# EXTENDED FIBONACCI ZECKENDORF THEORY

PETER G. ANDERSON

Abstract. We review the well-known and less well-known properties of the two-way infinite Fibonacci Zeckendorf array, namely every positive integer occurs exactly once in the right half (Zeckendorf's theorem), every integer occurs exactly once in a left portion with a ragged boundary (Bunder's theorem), and every pair of positive integers occurs as adjacent entries exactly once (Morrison's theorem). We refine the third statement and show how to locate the given pairs in the array.

## 1. The Extended Zeckendorf Array

Table 1 shows the upper left-hand corner of the infinite *Fibonacci Zeckendorf array* [4, 5, 6]. The top row consists of the Fibonacci numbers starting $1, 2, 3, \ldots$. Each subsequent row begins with the smallest positive integer that has not yet appeared (the numbers in each row are strictly increasing, so this poses no problem). If a number has Zeckendorf representation [7] $\sum c_i F_i$, then the number to its right in the table is $\sum c_i F_{i+1}$. Each row in this array clearly follows the Fibonacci recurrence rule: $a_{i,j} = a_{i,j-1} + a_{i,j-2}$. It follows immediately that the infinite table contains every positive integer once and only once.

This array is closely connected to the well-known Theorem of Zeckendorf [7].

**Theorem 1.1.** *Every positive integer $n$ is uniquely a finite sum $\sum_{k\geq 2} c_k F_k$ with $c_k \in \{0,1\}$ and $c_k + c_{k+1} \leq 1$, for all $k$.*

Each row of Table 1 can be extended arbitrarily far to the left via *precurrence*: $a_{i,j} = a_{i,j+2} - a_{i,j+1}$. Table 2 shows a fragment of Table 1 precursed several columns. The unshaded right two columns in Table 2 are the initial two columns of Table 1. The unshaded left portion of Table 2 corresponds to those numbers expressed in the table as sums consisting only of Fibonacci numbers with negative subscripts. Bunder [3] showed that every non-zero integer has a unique representation as a sum of Fibonacci numbers with negative subscripts, no two consecutive. He also provided an algorithm to produce such sums.

The present paper deals with *Extended Fibonacci Zeckendorf (EZ)* representations in which we express integers as sums of non-consecutive Fibonacci numbers without restriction on the signs of the subscripts. It is easy to see that without some rules there are an infinite number of ways to express any integer (if $k$ is even $F_k = -F_{-k}$, so there are an infinite number of representations of zero).

The **Main Result** (Theorem 2.11) is that given any pair of positive integers, $a$ and $b$, there is an Extended Zeckendorf representation $a = \sum c_k F_k$ such that $b = \sum c_k F_{k+1}$. Section 3 discusses methods of determining the $\{c_k\}$. (Theorem 2.11 is slightly stronger than stated here.)

Table 1. The Fibonacci Zeckendorf array.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 |
| 4 | 7 | 11 | 18 | 29 | 47 | 76 | 123 | 199 | 322 |
| 6 | 10 | 16 | 26 | 42 | 68 | 110 | 178 | 288 | 466 |
| 9 | 15 | 24 | 39 | 63 | 102 | 165 | 267 | 432 | 699 |
| 12 | 20 | 32 | 52 | 84 | 136 | 220 | 356 | 576 | 932 |
| 14 | 23 | 37 | 60 | 97 | 157 | 254 | 411 | 665 | 1076 |
| 17 | 28 | 45 | 73 | 118 | 191 | 309 | 500 | 809 | 1309 |
| 19 | 31 | 50 | 81 | 131 | 212 | 343 | 555 | 898 | 1453 |
| 22 | 36 | 58 | 94 | 152 | 246 | 398 | 644 | 1042 | 1686 |
| 25 | 41 | 66 | 107 | 173 | 280 | 453 | 733 | 1186 | 1919 |
| 27 | 44 | 71 | 115 | 186 | 301 | 487 | 788 | 1275 | 2063 |
| 30 | 49 | 79 | 128 | 207 | 335 | 542 | 877 | 1419 | 2296 |
| 33 | 54 | 87 | 141 | 228 | 369 | 597 | 966 | 1563 | 2529 |
| 35 | 57 | 92 | 149 | 241 | 390 | 631 | 1021 | 1652 | 2673 |
| 38 | 62 | 100 | 162 | 262 | 424 | 686 | 1110 | 1796 | 2906 |
| 40 | 65 | 105 | 170 | 275 | 445 | 720 | 1165 | 1885 | 3050 |
| 43 | 70 | 113 | 183 | 296 | 479 | 775 | 1254 | 2029 | 3283 |

Table 2. The precursed Fibonacci Zeckendorf array. The unshaded portion on the right repeats the first two columns of the array of Table 1. The unshaded portion on the left contains the numbers represented by sums of only negatively subscripted Fibonacci numbers.

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 89 | -55 | 34 | -21 | 13 | -8 | 5 | -3 | 2 | -1 | 1 | 0 | 1 | 1 | 2 |
| 123 | -76 | 47 | -29 | 18 | -11 | 7 | -4 | 3 | -1 | 2 | 1 | 3 | 4 | 7 |
| 68 | -42 | 26 | -16 | 10 | -6 | 4 | -2 | 2 | 0 | 2 | 2 | 4 | 6 | 10 |
| 102 | -63 | 39 | -24 | 15 | -9 | 6 | -3 | 3 | 0 | 3 | 3 | 6 | 9 | 15 |
| 136 | -84 | 52 | -32 | 20 | -12 | 8 | -4 | 4 | 0 | 4 | 4 | 8 | 12 | 20 |
| 81 | -50 | 31 | -19 | 12 | -7 | 5 | -2 | 3 | 1 | 4 | 5 | 9 | 14 | 23 |
| 115 | -71 | 44 | -27 | 17 | -10 | 7 | -3 | 4 | 1 | 5 | 6 | 11 | 17 | 28 |
| 60 | -37 | 23 | -14 | 9 | -5 | 4 | -1 | 3 | 2 | 5 | 7 | 12 | 19 | 31 |
| 94 | -58 | 36 | -22 | 14 | -8 | 6 | -2 | 4 | 2 | 6 | 8 | 14 | 22 | 36 |
| 128 | -79 | 49 | -30 | 19 | -11 | 8 | -3 | 5 | 2 | 7 | 9 | 16 | 25 | 41 |
| 73 | -45 | 28 | -17 | 11 | -6 | 5 | -1 | 4 | 3 | 7 | 10 | 17 | 27 | 44 |
| 107 | -66 | 41 | -25 | 16 | -9 | 7 | -2 | 5 | 3 | 8 | 11 | 19 | 30 | 49 |
| 141 | -87 | 54 | -33 | 21 | -12 | 9 | -3 | 6 | 3 | 9 | 12 | 21 | 33 | 54 |
| 86 | -53 | 33 | -20 | 13 | -7 | 6 | -1 | 5 | 4 | 9 | 13 | 22 | 35 | 57 |
| 120 | -74 | 46 | -28 | 18 | -10 | 8 | -2 | 6 | 4 | 10 | 14 | 24 | 38 | 62 |
| 65 | -40 | 25 | -15 | 10 | -5 | 5 | 0 | 5 | 5 | 10 | 15 | 25 | 40 | 65 |
| 99 | -61 | 38 | -23 | 15 | -8 | 7 | -1 | 6 | 5 | 11 | 16 | 27 | 43 | 70 |

Table 3. Bergman's $\phi$nary representation of some small integers. The $\bullet$ is the analog of a traditional radix point. Columns are labeled with powers of $\phi$ and also with Fibonacci numbers.

| | 5 | 3 | 2 | 1 | 1 | $\bullet$ | 0 | 1 | -1 | 2 |
|---|---|---|---|---|---|---|---|---|---|---|
| | $\phi^4$ | $\phi^3$ | $\phi^2$ | $\phi$ | 1 | $\bullet$ | $\phi^{-1}$ | $\phi^{-2}$ | $\phi^{-3}$ | $\phi^{-4}$ |
| 1: | | | | | 1 | $\bullet$ | | | | |
| 2: | | | | 1 | 0 | $\bullet$ | 0 | 1 | | |
| 3: | | | 1 | 0 | 0 | $\bullet$ | 0 | 1 | | |
| 4: | | | 1 | 0 | 1 | $\bullet$ | 0 | 1 | | |
| 5: | | 1 | 0 | 0 | 0 | $\bullet$ | 1 | 0 | 0 | 1 |
| 6: | | 1 | 0 | 1 | 0 | $\bullet$ | 0 | 0 | 0 | 1 |
| 7: | 1 | 0 | 0 | 0 | 0 | $\bullet$ | 0 | 0 | 0 | 1 |
| 8: | 1 | 0 | 0 | 0 | 1 | $\bullet$ | 0 | 0 | 0 | 1 |
| 9: | 1 | 0 | 0 | 1 | 0 | $\bullet$ | 0 | 1 | 0 | 1 |
| 10: | 1 | 0 | 1 | 0 | 0 | $\bullet$ | 0 | 1 | 0 | 1 |
| 11: | 1 | 0 | 1 | 0 | 1 | $\bullet$ | 0 | 1 | 0 | 1 |

## 2. Extended Zeckendorf representations

Bergman [2] introduced the representation of non-negative integers using the irrational base $\phi = \frac{1+\sqrt{5}}{2}$ where

$$n = \sum_{-\infty}^{\infty} c_k \phi^k \tag{2.1}$$

is a finite sum (i.e., Laurent polynomial) with $c_k \in \{0, 1\}$ and $c_k + c_{k+1} \leq 1$, for all $k$, as in the Zeckendorf and Bunder representations above. The relation

$$\phi^{k+1} = \phi^k + \phi^{k-1}, \text{ for all } k \tag{2.2}$$

yields carrying-and-borrowing rules for this notation. This is also known as the $\phi$nary number system.

The extension of Bergman's results to $\phi$nary representations of positive numbers $b\phi + a$, $a$ and $b$ integers, will yield our main result.

The process of determining $\phi$nary representations is based on two observations of Bergman's. (In the following observations, $n$ is a positive integer.)

**Observation 2.1.** *If there is a finite sum $n = \sum c_k \phi^k$ with $c_k \in \{0, 1\}$, for all $k$, then there is a finite sum $n = \sum d_k \phi^k$ with $d_k \in \{0, 1\}$, and $d_k + d_{k+1} \leq 1$, for all $k$.*

**Observation 2.2.** *If there is a finite sum $n = \sum c_k \phi^k$ with $c_k \in \{0, 1\}$, for all $k$, then there is a finite sum $n = \sum d_k \phi^k$ with $d_k \in \{0, 1\}$, for all $k$, and $d_0 = 0$.*

These are proved by straightforward applications of the carrying-and-borrowing principle.

Bergman's theorem, that every non-negative integer has a representation as in Equation (2.1), follows by induction from these two observations, starting with $\phi^0 = 1$. Table 2 shows the $\phi$nary representation of integers 1–11 written to mimic binary representation. Each column is labeled with the $\phi^k$ and also with $F_{k+1}$ per Proposition 2.5.

For Proposition 2.4, we need a third observation corresponding to Bergman's two:

**Observation 2.3.** *If there is a finite sum $n = \sum c_k \phi^k$, with $c_k \in \{0,1\}$, for all $k$, then there is a finite sum $n = \sum d_k \phi^k$ with $d_k \in \{0,1\}$, for all $k$, and $d_0 = 1$.*

**Proposition 2.4.** *Bergman's $\phi$nary representation of non-negative integers is unique.*

*Proof.* Bergman's mechanism of adding 1 to the representation of $n$ to get the representation of $n+1$ can be easily reversed using Observation 2.3, along with the trivial observation that for positive $n = \sum c_k \phi^k$ (the Bergman representation), for at least one $k \geq 0$, $c_k = 1$. □

**Proposition 2.5.** *When $n = \sum c_k \phi^k$ is Bergman's $\phi$nary representation of $n$ then*

$$n = \sum c_k F_{k+1} \tag{2.3}$$

*Proof.* Bergman's mechanism of adding $\phi^0 = 1$ to the representation of $n$ to get the representation of $n+1$ is unchanged when the powers of $\phi$ are replaced by the Fibonacci numbers. We replace $\phi^0$ with $F_1$ and, generally, $\phi^k$ with $F_{k+1}$. (However, it is important to regard numbers such as $\cdots, F_2, F_1, F_0, F_{-1}, \cdots$ as Fibonacci numbers with generic subscripts, $\cdots, F_{k+2}, F_{k+1}, F_k, F_{k-1}, \cdots$.) □

We call Eq. (2.3) the *Bergman-Zeckendorf (BZ) representation* of $n$.

**Proposition 2.6.** *If $n = \sum c_k F_{k+1}$ is the BZ representation of $n$, then $0 = \sum c_k F_k$.*

*Proof.* Replace the notion of repeatedly adding $F_1 = 1$ in the proof of Proposition 2.5 by adding $F_0 = 0$. □

In Proposition 2.6, we have Bergman's representation of $\phi^{-1}$ corresponding to an EZ representation of zero. We exploit this below.

**Proposition 2.7.** *The EZ representations of zero given in Theorem 2.6 are the* only *EZ representations of zero.*

*Proof.* Suppose $0 = \sum c_k F_k$ is an EZ, specifically a finite sum. For sufficiently large $m$, the number $u_m = \sum c_k F_{k+m}$ will be positive, because the Fibonacci numbers $\{F_{k+m}\}$ must all be positive. Consequently, the sequence $\{u_m\}_{m=0}^{\infty}$, which obeys the Fibonacci recurrence, must satisfy $u_1 = u_2 > 0$, and EZ representation we have for $u_1$ must be its BZ representation. □

The following propositions extend Bergman's representation to numbers of the form $b\phi + a$. In the following, $u_0$ and $u_1$ are integers, and $u_{n+1} = u_n + u_{n-1}$ for all $n$.

**Proposition 2.8.** *If $u_1\phi + u_0 > 0$, there exists $n_0$ such that for any $n \geq n_0$ we have $u_n > 0$.*

*Proof.* Use the matrix form of the Fibonacci recurrence, $(u_{n+1}, u_n) = (u_n, u_{n-1}) \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. The positive eigenvalue of the matrix, $(1+\sqrt{5})/2$, corresponds to the eigenvector $(\phi, 1)$. Consequently, if the scalar product $u_1\phi + u_0 = (u_1, u_0) \cdot (\phi, 1) > 0$ and $n$ is sufficiently large, then the components of the vector$(u_{n+1}, u_n) = (u_1, u_0) \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n$ will be positive. □

**Proposition 2.9.** *We have $(u_1\phi + u_0)\phi^n = u_{n+1}\phi + u_n$.*

*Proof.* This follows immediately from $\phi^2 = \phi + 1$. □

**Proposition 2.10.** *If $b\phi + a > 0$, $a$ and $b$ integers, there is a $\phi$nary representation of $b\phi + a$.*

*Proof.* Let $B\phi + A = (b\phi + a)\phi^n = b\phi^{n+1} + a\phi^n$ and $n$ be large enough so $A > 0, B > 0$. $B\phi + A$ has $\phi$nary representation $\sum c_k \phi^k$. Consequently, $b\phi + a = \sum c_{k-n}\phi^k$. □

We may now give the refined statement and proof of our main result.

**Theorem 2.11.** *If $b\phi+a > 0$, $a$ and $b$ integers, there is an Extended Zeckendorf representation $a = \sum c_k F_k$ such that $b = \sum c_k F_{k+1}$.*

*Proof.* Because $\phi$nary representations of $\phi^{-1}$ correspond to EZ representations of zero, and $F_1 = F_{-1} = 1$, the desired coefficients are in the $\phi$nary representation $b\phi^{-1} + a\phi^{-2} = \sum c_k \phi^k$. This can be achieved by a minor modification of the above. □

## 3. Illustrations and Finding the Coefficients

Below are terms of sequences defined using the Fibonacci recurrence with initial values $(a, b)$ of $(6, 5)$ and $(5, 6)$. The initial numbers in each list are expressed as sums of Fibonacci numbers using the usual Zeckendorf expansion. Eventually, at $(27, 43)$ in the left example, $(17, 28)$ in the right, the expansion of the second value in the pair is clearly the Fibonacci shift of the first. From that point on, the list reverses (using precursion) back to the starting pair of values. During this reversal, the Fibonacci numbers in the right-hand summations are also precursed, leading to the desired expansion of 5 as the shift of 6 and vice versa.

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 6 | = | 1 | + | 5 | | | | 5 | = | 5 | | | | |
| 5 | = | 5 | + | | | | | 6 | = | 1 | + | 5 | | |
| 11 | = | 3 | + | 8 | | | | 11 | = | 3 | + | 8 | | |
| 16 | = | 3 | + | 13 | | | | 17 | = | 1 | + | 3 | + | 13 |
| 27 | = | 1 | + | 5 | + | 21 | | 28 | = | 2 | + | 5 | + | 21 |
| 43 | = | 1 | + | 8 | + | 34 | | 17 | = | 1 | + | 3 | + | 13 |
| 27 | = | 1 | + | 5 | + | 21 | | 11 | = | 1 | + | 2 | + | 8 |
| 16 | = | 0 | + | 3 | + | 13 | | 6 | = | 0 | + | 1 | + | 5 |
| 11 | = | 1 | + | 2 | + | 8 | | 5 | = | 1 | + | 1 | + | 3 |
| 5 | = | -1 | + | 1 | + | 5 | | | | | | | | |
| 6 | = | 2 | + | 1 | + | 3 | | | | | | | | |

Notice that the pairs of values $(6, 5)$ and $(5, 6)$ are in the shaded regions of Table 2.

The above may be thought of as an algorithm—admittedly inefficient—for locating the coefficients of Theorem 2.11.

A second—also inefficient—algorithm is to determine the $\phi$nary representation

$$b\phi^{-1} + a\phi^{-2} = \sum c_i \phi^i \tag{3.1}$$

The coefficients $\{c_i\}$ of Eq. 3.1 are again those of Theorem 2.11.

A third algorithm, in the spirit of the greedy change-making algorithm to find the usual Zeckendorf coefficients, is as follows.

We are given $(a, b)$ such that $a + b\phi > 0$. Iteratively, locate the largest $n$ such that $[(a, b) - (F_n, F_{n+1})] \cdot (1, \phi) \geq 0$ and replace $(a, b)$ with the difference $(a, b) - (F_n, F_{n+1})$. Terminate the algorithm when $(a, b) = (0, 0)$.

Below, we use this algorithm on our example pairs $(6, 5)$ and $(5, 6)$.

| | | | $(F_n$ | $F_{n+1})$ | | $(a$ | $b)$ | $a+b\phi$ |
|---|---|---|---|---|---|---|---|---|
| | | | | | | (6 | 5) | 14.09017 |
| (6 | 5) | - | (3 | 5) | = | (3 | 0) | 3.00000 |
| (6 | 5) | - | (5 | 8) | = | (1 | -3) | -3.85410 |
| (3 | 0) | - | (1 | 1) | = | (2 | -1) | 0.38197 |
| (3 | 0) | - | (1 | 2) | = | (2 | -2) | -1.23607 |
| (2 | -1) | - | (2 | -1) | = | (0 | 0) | 0.00000 |
| (2 | -1) | - | (-1 | 1) | = | (3 | -2) | -0.23607 |

| | | | $(F_n$ | $F_{n+1})$ | | $(a$ | $b)$ | $a+b\phi$ |
|---|---|---|---|---|---|---|---|---|
| | | | | | | (5 | 6) | 14.70820 |
| (5 | 6) | - | (3 | 5) | = | (2 | 1) | 3.61803 |
| (5 | 6) | - | (5 | 8) | = | (0 | -2) | -3.23607 |
| (2 | 1) | - | (1 | 1) | = | (1 | 0) | 1.00000 |
| (2 | 1) | - | (1 | 2) | = | (1 | -1) | -0.61803 |
| (1 | 0) | - | (1 | 0) | = | (0 | 0) | 0.00000 |
| (1 | 0) | - | (0 | 1) | = | (1 | -1) | -0.61803 |

## 4. Generalizations to Other Recurrences: Success and Failure

Now consider $k$-th order "generalized Fibonacci sequences" of the form $u_n = \sum_{i=1}^{k} R_i u_{n-i}$, starting with $k$ initial values $0, \ldots, 0, 1$.

Zeckendorf representations and arrays exist, for these sequences, as above. That is, the initial row of the array is the sequence $a_{i,j} = u_j$ suitable shifted so the first two elements are 1 and an integer larger than 1. Subsequent rows begin with the smallest number that has not yet appeared, with the elements of that row being Zeckendorf shifts of the first element. These arrays contain each positive integer exactly once. (Zeckendorf representations based on this recurrence are, as with the Fibonacci case, determined by the greedy change-making algorithm.)

The $k$-bonacci numbers for which $\{R_i\} = (1, 1, \ldots, 1)$ were addressed in [1], which proved the analogy of Theorem 2.11: a sequence of $k$ positive numbers $(a_1, \ldots, a_k)$ possesses a $k$-bonacci extended Zeckendorf representation for each $a_i$ such that the representation of $a_{i+i}$ is the shift of that of $a_i$, for all $1 \leq i < k$.

However, for the case of $\{R_i\} = (1, 0, \ldots, 0, 1)$ the $\phi$nary representation theory does not apply, starting with $k = 4$: $u_n = u_{n-1} + u_{n-4}$. This sequence begins

$$0, 0, 0, 1, 1, 1, 1, 2, 3, 4, 5, 7, 10, 14, 19, 26, 36, 50, 69, 95, 131, 181, 250, 345, 476, 657, 907, \ldots$$

The Bergman/Zeckendorf coefficients $\{c_i\}$ for this sequence require

- $c_i \in \{0, 1\}$.
- Every non-zero $c_i$ is preceded by and followed by at least three zeros.

There is no finite Bergman representation for 2 or $1+\phi$. That is, the sequences $\{2u_i\}$ and $\{u_i + u_{i+1}\}$ are not in the Zeckendorf array.

For the $k = 5$, the periodic sequence, $\mathcal{S}$, with period five,

$$0, 1, 1, 0, -1, -1, \ldots$$

satisfies $u_n = u_{n-1} + u_{n-5}$. The Zeckendorf array for this sequence, as usual, contains every positive integer exactly once. The sum of two sequences that satisfies a given recurrence will also satisfy that recurrence, so the sum of $\mathcal{S}$ with any row, $\mathcal{R}$, of the Zeckendorf array will be eventually positive, yet will contain infinitely many values in common with $\mathcal{R}$.

## References

[1] P. G. Anderson and C. Cooper, *Every Positive k-bonacci-like Sequences Eventually Agrees with a Row of the k-Zeckendorf Array* The Fibonacci Quarterly, **49.4** (2011) 303–309.

[2] G. Bergman, *A Number System with an Irrational Base* Mathematics Magazine **31.2** (1957) 98–110.

[3] M. W. Bunder, *Zeckendorf representations using negative Fibonacci numbers*, The Fibonacci Quarterly, **30.2** (1992), 111–115.

[4] C. Cooper, *The k-Zeckendorf array*, Presented at the Fourteenth International Conference on Fibonacci Numbers and Their Applications, Morelia, Mexico, July 2011.

[5] C. Kimberling, *The Zeckendorf array equals the Wythoff array*, Fibonacci Quarterly **33.1** (1995) 3–8.

[6] D. R. Morrison, *A Stolarsky array of Wythoff pairs*, A Collection of Manuscripts Related to the Fibonacci Sequence, 18th Anniversary Volume, V. E. Hoggatt, Jr. and Marjorie Bicknell-Johnson, eds., 1980, 134–136.

[7] E. Zeckendorf, *Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas*, Bull. Soc. Roy. Liege, **41** (1972), 179–182.

MSC2010: 11B39, 33C05

Computer Science Department, Rochester Institute of Technology, Rochester, NY 14531
*E-mail address*: pga@cs.rit.edu