations and integer sequences. Many examples of these problems and their associated solution methods can be found in [6, 7, 12, 14, 15]. A complete and informative treatment of the one–dimensional case is contained in Benjamin and Quinn's wonderful book [2]. Often, restrictions are made on the types and orientations of the permissible tiles in order to model a particular combinatorial problem. For example, it is well known that the number of ways to tile a $1 \times n$ board with 1×1 squares and 1×2 dominoes is the n^{th} combinatorial Fibonacci number f_n, while generalized domino tilings have deep connections to questions in statistical mechanics [9, 13, 22].

The particular case when the tiles are restricted to be square was considered by Brigham et al. [3] and Hare [10]. In 1999, Heubach used the combinatorial method of counting indecomposable blocks to generalize these earlier results [11]. More recently, Calkin et al. showed that when the square tiles are restricted in dimension, the number of tilings can be calculated as the sum of the entries in the n^{th} power of a recursively defined matrix [4]. This solution is based on a method of Calkin and Wilf for counting grid tilings [5]. This problem is a specific case of the forbidden sub–matrix problem. Furthermore, Webb has shown that such problems always have a recurrence solution [23].

Date: April 20, 2015.

Key words and phrases. Tilings; Recurrence Relations; Integer Sequences.

Aside from their intrinsic interest and applications, tilings are also an effective combinatorial technique for proving identities. While most of the identities concerning combinatorial objects have straightforward proofs through mathematical induction or algebraic manipulation with Binet forms, these approaches do not provide intuition for the results or suggest avenues for further investigation. Thus, bijective proofs utilizing tilings and other combinatorial models are preferred. Indeed, many of the most common Fibonacci and Lucas identities have simple and elegant proofs using the $1 \times n$ tiling model mentioned above.

1.2. **Notation.** In this paper, the Fibonacci sequence will be indexed combinatorially as $f_0 = 1$ and $f_1 = 1$, in order to have a direct connection with the tiling interpretation. The primary object of interest in this paper are the sequences formed by counting the number of legitimate tilings of rectangular boards by some fixed sets of tiles. In particular, for any arbitrary fixed set of tiles T (note that we do not require that the tiles be connected) and fixed board height m we will let the sequence $\{T_n\}$ be the number of ways to tile a $m \times n$ board with tiles in T. More generally, we will also be interested in the collection of sequences $\{\{T_n\}^{(m)}\}$ as m ranges over the natural numbers. Throughout, d will represent the length of the longest tile in T.

We will frequently need to consider the number of ways to tile boards where some subset of the initial squares have been deleted. In these examples the set of tiles will be clear from context and we will use capital letters to represent the boards and lower case letters to represent the number of ways to tile the board (see Figure 1 in Section 2.1).

Following DeTemple and Webb, we will denote the successor operator on sequences by E. That is, for any sequence a_n we have $E(a_n) = a_{n+1}$. The successor operator offers an elegant way to express and prove many combinatorial identities [6]. Finally, throughout this paper the phrase "recurrence relation" will be used to refer to a linear, homogeneous, constant–coefficient recurrence relation.

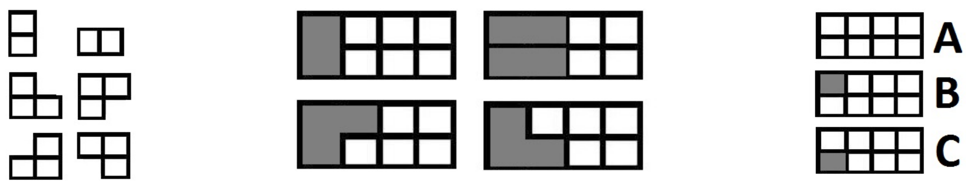
1.3. **Contributions.** In this paper we consider enumerating distinct tilings up to symmetry. These problems arise when trying to prove recurrence order bounds for standard tiling problems. We give a general formula for all $1 \times n$ tiling problems generalized from the standard Fibonacci tiling model. Finally, we show that for any fixed tile set T and number of rows m the sequence of distinct tilings of $m \times n$ boards satisfies a recurrence relation and give examples incorporating the Fibonacci numbers.

2. Tilings and Recurrence Relations

As discussed in [2], if we permit ourselves to consider weighted tilings with initial phases, we can realize any sequence satisfying a recurrence relation as tiling problem on a $1 \times n$ board. In 2004, Webb, Criddle, and DeTemple proved an interesting converse to this statement by showing that for any fixed set of tiles, T and any fixed board height, m, the sequence $\{T_n\}$ satisfies a recurrence relation, by conditioning on the number of ways to cover the leftmost column [24]. This proof and its generalizations rely on an algebraic lemma proved in [6] that any collection of arbitrary sequences that satisfy a homogeneous linear system in E are recurrent sequences annihilated by the determinant of that system.

Before proceeding, we provide a simple example using this methodology:

2.1. **Example: Tilings of a $2 \times n$ board with Dominoes and L–shaped Trominoes.** The tiles, endings, and necessary sub–boards are shown below in Figure 1:



(a) Tiles (b) Endings (c) Subboards

FIGURE 1. Figures for Example 1

Considering the number of ways to fill the initial column of board A, we see that we can either use one vertical domino, two horizontal dominoes, or an L–shaped tile in either orientation. The remaining boards, B and C, are simpler, because each tile may only be placed in one orientation. This leads to the following system of sequences:

$$a_n = a_{n-1} + a_{n-2} + b_{n-1} + c_{n-1} \tag{2.1}$$

$$b_n = c_{n-1} + a_{n-2} \tag{2.2}$$

$$c_n = b_{n-1} + a_{n-2}. \tag{2.3}$$

As an example of how these equations are obtained, consider (2.3). In order to tile a C board of length n we may either place a horizontal domino in the top row, leaving a B board of length $n-1$, or we may place a tromino that covers the remaining squares in the first two columns, leaving an A board of length $n-2$. Rewriting these as a linear system in E we obtain:

$$\begin{bmatrix} E^2 - E - 1 & -E & -E \\ -1 & E^2 & -E \\ -1 & -E & E^2 \end{bmatrix} \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{2.4}$$

The determinant of this matrix $E^2(E^4 - E^3 - 2E^2 - E - 1) = E^2(E+1)(E^3 - 2E^2 - 1)$, and indeed we can check the initial conditions, $T_1 = 1$, $T_2 = 2$, $T_3 = 5$, $T_4 = 11$ and $T_5 = 24$ to see that our desired sequence satisfies the recurrence relation $T_n = 2T_{n-1} + T_{n-3}$ corresponding to the irreducible cubic factor. This appears in the OEIS as A052980, although this tiling interpretation is not yet included [19]. In a sense that will be made precise later, the matrix obtained in (2.4) is typical of such problems. The fact that the determinant is a degree six polynomial highlights the important fact that this method does not always directly return the polynomial corresponding to the minimal recurrence relation satisfied by the sequence, which we will discuss in the next section.

3. Recurrence Order Bounds

3.1. **Motivation.** The order of a sequence is defined as the degree of the characteristic polynomial of the minimal recurrence relation that the sequence satisfies. As discussed in Chapter 7 of [6] knowing the order or even an upper bound for the order of a sequence can allow us to prove identities and results without actually computing the coefficients themselves. In the case of tiling problems, where all of the sequences we are interested in satisfy some recurrence relation, having a bound on the recurrence order is particularly valuable.

The mechanical method for proving identities contained in [6] shows that the upper bound on sequence order describes how many initial conditions are necessary to compute in order to prove a desired identity. Additionally, it is possible compute the coefficients from initial conditions by solving a simple linear system of size equal to twice the order bound. Thus, providing a better upper bound limits the amount of computation necessary to make use of a particular tiling model. This is particularly important, because it has been shown that enumerating the number of tilings can be #P–complete in some cases [20].

3.2. **Sequences of Sequences.** For a given tile set T, we can form a family of sequences $T_n^{(m)}$, each of which satisfies some recurrence relation, by letting the number of rows, m, range over the positive integers. It is natural to investigate the relationships between these sequences. For example, matrix methods of Calkin and Wilf [5] as well as those of Anderson [1], show that for some fixed sets of tiles, the recurrence relations can be calculated for any n by constructing a particular recursively constructed matrix. Similarly, families of tilings with dominoes or with the tiles restricted to be square can generate divisibility sequences [25].

In this paper, we are particularly interested in the growth rate of the order of the sequences. That is, let $\mathcal{O}(T_n)$ be the order of the minimal recurrence relation that T_n satisfies. Then, we can construct a sequence $\{\mathcal{O}(T_n^{(m)})\}$ of these orders, and in particular, consider the growth rate of the sequence. As discussed previously, this measure provides important information about the sequence without excessive computation.

3.3. **Trivial Bounds.** The proof that every tiling sequence satisfies a recurrence relation proceeds by constructing the characteristic polynomial of such a recurrence relation, as the determinant of a matrix whose entries are polyomials in E. As noted above, this recurrence relation is rarely minimal, but does provide an upper bound on the order. In order to compute this bound in general, we can consider the number of rows in the matrix and the maximum degree in E of the entries in each row.

Consider the case for a fixed number of rows m and maximum tile length d. Because we are considering arbitrary, possibly disconnected tiles, there are $(2^m-1)2^{m(d-1)}$ legitimate tiles to choose from, and a maximum of $2^{m(d-1)}$ states remaining of the board after the initial column is tiled. Thus, our matrix could have up to $2^{m(d-1)}+1$ rows, one for each board state and one for the initial board. Each of these rows can have exponent at most d, which can always be achieved along the main diagonal when all of the tiles are used. The product of the main diagonal entries is one summand of the determinant and thus, we obtain our first trivial upper bound on $\mathcal{O}(T_n^{(m)}) \leq d2^{m(d-1)+1} \sim \mathsf{O}(d2^{md})$.

There is some additional structure of the constructed matrix that can be used to reduce this bound. For example, even in the worst case where T contains all $(2^m-1)2^{m(d-1)}$ tiles, the E^d factors will only occur along the main diagonal and the only polynomials with non–zero constant term will appear in the initial column as in the matrix in (2.4). Hence, expanding down the initial column shows that there will be extraneous factors of E corresponding to sequence eigenvalues of 0 that may be discarded. Moreover, some of the states are translates of each other, and could thus be combined in order to further reduce the order. However, these improvements do not significantly impact the asymptotic behavior of the upper bound.

In general, this bound grows much too fast to be useful either combinatorially or computationally. For example, even for tilings with dominoes and squares the bound grows like $2 \cdot 2^{m(2-1)} = 2^{m+1}$. However, the actual recurrence orders are much smaller, as can be seen in Table 1 below. Thus, the trivial bound obtained from the proof is too inefficient for practical use.

m	OEIS	Upper Bound	Observed Order
1	A000045	4	2
2	A030186	8	3
3	A033506	16	6
4	A033507	32	9
5	A033508	64	20
6	A033509	127	36

Table 1. Enumerating tilings with squares and dominoes. The data in column 4 is from the OEIS [19]. Most of the computations were performed by Lundow [16]. The observed orders may not be minimal in all cases.

4. Motivating Example

In this section, we present a simple and well–studied counting problem as a case study suggesting some approaches to obtaining more reasonable recurrence order bounds for fixed sets of tiles. For the remainder of this section T will consist of 1×1 and 2×2 squares with m arbitrary. We will let A_n be the whole $m \times n$ board and hence the sequence a_n is equivalent to the desired sequence T_n. It is well known that the number of ways to tile a $2 \times n$ strip with 1×1 and 2×2 squares is equal to f_n [7]. Thus, there are f_m possible beginnings for a tiling of A_n.

This implies that the associated successor matrix has size bounded by $f_m \times f_m$. The maximum exponent of E in each row is one, except for the row corresponding to a_n which has a quadratic term from the all 2×2 tiling, balanced by the all 1×1 ending which is counted by a_{n-1}. Combined, this analysis provides us with f_m as an upper bound on the order of the recurrence. This is an asymptotic improvement, since $f_m \sim \varphi^m$. Moreover, we note that we can further restrict the size of the matrix by only considering the distinct endings up to symmetry. This was also true in Example 1, as the sequences b_n and c_n are clearly identical as C can be obtained from B by a reflection.

Thus, we need to compute the number of distinct Fibonacci tilings, relying on Burnside's Lemma[1] (see Theorem 8.7 in [6]). The next result is a specific case of the general formula presented in the next section, with $a_1 = a_2 = 1$ and $a_j = 0$ for $j > 2$. Similarly, Lemma 4.3 corresponds to $a_1 = 0, a_2 = a_3 = 1$ and $a_j = 0$ for $j > 3$.

Lemma 4.1. *The number of distinct Fibonacci tilings of order n up to symmetry is equal to $\frac{1}{2}(f_{2k} + f_{k+1})$ when $n = 2k$ and $\frac{1}{2}(f_{2k+1} + f_k)$ when $n = 2k + 1$.*

Proof. Let $n = 2k$ and consider the tilings of an $1 \times n$ board with squares and dominoes. Any reflection of a tiling across the line of symmetry between the k^{th} and $(k+1)^{st}$ squares produces another legitimate tiling. However, some tilings are self–similar under reflection, hence we cannot simply take $\frac{1}{2}f_n$ as our answer. The number of self–similar tilings can be computed by considering that the line of symmetry may either be covered by a domino, or uncovered. There are f_{k-1} self–similar tilings whose center tile is a domino and f_k self–similar tilings where the line of symmetry is uncovered.

Thus, there are $f_{k-1} + f_k = f_{k+1}$ self–similar tilings. Figure 2 shows examples of tilings with this property. By adding this quantity to the total number of tilings of length n, we have exactly twice the number of distinct classes of tilings up to symmetry. Hence, the number of classes of tilings is $\frac{1}{2}(f_{2k} + f_{k+1})$ and this case is complete.

When $n = 2k + 1$ we can apply a similar argument. In this case however, the line of symmetry passes directly through the $(k+1)^{st}$ square and thus must be covered by a square to create a self–similar tiling. Hence, there are exactly f_k self–similar tilings, and by applying Burnside's lemma as above we see that there must be exactly $\frac{1}{2}(f_{2k+1} + f_k)$ distinct tilings which completes the proof. $\square$

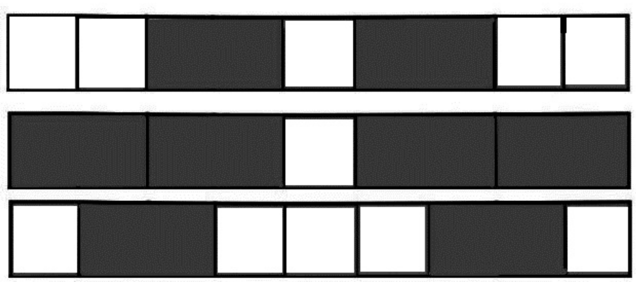


FIGURE 2. Self–Similar Fibonacci Tilings

The number of distinct classes of tilings provides a better bound on the order of our recurrence by limiting the number of rows in our successor operator matrix. However, we can offer another improvement by noticing that any ending that contains no consecutive 1×1 squares has exactly as many remaining tilings as a_{n-2} since the remaining un–tiled squares in the second column must also be covered by 1×1 squares. This implies that we can subtract the number of such endings, since they do not need to be represented

[1]Or rather, the lemma that is not Burnside's [26].

in our successor matrix. The number of endings that satisfy this condition is given by P_{n+2}, where P_n is the n^{th} Padovan number, which counts the number of ways to tile a $1 \times n$ board with 1×2 dominoes and 1×3 trominoes, satisfying the recurrence relation $P_n = P_{n-2} + P_{n-3}$. More interpretations of the Padovan sequence are given in the OEIS as sequence A000931 [19]. We prove this statement as the following lemma.

Lemma 4.2. *The number of endings with no consecutive* 1×1 *tiles is equal to* P_{n+2}.

Proof. We may construct a bijection between endings and tilings by associating every 2×2 square followed by a 1×1 square with a tromino in the Padovan tiling, while each 2×2 square not followed by a 1×1 square is associated with a domino. Then, since we need to count separately the cases when the tiling begins with a square or a domino, we have that the number of endings with no consecutive 1×1 squares is equal to $P_n + P_{n-1} = P_{n+2}$, by the Padovan recurrence. This completes our proof. □

Thus, we may subtract the number of distinct Padovan tilings from our previous bound to obtain a better order approximation. In order to calculate the number of distinct Padovan tilings we follow the methodology introduced in Lemma 1.

Lemma 4.3. *The number of distinct Padovan tilings of order* n *up to symmetry is equal to* $\frac{1}{2}(P_{2k} + P_{k+2})$ *when* $n = 2k$ *and* $\frac{1}{2}(P_{2k+1} + P_{k-1})$ *when* $n = 2k + 1$.

Proof. We may argue as in Lemma 4.1. Notice that we again have exactly one odd length and one even length tile, so the cases proceed exactly as in Lemma 4.1. Replacing the square by a tromino gives a third order recurrence, which now satisfies the defining relation of the Padovan numbers. It is then a straightforward calculation to verify the result.

□

The preceding discussion suffices to prove the following theorem:

Theorem 4.4. *The number of tilings of an* $m \times n$ *chessboard with* 1×1 *and* 2×2 *squares when* m *is fixed and* n *varies is not greater than:*

$$\frac{1}{2}(f_{2k} + f_{k+1} - P_{2k+2} - P_{k+3}) + 1$$

when $m = 2k$, *and*

$$\frac{1}{2}(f_{2k+1} + f_k - P_{2k+3} - P_k) + 1$$

when $m = 2k + 1$.

Table 2 below shows the differences between the bound and the actual order of the computed recurrence for the first several cases. Neither of these sequences appear to be contained in the OEIS. The values in the table row labelled $\mathcal{O}(a_n)$ are the orders of recurrences given in the OEIS for the solutions of these problems [19]. Computing the order of recurrences for other sets of tiles can be done in a similar fashion. For any particular case, analyzing the symmetry classes of the tiling endings can lead to greatly improved upper bounds.

n	2	3	4	5	6	7	8	9	10
$\mathcal{O}(a_n)$	2	2	3	4	6	8	14	19	32
Bound	2	2	3	4	7	10	17	26	44

TABLE 2. Comparison between the derived bound and the actual order

This theorem demonstrates the usefulness of our contributions. Using the successor operator method we may bound the order of the recurrence for a tiling problem by decomposing its endings into separate smaller problems of determining the number of distinct tilings of a simpler tile set. Since every chessboard tiling problem has an associated recurrence relation this is a very general method. Figure 3 below shows the possible endings of a $5 \times n$ board grouped in rows by equivalence class.

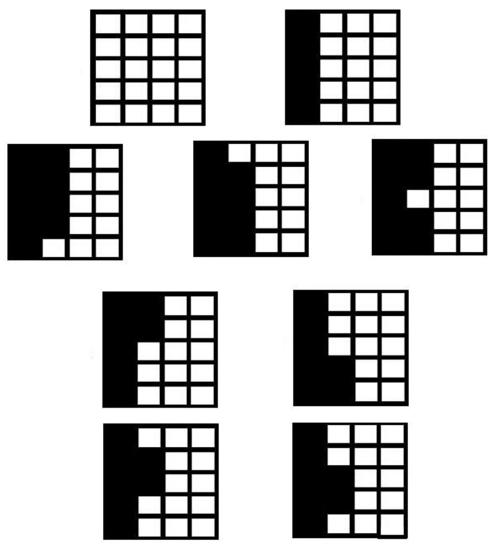


FIGURE 3. The $5 \times n$ endings

5. One Dimensional Tilings

In this section we provide a complete characterization of the number of distinct tilings of a $1 \times n$ rectangle with colored tiles of fixed lengths. We also use the Pòlya Enumeration Theorem (see Theorem 8.15 in [6]) to prove a similar result for $1 \times n$ bracelet tilings.

5.1. **Generalized Fibonacci Tilings.** Tilings of $1 \times n$ rectangles have been inextricably linked to the Fibonacci numbers by Benjamin and Quinn's classic book [2]. They give an interpretation of every (positive) linear homogeneous constant coefficient recurrence relation in terms of a generalization of the standard Fibonacci tiling model. Here, we prove a complementary theorem counting the number of distinct tilings for any possible collection of colors and tiles.

We begin by defining some convenient notation. Since we are covering boards of dimension $1 \times n$ we will consider tiling sets consisting of colored k–dominoes. We will represent the tile set as a vector, $T = (a_1, a_2, a_3, \ldots)$, where a_k represents the number of distinctly colored k–length dominoes available. Further, we will let T_n represent the number of ways to tile a $1 \times n$ rectangle with the tiles in T.

Finally, let α_j represent the number of self–symmetric $1 \times n$ T–tilings where the line of symmetry is covered by a j–domino. This gives the following piecewise definition for a_j:

$$a_j = \begin{cases} T_{\frac{n-j}{2}} & j \equiv n \equiv 0 \pmod 2 \\ 0 & j \equiv 0, n \equiv 1 \pmod 2 \\ 0 & j \equiv 1, n \equiv 0 \pmod 2 \\ T_{\frac{n-j}{2}} & j \equiv n \equiv 1 \pmod 2 \end{cases}. \tag{5.1}$$

Methods for calculating numerical values for T_k and by extension α_k are given in [2]. Now we may give the statement of our theorem.

Theorem 5.1. *Let T be any set of colored k–length dominoes. Then the number of distinct tilings up to symmetry of a $1 \times n$ rectangle is equal to*

$$\frac{1}{2}\left(T_n + \sum_{i=1}^{\infty} a_i \alpha_i + \frac{T_{\frac{n}{2}}}{2} + \frac{(-1)^n T_{\frac{n}{2}}}{2}\right). \tag{5.2}$$

Proof. We proceed again using Burnside's lemma. Since our board is one–dimensional the only symmetry we are concerned with is the reflection across the vertical line of symmetry. Notice that the set of tilings is closed under reflection which implies that it is sufficient to add the number of self–symmetric tilings to T_n to obtain the number of distinct tilings.

As in the proof of Lemma 1, we begin with the even case so let $n = 2k$. Since n is even the symmetry line falls between two units of our board. Thus, there are T_k tilings where the line is uncovered. This accounts for the final two terms in our sum. Additionally, it is easy to see that when n is odd these terms annihilate leaving us a single closed–form expression instead of a piecewise representation. This fits the combinatorial interpretation since when n is odd the line of symmetry bisects some unit square and must be covered by some tile.

Finally, for each j–length domino in T we must consider the case where the line of symmetry is covered by a tile of length j. These cases separately naturally into four parts, conditioning on the parity of j and n, as represented in Figure 4.

I) Both j and n are even:
In this case the line of symmetry must pass through the center of the j–domino. This leaves $\frac{j}{2}$ units covered in each half of the board. In order to construct a self–symmetric tiling, we must have both halves equivalent. Since there are no other restrictions on the tiling, there are $T_{\frac{n-j}{2}}$ such coverings and this case is complete.

II) When j is even and n is odd:
In this case, the center of the j–domino does not correspond to the line of symmetry. Hence, there can be no self–symmetric tilings with these conditions.

III) When j is odd and n is even:
As in case II we are unable to construct such a self–symmetric tiling since the domino covers a different number of squares on each half of the board.

IV) Both j and k are odd:
Here we may place the j–domino such that exactly $\frac{j-1}{2}$ squares are covered on each side. As in case I this implies that there are $T_{\frac{n-j}{2}}$ such coverings and no more.

Since for each j there are a_j colors, summing over $a_j\alpha_j$ for all $j \in \mathbb{N}$ counts all self–symmetric tilings where the line of symmetry is covered. Since all self–symmetric tilings have the line of symmetry either covered or uncovered, this completes the proof. $\square$

This result is particularly valuable in light of our work presented in the previous section. Notice, that to produce the bounds on our recurrence relation we only needed to apply this theorem twice, even though the number of rows, m, could be selected arbitrarily. This is because using the successor operator method, we need only consider the initial columns, and frequently a bijection can be constructed between tilings of the initial columns and colored $1 \times n$ tilings. Thus, this theorem is sufficient to provide recurrence order bounds on most traditional tiling problems.

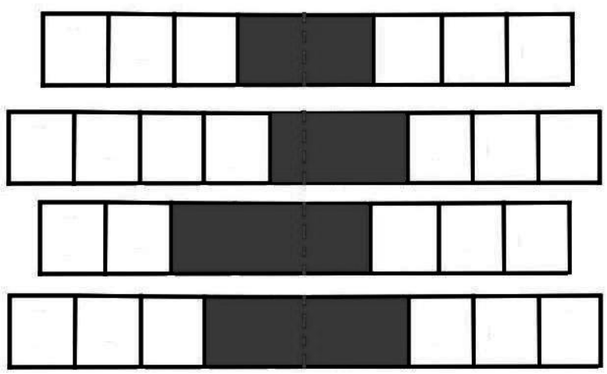


FIGURE 4. $1 \times n$ Self Symmetric Centers

5.2. **Distinct Lucas Tilings.** In addition to considering generalized Fibonacci relations, Benjamin and Quinn also provide a combinatorial interpretation of the Lucas numbers as tilings of a $1 \times n$ bracelet. We now show that the number of distinct Lucas tilings can be given by a number–theoretic formula, using the Pòlya Enumeration Theorem. The sequence generated by (5.3) occurs in the OEIS as A032190 [19].

Theorem 5.2. *The number of distinct Lucas tilings of a $1 \times n$ bracelet up to symmetry is:*

$$\sum_{i=0}^{\lceil \frac{n-1}{2} \rceil} \left[\frac{1}{n-i} \sum_{d|(i,n-i)} \varphi(d) \binom{\frac{n-i}{d}}{\frac{i}{d}} \right]. \tag{5.3}$$

Proof. In order to apply the Pòlya Enumeration Theorem, we must first calculate c_k for each bracelet B_n. Since the group acting on each bracelet is the n^{th} cyclic group we have that $c_k(B_n) = \varphi((n,k))$ elements of order k where φ represents the Euler totient function [21]. With this representation in hand, it follows that by the Pòlya Enumeration Theorem there are exactly

$$f(n,k) = \frac{1}{n} \sum_{d|(n,k)} \varphi(d) \binom{\frac{n}{d}}{\frac{k}{d}} \tag{5.4}$$

binary colorings of a n–bracelet with exactly k black units [14].

In order to enumerate the Lucas tilings we must condition on the number of dominoes in each tiling. Let each black unit in a distinct bracelet coloring represent a domino, and let each white unit in a distinct bracelet coloring represent a square. There can be at most $\lceil \frac{n}{2} \rceil$ dominoes in such a covering, since each domino covers two units. Replacing each domino with two squares, increases the number of available units by one up to n.

Each of these different combinations of tiles represents a unique distribution of the colors in a binary bracelet coloring of order $n-d$, with d representing the number of dominoes. Summing over all possible values for d gives:

$$\sum_{i=0}^{\lceil \frac{n}{2} \rceil} f(n-i, i). \tag{5.5}$$

Finally, substituting (5.4) for $f(n,k)$ gives the desired result completing this proof. $\square$

This result demonstrates the difficulties and complexities involved in employing the techniques of Burnside and Pòlya in more complex domains. While the number of distinct bracelet colorings has a convenient closed form expression [15], the techniques needed to catalog even the simplest cases of distinct Lucas tilings are much more significant. Consider extending Theorem 5.2 by adding curved trominoes to the tile–set. The resulting expression is a triple sum over multinomial coefficients. Similarly, adding colored dominoes or squares again increases the complexity of the expression exponentially.

6. Larger Rectangular Tilings

6.1. **Recurrence Relations for Distinct Tilings.** In this section we consider more generally the problem of enumerating the number of distinct tilings of an $m \times n$ chessboard. We prove a complementary result to the result of Webb et al. showing that every such sequence satisfies a recurrence relation and conclude with some examples of this method applied to some well known tiling problems.

Theorem 6.1. *Let T be a fixed set of tiles with maximum length d, and $m > 0$ be a fixed number of rows. The sequence $\{D_n\}$ of distinct tilings of an $m \times n$ board satisfies a recurrence relation.*

Proof. Since our chessboards are rectangular, the group of symmetries is isomorphic to the Klein group. We will use the notation $G = \{e, h, v, r\}$, where e is the identity element, h and v represent the horizontal and vertical reflections respectively, and r is the 180° rotation. Then, letting e_n, h_n, v_n, and r_n be the number of tilings of a $m \times n$ board with tiles in T fixed by each respective group element, by Burnside's Lemma, we have that

$$D_n = \frac{1}{4}(e_n + h_n + v_n + r_n). \tag{6.1}$$

Since the set of all sequences satisfying some recurrence relation is a vector space, any finite linear combination of such sequences also satisfies a recurrence relation. Thus, it suffices to show that e_n, h_n, v_n, and r_n are all recurrent sequences. Note that this together with Theorem 5.1 imply the case for $m = 1$ since the even and odd cases each separately are a finite linear combination of recurrent sequences (the T_i). In the general case, the theorem of Webb guarantees that e_n satisfies a recurrence relation since e fixes all T_n tilings.

We consider the remaining three cases in turn, following the idea in [24]. The case of h_n is simplest after the identity. Let $\mathcal{S}$ be the set of all possible boards formed from A by deleting some (possibly empty) collection of squares in the first $d-1$ columns and let $\mathcal{S}_h^*$ represent the corresponding sequences counting the number of ways to distinctly tile a $m \times n$ board with initial columns in $\mathcal{S}$. Note that we actually need only include those endings that are fixed under h in $\mathcal{S}$ since the corresponding sequences are 0 for all other endings.

For each board $B \in \mathcal{S}$ we may form a linear equation in E for b_n in terms of sequences in $\mathcal{S}_h^*$ by considering the number of distinct ways, up to symmetry, to tile the initial column of the board, since any such covering of the initial squares will leave another board in $\mathcal{S}$ of shorter length. Hence, each sequence in $\mathcal{S}_h^*$ (including h_n) can be represented as a linear combination in E of other sequences. Then, the determinant of this system is the characteristic polynomial of a recurrence that annihilates h_n.

We may proceed similarly for v and r, defining $\mathcal{S}_v^*$ and $\mathcal{S}_r^*$ to enumerate corresponding sequences counting the number of ways to distinctly tile a $m \times n$ board with initial columns in $\mathcal{S}$ fixed by v and r respectively. Again, by considering the number of ways to distinctly cover the initial column of each board in $\mathcal{S}$ we may form linear systems whose determinants give recurrences annihilating the sequences v_n and r_n. Hence, we have shown that D_n is a linear combination of sequences satisfying recurrence relations and so D_n must also be a recurrent sequence as desired.

□

6.2. **Examples.** We conclude this paper by presenting some simple, discrete examples of enumerating distinct two–dimensional tilings. These examples are meant to be representative of the solution methods necessary to approach more general problems.

6.2.1. *Tilings with Dominoes.* In this example we consider the distinct tilings of a $2 \times n$ rectangle with dominoes. Recall that the total number of ways to tile a $2 \times n$ rectangle with dominoes is f_n. For m up to 9 these distinct domino tiling values have been computed numerically by Mathar [17]. In [18], Mathar computes generating functions for several generalizations of this problem, including using larger dominoes and three dimensional tilings, using the transfer matrix method on a digraph constructed to represent possible endings. The sequence presented in the following example occurs in the OEIS as A060312 [19].

Proposition 6.2. *The number of distinct tilings of a $2\times n$ rectangle with 1×2 dominoes is*

$$\frac{1}{2}\left(f_{2k}+f_{k+1}\right) \tag{6.2}$$

when $n=2k$ and

$$\frac{1}{2}\left(f_{2k+1}+f_{k}\right) \tag{6.3}$$

when $n=2k+1$.

Proof. In order to apply Burnside's Lemma, we must count the number of elements fixed by each group action.

Since the identity element e fixes all tilings, it contributes f_n to the sum regardless of the parity of n. To see that h accounts for f_n regardless of parity, consider the bijection between $1\times n$ squares and dominoes and the Fibonacci recurrence [7]. Since applying h to a $2\times n$ board leaves a $1\times n$ board this is sufficient.

The last two group actions are parity dependent, so first let $n=2k$ and consider the actions of r and v. In both cases either the line of symmetry is covered by two horizontal dominoes or it is not covered at all. These observations add the final terms to the even case: $2f_k$ and $2f_{k-1}$ respectively. This completes the example when n is even. Figure 5 shows the symmetric centers under r and v for both parities.

When $n=2k+1$ is odd, under both v and r in order for a tiling to be self–similar the symmetric line must be covered by a single vertical domino leaving only $2f_k$ remaining tilings fixed by these actions. Since we have considered all of the elements of V and $|V|=4$ by Burnside's Lemma we have that the number of distinct tilings is equal to:

$$\frac{1}{4}\left(f_{2k}+f_{2k}+2f_k+2f_{k-1}\right) \tag{6.4}$$

when $n=2k$ and

$$\frac{1}{4}\left(f_{2k+1}+f_{2k+1}+2f_k\right) \tag{6.5}$$

when $n=2k+1$. Simplifying with the Fibonacci recurrence then gives the result. $\square$

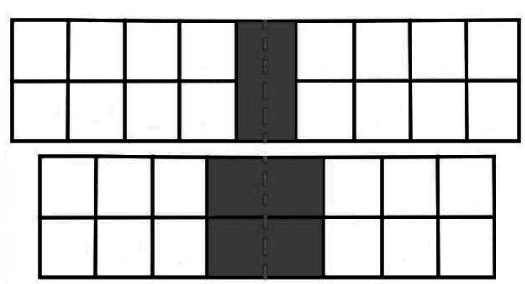


FIGURE 5. Legitimate Symmetric Centers for $2\times n$ Domino Tilings

6.2.2. *Tilings with Squares.* In this final example we extend the motivating example of Section 4, tiling with 1×1 squares and 2×2 squares.

Proposition 6.3. *The number of distinct tilings of a $3 \times n$ rectangle with squares of size 1×1 and 2×2 is*

$$\frac{1}{3}\left(2^{2n-1} + 2^n + 2^{n-1} + \frac{1+(-1)^n}{2}\right) \tag{6.6}$$

when n is odd, and

$$\frac{1}{3}\left(2^{2n} + 2^n + 2^{n-1} + 1\right) \tag{6.7}$$

when n is even.

Proof. Since our group of symmetric actions again has four elements, by Burnside's Lemma we need only compute the tilings that are fixed by each symmetric transformation. Using the notation of Heubach [11], let $T_{3,a}$ represent the number of traditional tilings of a $3 \times a$ rectangle.

The identity transformation fixes every tiling, which contributes a term of $T_{3,n}$. Similarly, the horizontal reflection fixes only the tiling with all 1×1 squares since any 2×2 square cannot be centered across the horizontal line of symmetry.

A rotation of $180°$ fixes exactly $T_{3,\lfloor \frac{n}{2} \rfloor}$ tilings since when n is odd the center column must be covered with 1×1 tiles and when n is even the center two columns must be covered with 1×1 tiles. If a 2×2 square infringes on one of these areas, it would overlap itself under r and hence cannot be self–symmetric.

The vertical line of symmetry separates the two parities. When $n = 2k + 1$ the symmetric line crosses the central units and must be covered by 1×1 squares contributing $T_{3,k}$ to the final sum. When $n = 2k$ is even the line of symmetry may be covered in one of two ways by a single 2×2 square or be surrounded but not covered by squares on both sides. These terms are $2T_{3,k-1}$ and $T_{3,k}$ respectively which completes the even case.

Applying Burnside's Lemma to these terms gives a representation of the number of tilings in terms of Heubach's recurrence relation:

$$\frac{1}{4}\left(T_{3,2k+1} + 1 + 2T_{3,k}\right) \tag{6.8}$$

when $n = 2k + 1$ and

$$\frac{1}{4}\left(T_{3,2n} + 1 + 2T_{3,n} + 2T_{3n+1}\right) \tag{6.9}$$

when $n = 2k$.

Constructing a generalized power sum for $T_{3,a}$ gives the following closed form expression [19],

$$T_{3,a} = \frac{2^{a+1} - (-1)^{a+1}}{3}. \tag{6.10}$$

Substituting (6.10) into (6.7) and (6.8) respectively gives the desired result and completes this example. $\square$

References

[1] P. Anderson: *Tiling Rectangles, Cylinders, and Mobius Strips*, Proceedings of the Fourteenth International Conference on Fibonacci Numbers and their Applications, (2010).

[2] A. Benjamin and J. Quinn: *Proofs that Really Count*, MAA, Washington D.C., 2003.

[3] R. Brigham, R. Caron, P Chinn, and R. Grimaldi: *A Tilings Scheme for the Fibonacci Numbers*, Journal of Recreational Mathematics, **28 (1)**, (1997), 10–17.

[4] N. Calkin, K. James, S. Purvis, S. Race, K. Schneider, and M. Yancey: *Counting Kings: Explicit Formulas, Recurrence Relations, and Generating Functions! Oh My!*, Congressus Numerantium **182**, (2006), 41–51.

[5] N. Calkin and H. Wilf: *The Number of Independent Sets in a Grid Graph*, SIAM J. Discrete Math, **11** (1998), 54–60.

[6] D. DeTemple and W. Webb: *Combinatorial Reasoning: An Interoduction to the Art of Counting*, Wiley, Hoboken, 2014.

[7] R. Graham, D. Knuth, and O. Patashnik: *Concrete Mathematics*, Addison-Wesley, Reading, 1994.

[8] F. Harary: *Determinants, Permanents and Bipartite Graphs*, Mathematics Magazine, **42(3)**, (1969), 146-148.

[9] F. Harary: *Graph Theory and Theoretical Physics*, Academic P., New York, (1967).

[10] E. Hare: *Tiling a $3 \times n$ Area with Cuisinaire Rods of Length Less Than or Equal to k*, Congressus Numerantium **105**, (1994), 34–55.

[11] S. Heubach: *Tiling an m–by–n Area with Squares of Size up to k–by–k ($m \leq 5$)*, Congrssus Numerantium **140**, (1999), 43–64.

[12] R. Honsberger: *In Pólya's Footsteps*, MAA, New York, 1997.

[13] P. Kasteleyn: *The Statistics of Dimers on a Lattice: I. The Number of Dimer Arrangements on a Quadratic Lattice*, Physica, **27(12)**, (1961), 1209–1225.

[14] J. van Lint and R. Wilson: *A Course in Combinatorics*, Cambridge University Press, Cambridge, 2001.

[15] N. Loehr: *Bijective Combinatorics*, CRC Press, Boca Raton, 2011.

[16] P. Lundow: *Computation of matching polynomials and the number of 1-factors in polygraphs*, Research Reports Umeå, **12**, (1996).

[17] R. Mathar: *Paving Rectangular Region With Rectangular Tiles: Tatami And Non-Tatami Tilings*, arXiv:1311.6135 [math.CO], (2013).

[18] R. Mathar: *Tilings of Rectangular Regions by Rectangular Tiles: Counts Derived from Transfer Matrices*, arXiv:1406.7788 [math.CO], (2014).

[19] OEIS Foundation Inc.: *The On-Line Encyclopedia of Integer Sequences*, http://oeis.org, (2012).

[20] I. Pak and J. Yang: *The complexity of generalized domino tilings*, Electron. J. Combin., 20(4):Paper 12, 23, (2013).

[21] A. Slomson: *An Introduction to Combinatorics*, Chapman and Hall, London, 1991.

[22] H. Temperley and M. Fischer: *Dimer Problem in Statistical Mechanics-An Exact Result*, Philosophical Magazine, **6(68)**, (1961), 1061–1063.

[23] W. Webb: *Matrices with Forbidden Submatrics*, Aportaciones Mathemàticas, **20**, (2011), 275–278.

[24] W. Webb, N. Criddle, and D. DeTemple: *Combinatorial Chessboard Tilings*, Congressus Numerantium **194**, (2004), 257–262.

[25] H. Williams and R. Guy: *Some Fourth–Order Linear Divisibility Sequences*, Int. J. Number Theory **07**, 1255 (2011).

[26] E. Wright *Burnside's lemma: A Historical Note*, Journal of Combinatorial Theory, Series B **30(1)**, (1981), 89–90.

Department of Mathematics, Dartmouth College, Hanover, NH, 03755, USA
E-mail address: `ddeford@math.dartmouth.edu`